Dr. S. Eichmann
Winter 2022/23

Solutions to exercise 5

Solution to exercise 5.1

We multiply the equation with a test function $\varphi \in C_0^\infty(\Omega)$ and integrate. By partial integration we obtain

$$\int_\Omega f\varphi\,dx = -\int_\Omega \operatorname{div}(|\nabla u|^{p-2}\nabla u)\varphi = \int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla\varphi\rangle\,dx.$$

Hence we say, that $u \in W_0^{1,p}(\Omega)$ solves the Dirichletproblem weakly, iff for all $\varphi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla\varphi\rangle\,dx = \int_\Omega f\varphi\,dx.$$

We define the following energy functional $E : W_0^{1,p}(\Omega) \to \mathbb{R}$ by:

$$E(v) := \frac{1}{p}\int_\Omega |\nabla v|^p\,dx - \int_\Omega fv\,dx.$$

Since $1 < p < \infty$, the space $W_0^{1,p}(\Omega)$ is a reflexive Banachspace. Next we fixate the integrability of f. Therefore we like the last term in the energy to be well defined:
By Sobolevembedding we have that $W_0^{1,p}(\Omega)$ is continuously embedded in $L^{\frac{np}{n-p}}(\Omega)$. The dual exponent $r \in \mathbb{R}$ to $\frac{np}{n-p}$ is given by

$$\frac{1}{r} + \frac{1}{\frac{np}{n-p}} = 1 \;\Rightarrow\; \frac{1}{r} = \frac{np-(n-p)}{np}.$$

Hence we set $r := \frac{np}{np-n+p}$ and require $f \in L^r(\Omega)$. This yields for every $v \in W_0^{1,p}(\Omega)$ by Hölders inequality

$$\int_\Omega |fv|\,dx \le \|f\|_{L^r(\Omega)}\|v\|_{L^{\frac{np}{n-p}}(\Omega)} \le C_{Sob}\|f\|_{L^r(\Omega)}\|v\|_{W_0^{1,p}(\Omega)} < \infty.$$

Therefore this last term is well defined. Before we proceed with the minimisation, we calculate the Euler-Lagrange equation of E:
We assume E has a minimum in $u \in W_0^{1,p}(\Omega)$. Then for every $\varphi \in W_0^{1,p}(\Omega)$ we have

$$0 = \frac{d}{dt}E(u+t\varphi)|_{t=0} = \frac{d}{dt}\frac{1}{p}\int_\Omega |\nabla u + t\nabla\varphi|^p\,dx|_{t=0} - \int_\Omega f\varphi\,dx.$$

First we formally interchange integral and derivative. If we then see that the integrand has a majorant for $t \in [-1,1]$ independent of t, we justified, that we can interchange integral and derivative.

$$\frac{d}{dt}\frac{1}{p}\int_\Omega |\nabla u + t\nabla\varphi|^p\,dx = \frac{1}{p}\int_\Omega \frac{d}{dt}|\nabla u + t\nabla\varphi|^p\,dx$$
$$= \frac{1}{p}\int_\Omega p|\nabla u + t\nabla\varphi|^{p-2}\langle \nabla u + t\nabla\varphi, \nabla\varphi\rangle\,dx.$$

The integrand can be estimated by e.g. Ex. 4.3

$$\begin{aligned} &\big||\nabla u + t\nabla\varphi|^{p-2}\langle \nabla u + t\nabla\varphi, \nabla\varphi\rangle\big| \le |\nabla u + t\nabla\varphi|^{p-1}|\nabla\varphi| \\ \le&(|\nabla u| + |t||\nabla\varphi|)^{p-1}(|\nabla u| + |t|\nabla\varphi|) = (|\nabla u| + |t||\nabla\varphi|)^p \\ \le& C(p)(|\nabla u|^p + |\nabla\varphi|^p) \in L^1(\Omega). \end{aligned}$$

Hence we are allowed to interchange integration and derivation. This yields

$$0 = \frac{d}{dt}E(u + t\varphi)|_{t=0} = \int_\Omega |\nabla u|^{p-2}\langle\nabla u, \nabla\varphi\rangle\, dx - \int_\Omega f\varphi\, dx,$$

which is exactly our weak formulation of the Dirichletproblem.
Now we proceed to show the existence of a minimiser: First we show a coercitivity estimate. Let $v \in W_0^{1,p}(\Omega)$. Then Ex. 4.3 and the Sobolev embedding yields with $\frac{1}{q} + \frac{1}{p} = 1$

$$\begin{aligned} E(v) \ge& \frac{1}{p}\|v\|^p_{W_0^{1,p}(\Omega)} - C_{Sob}\|f\|_{L^r(\Omega)}\|v\|_{W_0^{1,p}(\Omega)} \\ \ge& \frac{1}{p}\|v\|^p_{W_0^{1,p}(\Omega)} - \frac{1}{2p}\|v\|^p_{W_0^{1,p}(\Omega)} - C(\Omega, p)\|f\|^q_{L^r(\Omega)} \\ =& \frac{1}{2p}\|v\|^p_{W_0^{1,p}(\Omega)} - C(\Omega, p, f). \end{aligned}$$

Hence if $\|v\|_{W_0^{1,p}(\Omega)} \to \infty$, then also $E(v) \to \infty$ and E is coercive. Furthermore E is weakly lower semicontinuous, because

$$v \mapsto \|v\|^p_{W_0^{1,p}(\Omega)} = \int_\Omega |\nabla v|^p\, dx$$

is weakly lower semicontinuous by Thm. 2.16, because it is the power of a norm. Furthermore

$$v \mapsto \int_\Omega fv\, dx$$

is weakly continuous w.r.t. the weak convergence in $W_0^{1,p}(\Omega)$, because by the above calculation, it is a linear and continuous functional in $W_0^{1,p}(\Omega)$. Since $W_0^{1,p}(\Omega)$ is reflexive, Thm. 2.17 yields the existence of a minimiser u, which satisfies the Dirichletproblem weakly.

Solution to exercise 5.2

We proceed by contradiction and assume no such constant exists. Hence we find a sequence $u_m \in W^{1,2}(\Omega)$ with

$$\int_\Omega |u_m - \overline{u_m}|^2\, dx > m\int_\Omega |\nabla u_m|^2\, dx.$$

We define

$$v_m := \frac{1}{\|u_m - \overline{u_m}\|_{L^2(\Omega)}}(u_m - \overline{u_m}).$$

Hence

$$\begin{aligned}\int_\Omega |v_m|^2\,dx &= \frac{1}{\|u_m-\overline{u_m}\|^2_{L^2(\Omega)}}\int_\Omega |u_m-\overline{u_m}|^2\,dx\\ &> \frac{m}{\|u_m-\overline{u_m}\|^2_{L^2(\Omega)}}\int_\Omega |\nabla(u_m-\overline{u_m})|^2\,dx\\ &= m\int_\Omega |\nabla v_m|^2\,dx.\end{aligned}$$

Furthermore we have $\|v_m\|_{L^2(\Omega)} = 1$. This yields

$$\frac{1}{m} > \int_\Omega |\nabla v_m|^2\,dx.$$

Hence the Sobolevnorm $\|v_m\|_{W^{1,2}(\Omega)}$ is bounded. The Sobolevembedding now yields a subsequence and a $v \in W^{1,2}(\Omega)$, such that

$$v_m \to v \text{ in } L^2(\Omega),\ v_m \to v \text{ weakly in } W^{1,2}(\Omega).$$

Therefore $\|v\|_{L^2(\Omega)} = 1$. Furthermore the weak lower semicontinuity of a seminorm (i.e. a convex functional, see Ex. 3.1) yields

$$\int_\Omega |\nabla v|^2\,dx \le \liminf_{m\to\infty}\int_\Omega |\nabla v_m|^2\,dx \le \liminf_{m\to\infty}\frac{1}{m} = 0.$$

Hence $\nabla v = 0$ and therefore $v = const$. Since

$$0 = \int_\Omega v_m\,dx \to \int_\Omega v\,dx = const\mathcal{L}^n(\Omega),$$

we have $v = 0$. This is a contradiction to $\|v\|_{L^2(\Omega)} = 1$.

Solution to exercise 5.3

1. Let $v \in W^{1,2}(\Omega)$ and $c \in \mathbb{R}$. Then

$$\begin{aligned}E(v+c) &= \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega f(v+c)\,dx\\ &= \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega fv\,dx - c\int_\Omega f\,dx = E(v).\end{aligned}$$

This allows us to manipulate a minimising sequence. Let $u_m \in W^{1,2}(\Omega)$ be such that

$$E(u_m) \to \inf_{v\in W^{1,2}} E(v).$$

Then the above calulation yields for

$$v_m := u_m - \int_\Omega u_m\,dx$$

that we have

$$E(v_m) = E(u_m),$$

i.e. it is also a minimising sequence. Since

$$\int_\Omega v_m \, dx = 0$$

Exercise 5.2 yields

$$\int_\Omega |v_m|^2 \, dx \leq C \int_\Omega |\nabla v_m|^2 \, dx.$$

As in the lecture with zero boundary values, we have

$$C \geq E(v_m) \geq \frac{1}{2} \int_\Omega |\nabla v_m|^2 \, dx - \varepsilon \|v_m\|^2_{L^2(\Omega)} - C_\varepsilon \|f\|^2_{L^2(\Omega)}$$

for every $\varepsilon > 0$. Hence choosing $\varepsilon > 0$ small enough yields

$$\begin{aligned} E(v_m) \geq & \frac{1}{2} \int_\Omega |\nabla v_m|^2 \, dx - \varepsilon C \int_\Omega |\nabla v_m|^2 \, dx - C_\varepsilon \int_\Omega |f|^2 \, dx \\ \geq & \frac{1}{4} \int_\Omega |\nabla v_m|^2 \, dx - C(\Omega, f). \end{aligned}$$

Putting everything together we obtain

$$\|v_m\|^2_{W^{1,2}(\Omega)} = \int_\Omega |v_m|^2 \, dx + \int_\Omega |\nabla v_m|^2 \, dx \leq C.$$

Since $W^{1,2}(\Omega)$ is a Hilbertspace, we then obtain a weakly converging subsequence $v_m \to v \in W^{1,2}(\Omega)$. The term

$$u \mapsto \int_\Omega |\nabla u|^2 \, dx$$

is weakly lower semicontinuous because it is continuous and convex (it is a seminorm). Furthermore

$$u \mapsto \int f u \, dx$$

is by Cauchy-Schwartz a linear continuous functional on $W^{1,2}(\Omega)$, hence weakly continuous. All in all we have that, E is weakly lower semicontinuous. Therefore v is a desired minimiser.

2. Let $\varphi \in W^{1,2}(\Omega)$ and u a smooth minimiser. Then

$$0 = \frac{d}{dt} E(u + t\varphi)|_{t=0} = \int_\Omega \langle \nabla u, \nabla \varphi \rangle \, dx - \int_\Omega f \varphi \, dx.$$

The difference is now, that the testfunctions φ are allowed to have nonzero boundary values. Take $\varphi \in C^\infty(\mathbb{R}^n)$ and assume u to be smooth. Then partial integration, i.e. Gauss Theorem yields with ν being the outer normal of $\partial\Omega$

$$\int_\Omega \langle \nabla u, \nabla \varphi \rangle \, dx = \int_{\partial\Omega} \varphi \langle \nabla u, \nu \rangle \, darea_{\partial\Omega} - \int_\Omega \Delta u \varphi \, dx.$$

Choosing first $\varphi \in C_0^\infty(\Omega)$ yields the boundary term to disappear and therefore by the fundamental lemma of variational calculus we have $-\Delta u = f$ in Ω. By applying the fundamental lemma of variational calculus in charts on $\partial\Omega$ we also obtain

$$\langle \nabla u, \nu \rangle = 0 \text{ on } \partial\Omega.$$

Hence u satisfies

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \langle \nabla u, \nu \rangle = 0 & \text{on } \partial\Omega. \end{cases}$$

Since we minimised E without any further conditions, the resulting boundary value $\langle \nabla u, \nu \rangle = 0$ is also called natural boundary condition for E.

3. If $\int_\Omega f\, dx \neq 0$ we do not have a minimiser. The above calculation shows for any $c \in \mathbb{R}$ and any $v \in W^{1,2}(\Omega)$

$$E(v + c) = E(v) - c \int_\Omega f\, dx.$$

Hence choosing c very small (or big), yields E to not have a finite infimum or supremum.