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Winter 2022/23

Solutions to exercise 6

Solution to exercise 6.1

We start with the length: The substitution formula for one-dimensional integrals and the chain rule yield

$$\int_d^e |(c\circ\varphi)'|\,dt = \int_d^e |\varphi'||c'\circ\varphi|\,dt = \operatorname{sign}(\varphi')\int_d^e |c'\circ\varphi|\varphi'\,dt$$
$$= \operatorname{sign}(\varphi')\int_{\varphi(d)}^{\varphi(e)} |c'|\,dt = \int_a^b |c'(t)|\,dt$$

The last part follows by a distinction of two cases. Since φ is a diffeomorphism, we can either have $\varphi' > 0$ or $\varphi' < 0$. In the first case the above solution is clear, in the second case we have $\varphi(d) = b$ and $\varphi(e) = a$. Interchanging the limits of the integral and using $\varphi' < 0$ yields the desired result.
Now we proceed with the elastic energy. The sign of φ' can be handled as has been done with the length functional:

$$\int_d^e \frac{1}{|(c\circ\varphi)'|}\left|\frac{d}{dt}\left(\frac{(c\circ\varphi)'}{|(c\circ\varphi)'|}\right)\right|^2 dt = \int_d^e \frac{1}{|\varphi'||c'\circ\varphi|}\left(\frac{d}{dt}\frac{\varphi' c'\circ\varphi}{|\varphi'||c'\circ\varphi|}\right)^2 dt$$
$$= \int_d^e \frac{1}{|\varphi'||c'\circ\varphi|}\left(\operatorname{sign}(\varphi')\frac{d}{dt}\frac{c'\circ\varphi}{|c'\circ\varphi|}\right)^2 dt = \int_d^e \frac{1}{|\varphi'||c'\circ\varphi|}\left(\frac{d}{dt}\frac{c'\circ\varphi}{|c'\circ\varphi|}\right)^2 dt$$
$$= \int_d^e \frac{1}{|\varphi'||c'\circ\varphi|}\left(\frac{\varphi' c''\circ\varphi}{|c'\circ\varphi|} - \varphi'\frac{\langle c'\circ\varphi, c''\circ\varphi\rangle}{|c'\circ\varphi|^2}\right)^2 dt$$
$$= \int_d^e \frac{|\varphi'|}{|c'\circ\varphi|}\left(\frac{c''\circ\varphi}{|c'\circ\varphi|} - \frac{\langle c'\circ\varphi, c''\circ\varphi\rangle}{|c'\circ\varphi|^2}\right)^2 dt = \int_a^b \frac{1}{|c'|}\left(\frac{c''}{|c'|} - \frac{\langle c', c''\rangle}{|c'|^2}\right)^2 dt$$
$$= \int_a^b \frac{1}{|c'|}\left|\frac{d}{dt}\left(\frac{c'}{|c'|}\right)\right|^2 dt.$$

Solution to exercise 6.2

By Sobolevembedding the function $t \mapsto |c'(t)|$ is Hölder continuous. Hence the map

$$\psi(s) := \int_a^s |c'(t)|\,dt$$

is well defined and continuously differentiable. Furthermore

$$\psi'(s) = |c'(s)| > 0,$$

hence $\psi : [a,b] \to [0, L(P)]$ is strictly increasing and therefore bijective and a diffeomorphism. We set

$$\tilde{\varphi} := \psi^{-1}.$$

and obtain

$$\tilde{\varphi}' = \frac{1}{\psi' \circ \tilde{\varphi}} = \frac{1}{|c' \circ \tilde{\varphi}|}$$

Then we have

$$|(c \circ \tilde{\varphi})'| = |\tilde{\varphi}'||c' \circ \tilde{\varphi}| = 1.$$

We define

$$\varphi(t) := \tilde{\varphi}(L(P) \cdot t).$$

Then $\varphi : [0,1] \to [a,b]$ is a diffeomorphism with $\varphi' = L(P)\tilde{\varphi}' > 0$. Furthermore

$$|(c \circ \varphi)'| = |L(P)\tilde{\varphi}' c' \circ \varphi| = L(P).$$

Solution to exercise 6.3

Let $P_k \in M$ be a minimising sequence for W_λ. By Exercise 6.2 we find $c_k \in W^{2,2}((0,1),\mathbb{R}^2)$ parametrising P_k such that

$$|c_k'| = L(P_k).$$

Since P_k is a minimising sequence and $\lambda > 0$, we have

$$E(P_k), L(P_k) \leq C.$$

Hence

$$E(P_k) = \int_0^1 \frac{1}{|c_k'|} \left| \frac{d}{dt} \left(\frac{c_k'}{|c_k'|} \right) \right|^2 dt = \int_0^1 \frac{1}{L(P_k)^3} |c_k''|^2 \, dt.$$

Hence

$$\|c_k\|^2_{W_0^{2,2}((0,1),\mathbb{R}^2)} = \int_0^1 |c_k''|^2 \, dt = E(P_k) L(P_K)^3 \leq C.$$

Furthermore we have

$$\|c_k\|^2_{W_0^{1,2}((0,1),\mathbb{R}^2)} = \int_0^1 |c_k'|^2 \, dt = L(P_k)^2 \leq C.$$

The mean value theorem in integral form further yields for $t \in [0,1]$

$$|c_k(t)| \leq |c_k(t) - c_k(0)| + |c_k(0)| = \left| \int_0^1 \frac{d}{ds}(c_k(st)) \, ds \right| + |a|$$
$$= \left| \int_0^1 s c_k'(st) \, ds \right| + |a| \leq \int_0^1 s|c_k'(st)| \, ds + |a| = \frac{L(P_k)}{2} + |a| \leq C.$$

This yields

$$\|c_k\|^2_{L^2((0,1),\mathbb{R}^2)} = \int_0^1 |c_k|^2 \, dt \leq C.$$

All in all we have

$$\|c_k\|_{W^{2,2}((0,1),\mathbb{R}^2)} \leq C.$$

Since $W^{2,2}((0,1),\mathbb{R}^2)$ is a Hilbertspace, we find a weakly converging subsequence. Hence using the Sobolevembedding theorem we can assume after relabeling, that there exists a $c \in W^{2,2}((0,1),\mathbb{R}^2)$ with

$$c_k \to c \text{ weakly in } W^{2,2}((0,1),\mathbb{R}^2)$$

and

$$c_k \to c \text{ in } C^{1,\alpha}((0,1),\mathbb{R}^2)$$

for a given $0 < \alpha < \frac{1}{2}$. The last convergence yields

$$|c'| = \lim_{k\to\infty} |c_k'| = \lim_{k\to\infty} L(P_k).$$

Since the straight line from a to b always has lower length than any c_k, we have

$$L(P_k) \geq C$$

and therefore

$$|c'| > 0.$$

Hence $P := c(0,1)$ is a $W^{2,2}$-regular curve. We also have

$$L(P) = \int_0^1 |c'|\,dt = \lim_{k\to\infty} L(P_k),$$

hence $|c'| = L(P) > 0$. Since $\|\cdot\|^2_{W_0^{2,2}((0,1),\mathbb{R}^2)}$ is weakly lower semicontinuous and $L(\cdot)$ is continuous, we therefore have

$$\begin{aligned}\liminf_{k\to\infty} W_\lambda(P_k) &= \liminf_{k\to\infty} \frac{1}{L(P_k)^3}\int_0^1 |c_k''|^2\,dt + \lambda \lim_{k\to\infty} L(P_k)\\ &\geq \frac{1}{L(P)^3}\int_0^1 |c''|^2\,dt + \lambda L(P) = E(P) + \lambda L(P) = W_\lambda(P).\end{aligned}$$

Hence W_λ is weakly lower semicontinuous. The $C^{1,\alpha}$-convergence further yields, that P satisfies the boundary values. Hence $P \in M$ and it is therefore a minimiser.