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Solutions to exercise 4

Solution to exercise 4.1

1. Let $x_k \in B_1(0) \subseteq B$ such that

$$\|T(x_k)\|_V \to \|T\|_{B,V} = \sup_{x \in B_1(0)} \|T(x)\|_V.$$

Since x_k is a bounded sequence, the compactness of T yields a subsequence and an $y \in V$, such that

$$\infty > \|y\|_V \leftarrow \|T(x_k)\|_V.$$

Hence T is bounded and therefore continuous.

2. By Remark 2.9 the sequence $x_k \in B$ is bounded, hence we find a subsequence $x_k \in B$ and $y \in V$ with

$$\lim_{k\to\infty} T(x_k) = y.$$

Let $L \in V^*$. Then $L \circ T \in B^*$, because T is continuous. Hence the weak convergence yields

$$L \circ T(x_k) \to L \circ T(x).$$

Hence $T(x_k) \to T(x)$ weakly. Since norm convergence also induces weak convergence and weak limits are unique, we have

$$T(x) = y.$$

Solution to exercise 4.2

First we deal with the boundary data: Let $m(t) := (1-t)a + bt$. Then m is smooth and therefore $m \in W^{1,2}((0,1))$. Furthermore $m(0) = a$ and $m(1) = b$. Hence we say, that a function $u \in W^{1,2}(0,1)$ satisfies the boundary condition

$$u(0) = a \text{ and } u(1) = b,$$

if and only if

$$u - m \in W_0^{1,2}((0,1)).$$

Now we turn to the variational, i.e. weak formulation of the differential equation itself: Let $\varphi \in C_0^\infty(\Omega)$. Then if u solves $u'' + \sin(u) = 0$, we have by partial integration.

$$0 = \int_0^1 -u''\varphi - \sin(u)\varphi \, dt = \int_0^1 u'\varphi' - \sin(u)\varphi \, dt.$$

Hence we say that $u \in W^{1,2}((0,1))$ solves the boundary value problem weakly, if and only if for all $\varphi \in C_0^\infty(\Omega)$ we have

$$u - m \in W_0^{1,2}((0,1)) \text{ and } \int_0^1 u'\varphi' - \sin(u)\varphi \, dt = 0.$$

We now define an appropriate energy $E : W^{1,2}((0,1)) \to \mathbb{R}$ by

$$E(u) := \frac{1}{2} \int_0^1 (u')^2 \, dt + \int_0^1 \cos(u) \, dt.$$

We calculate for $u \in W^{1,2}((0,1))$ and $\varphi \in C_0^\infty((0,1))$

$$\begin{aligned}
&\frac{d}{ds} E(u + s\varphi)|_{s=0} = \int_0^1 u'\varphi' \, dt + \frac{d}{ds} \int_0^1 \cos(u + s\varphi) \, dt|_{s=0} \\
&= \int_0^1 u'\varphi' \, dt + \int_0^1 \frac{d}{ds} \cos(u + s\varphi)|_{s=0} \, dt \\
&= \int_0^1 u'\varphi' \, dt - \int_0^1 \sin(u)\varphi \, dt.
\end{aligned}$$

Interchanging the derivative and the integral is by

$$|\frac{d}{ds} \cos(u + s\varphi)| = |-\sin(u + s\varphi)\varphi| \le |\varphi|.$$

Since $\varphi \in L^1((0,1))$ the dominated convergence theorem allows this interchanging.

Now we show existence of a weak solution:

We define

$$M := \{u \in W^{1,2}((0,1)) |\; u - m \in W_0^{1,2}((0,1))\}.$$

Let $\lambda \in [0,1]$ and $u_1, u_2 \in M$. Then

$$\lambda u_1 + (1 - \lambda)u_2 - m = \lambda(u_1 - m) + (1 - \lambda)(u_2 - m) \in W_0^{1,2}((0,1)),$$

hence M is convex. Furthermore M is closed w.r.t. $\|\cdot\|_{W^{1,2}((0,1))}$ because if $u_k \in M$ with $u_k \to u$ in $W^{1,2}((0,1))$, then

$$u_k - m \in W_0^{1,2}((0,1)) \;\Rightarrow\; u - m \in W_0^{1,2}((0,1)),$$

since by Definition $W_0^{1,2}((0,1))$ is closed. Hence M is weakly closed by Thm. 2.18.

Now let $u_k \in M$ be a minimising sequence for E in M, i.e.

$$E(u_k) \to \inf_{v \in M} E(v).$$

This yields

$$C > E(u_k) = \int_0^1 |u'|^2 \, dt - \int_0^1 \cos(u_k) \, dt \ge \int_0^1 |u'|^2 \, dt - 1.$$

Hence

$$\|u_k - m\|_{W_0^{1,2}((0,1))} \le \|u_k\|_{W_0^{1,2}((0,1))} + \|m\|_{W_0^{1,2}((0,1))} < C.$$

Therefore we find a weakly convergent subsequence and a limit $\tilde{u} \in W_0^{1,2}((0,1))$, i.e.

$$u_k - m \to \tilde{u} \text{ weakly.}$$

Since m does not depend on k, we also have

$$u_k \to \tilde{u} + m =: u \in M \text{ weakly.}$$

By the Sobolev embedding we have that after extracting another subsequence, that

$$u_k \to u \text{ uniformely.}$$

Hence the lower semicontinuity of the norm and the continuity of $\cos(\cdot)$ yield

$$E(u) \le \liminf_{k\to\infty} E(u_k).$$

Hence u is a minimum.
Let $\varphi \in C_0^\infty((0,1))$. Then $u + s\varphi \in M$ for all $s \in \mathbb{R}$, because by definition we find $\varphi_k \in C_0^\infty((0,1))$ with

$$\varphi_k \to u - m \text{ in } W^{1,2}((0,1)).$$

Then $\varphi_k + s\varphi \in C_0^\infty(\Omega)$ and

$$\varphi_k + s\varphi \to u - m + s\varphi = u + s\varphi - m \in W_0^{1,2}((0,1)).$$

Hence $u + s\varphi \in M$ and the calculation above for the Euler-Lagrange equation applies. Hence u indeed solves our problem weakly.

Solution to exercise 4.3

1. Let $1 < q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, i.e.

   $$\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}.$$

   Then Hölders inequality in $\mathbb{R}^n$ yields

   $$\sum_{j=1}^n |x_j| = \sum_{j=1}^n 1 \cdot |x_j| \le \left(\sum_{j=1}^n 1^q\right)^{\frac{1}{q}} \left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}} = n^{\frac{p-1}{p}} \left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}}.$$

   Hence we have

   $$\left(\sum_{j=1}^n |x_j|\right)^p \le n^{p-1} \sum_{j=1}^n |x_j|^p.$$

2. The weighted inequality of arithmetic and geometric means is as follows: Let $x, y \ge 0$ and $w_1, w_2 \ge 0$ with $w := w_1 + w_2$. Then

   $$\frac{w_1 x + w_2 y}{w} \ge \left(x^{w_1} y^{w_2}\right)^{\frac{1}{w}}.$$

   This inequality can be shown with Jensens inequality and the concavity of the log.

Now we set $x = a^p$, $y = b^q$, $w_1 = \frac{1}{p}$ and $w_2 = \frac{1}{q}$. Then we have that $w = 1$ and therefore

$$\frac{w_1 x + w_2 y}{w} = \frac{1}{p} a^p + \frac{1}{q} b^q.$$

On the other hand we have

$$(x^{w_1} y^{w_2})^{\frac{1}{w}} = (a^p)^{\frac{1}{p}} (b^q)^{\frac{1}{q}} = ab$$

and we have

$$ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q.$$

Now we define

$$\tilde{a} := (\varepsilon p)^{\frac{1}{p}} a, \;\; \tilde{b} = (\varepsilon p)^{-\frac{1}{p}} b.$$

Then we have

$$ab = \tilde{a}\tilde{b} \leq \frac{1}{p} \tilde{a}^p + \frac{1}{q} \tilde{b}^q = \varepsilon a^p + \frac{1}{(\varepsilon p)^{\frac{q}{p}} q} b^q = \varepsilon a^p + \frac{(\varepsilon p)^{1-q}}{q} b^q.$$