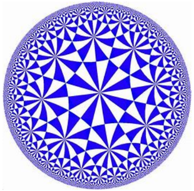
Laplace and Length Spectra for Closed Hyperbolic Surfaces of Large Volume

Giacomo Gavelli

joint work with Claudius Kamp

Universität Tübingen

10/06/2025



EBERHARD KARLS
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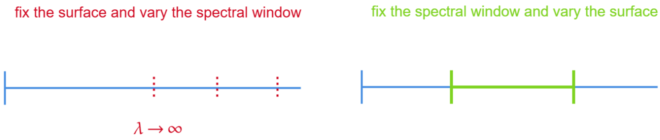

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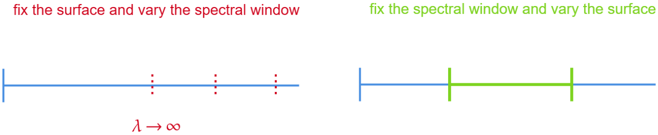

Question: How to obtain sequences of surfaces with increasing volume which give good asymptotics?

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$$\updownarrow$$
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Plancherel Convergence (spectral)

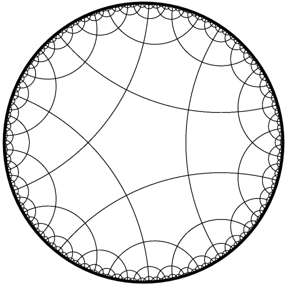
- The goal of this talk is to introduce these two notions, discuss their relation and (some of) the asymptotics they carry.

The hyperbolic plane

We represent the hyperbolic plane with the disc model

$$\mathbb{H} := \{z \in \mathbb{C} \ ; \ |z| < 1\}$$

equipped with the Riemannian metric $ds^2 := 4\frac{dz^2}{(1-|z|^2)^2}$.



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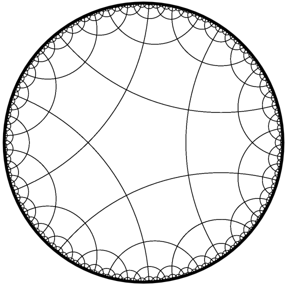

$SL_2(\mathbb{R})$ is the group of orientation preserving isometries of $\mathbb{H}$.

Hyperbolic surfaces

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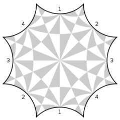
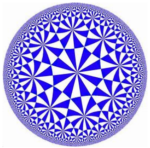
Fact

Any (closed) hyperbolic surface S is of the form $S = \Gamma \setminus \mathbb{H}$ for some (cocompact) lattice $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$. For such a group Γ we have an isomorphism $\Gamma \cong \pi_1(S)$.





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Theorem (Weyl's Asymptotic Law)

Let $S = \Gamma \setminus \mathbb{H}$ be a closed hyperbolic surface. Denote by $N(\Delta, S, x)$ the number of eigenvalues of the Laplacian on S which are $\leq x$. Then

$$\frac{N(\Delta, S, x)}{\mathrm{vol}(S)} \sim \frac{1}{4\pi} x, \quad \text{as } x \to \infty.$$

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Theorem (Prime Number Theorem for closed hyperbolic surfaces)

Denote by $N(\mathcal{C}(S), x)$ the number of closed geodesics on S with length $\leq \ln x$. Then

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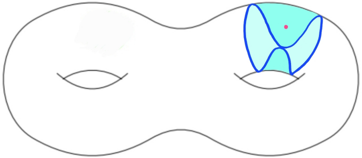
Notice that there exist isospectral surfaces which are not isometric.

Benjamini-Schramm convergence

Let $S = \Gamma \setminus \mathbb{H}$. The injectivity radius at a point p is the radius of the largest ball in S centered at p which is isometric to a ball in $\mathbb{H}$.



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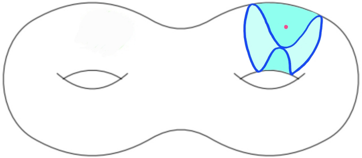
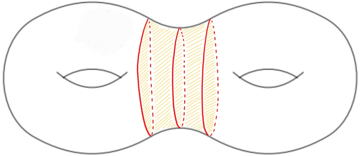


For $R > 0$, the *R-thin part* of S is

$$S_{<R} = \Big\{ p \in X \ : \ \mathrm{InjRad}(p) < R \Big\}.$$



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We say that a sequence $(S_n)_{n\in\mathbb{N}}$ of closed hyperbolic surfaces (given by $S_n = \Gamma_n \setminus \mathbb{H}$) is *Benjamini–Schramm convergent* to the hyperbolic plane $\mathbb{H}$ if for every $R > 0$

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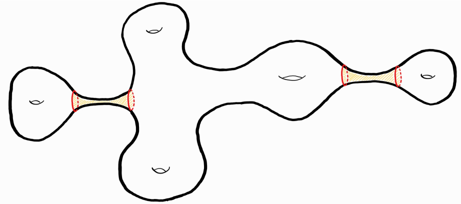
Equivalently, $(S_n)_{n\in\mathbb{N}}$ is BS-convergent if for any $R > 0$, the probability of a ball in S_n of radius R being isometric to a ball in $\mathbb{H}$ is asimptotically 1.



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$\mathrm{InjRad}(S_n) \xrightarrow{n\to\infty} \infty \implies$ the R-thin part of S_n is empty for large enough $n \implies S_n$ is Benjamini-Schramm convergent to $\mathbb{H}$.

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$\mu_{Pl}(\hat{f}) := \int_{\widehat{G}} \hat{f}(\pi) d\mu_{Pl}(\pi) = f(e)$	$\int_{\mathbb{R}} \hat{f}(\xi) d\xi = f(0)$

Plancherel convergence

Definition (Plancherel convergence)

A sequence $(S_n)_{n\in\mathbb{N}}$ of closed hyperbolic surfaces is called a *Plancherel sequence* if for every $f \in C_c^\infty(G)$

$$\frac{1}{\mathrm{vol}(S_n)}\mu_{\Gamma_n}(\hat{f}) \longrightarrow \mu_{Pl}(\hat{f}), \qquad \text{as } n \to \infty.$$

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Proposition

Let $(S_n)_{n\in\mathbb{N}}$ be a Plancherel sequence, and **fix** $I = [a, b]$. Then

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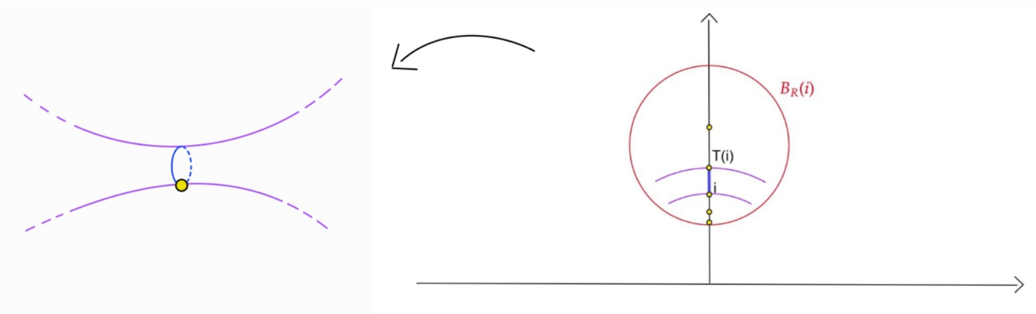
$$\lim_{n\to\infty} \frac{N(\Delta, S_n, I)}{\mathrm{vol}(S_n)} = \mu_{Pl}(I) = \frac{1}{8\pi}\int_a^b \tanh\left(\pi\sqrt{y - \frac{1}{4}}\right) dy.$$

Geometrical interpretation

Theorem (Deitmar, 2018)

$(S_n)_{n\in\mathbb{N}}$ is a Plancherel sequence if and only if for every $R > 0$

$$\frac{1}{\mathrm{vol}(S_n)} \int_{\mathcal{F}_n} \sharp\left(\Gamma_n^\star \cdot x \cap \overline{B_R(x)}\right) dx \xrightarrow{n\to\infty} 0.$$



Benjamini-Schramm vs Plancherel convergence

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Definition (Uniform Discreteness)

We call a sequence of closed hyperbolic surfaces $(S_n)_{n\in\mathbb{N}}$ *uniformly discrete* if there is a uniform lower bound on the injectivity radius of $S_n = \Gamma_n \setminus \mathbb{H}$ for $n \in \mathbb{N}$.

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Theorem (ABBGNRS, 2017)

A uniformly discrete Benjamini-Schramm convergent sequence is Plancherel.

Benjamini-Schramm vs Plancherel convergence

$$
\begin{tikzcd}
\text{Benjamini-Schramm} + \text{Uniformly Discrete} \arrow[d, Rightarrow, shift right=2] \\
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Theorem (G., Kamp, 2024)

There exists a Benjamini-Schramm convergent sequence of closed hyperbolic surfaces which is <u>not</u> Plancherel convergent.

Asymptotics for the Length Spectrum

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Theorem (Raimbault, 2018)

- $(S_n)_{n\in\mathbb{N}}$ has the closed geodesics property $\implies$ $(S_n)_{n\in\mathbb{N}}$ is Benjamini-Schramm.

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$(S_n)_{n\in\mathbb{N}}$ has the *simple closed geodesics property* if for all $R > 0$

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G semisimple, rank 1

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 & \text{Benjamini-Schramm} + \text{Uniform Discreteness} \arrow[dl, Rightarrow] \arrow[d, Rightarrow, shift right] \\
\text{Closed geodesics property} \arrow[dr, Rightarrow] & \text{Plancherel} \arrow[u, Rightarrow, shift right, "-" marking] \arrow[d, Rightarrow] \\
\text{Simple closed geodesics property} & \text{Benjamini-Schramm}
\end{tikzcd}
$$

G semisimple, rank 1

$$
\begin{tikzcd}
 & & \text{Benjamini-Schramm} + \text{Uniformly Discrete} \arrow[dll, Rightarrow] \arrow[d, Rightarrow, shift right=2] \\
\text{Closed geodesics property} \arrow[drr, Rightarrow] & & \text{Plancherel} \arrow[u, Rightarrow, shift right=2, "/" marking] \arrow[d, Rightarrow, shift right=2] \\
\text{Simple closed geodesics property} & & \text{Benjamini-Schramm} \arrow[u, Rightarrow, shift right=2, red, "??"']
\end{tikzcd}
$$

$\mathrm{SL}_2(\mathbb{R})$

$$
\begin{tikzcd}
 & \text{Benjamini-Schramm} + \text{Uniform Discreteness} \arrow[dl, Rightarrow] \arrow[d, Rightarrow, shift right=2] \\
\text{Closed geodesics property} \arrow[r, Rightarrow, blue] & \text{Plancherel} \arrow[u, Rightarrow, shift right=2, "|" marking] \arrow[dl, Rightarrow, blue] \arrow[d, Rightarrow, shift right=2] \\
\text{Simple closed geodesics property} \arrow[r, Rightarrow, blue] & \text{Benjamini-Schramm} \arrow[u, Rightarrow, shift right=2, red, "|" marking]
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\begin{tikzcd}[column sep=huge, row sep=huge]
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\end{tikzcd}
$$

Construction of the example

Consider the principal congruence subgroups

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \; : \; a, d \equiv 1 \pmod{N}, \; b, c \equiv 0 \pmod{N} \right\}.$$

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We denote by $S(N) = \Gamma(N) \setminus \mathbb{H}$ the congruence surface of level N.

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The number of cusps of $S(N)$ is always even for $N \geq 3$

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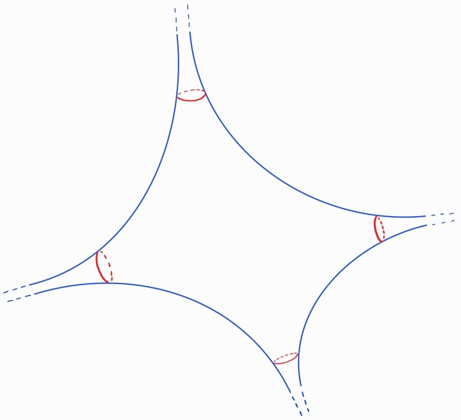
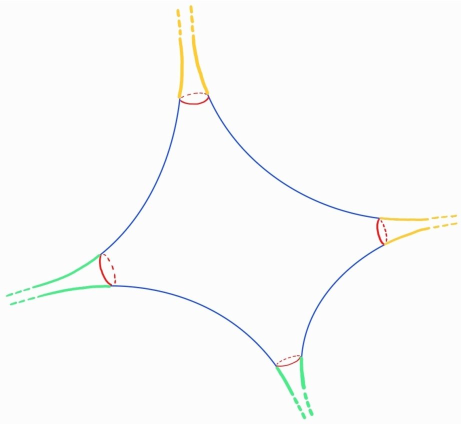


Construction of the example

Identify the cusps in pairs and substitute any cusp with a boundary geodesic of length $t > 0$.



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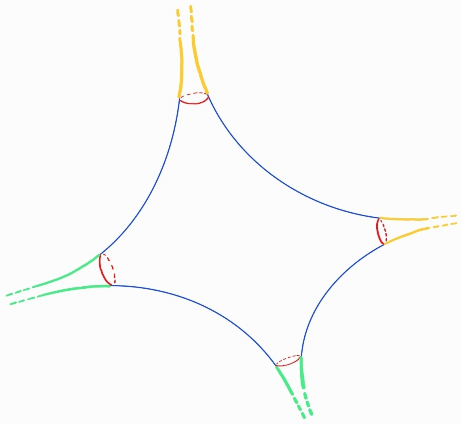
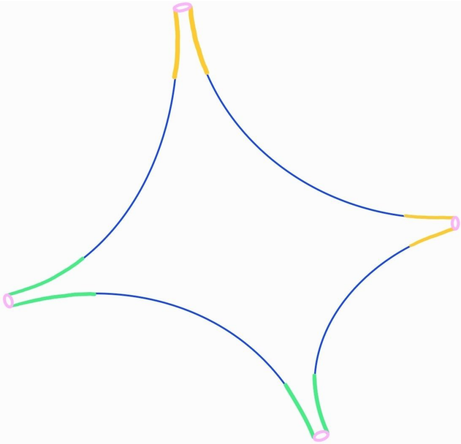
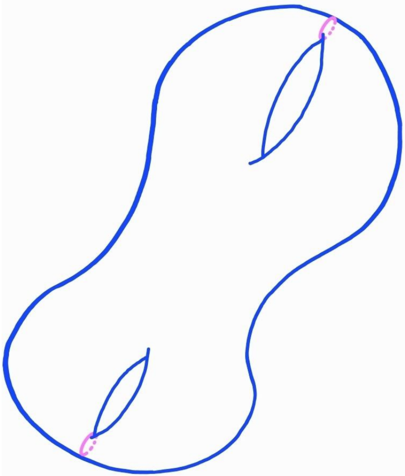




Construction of the example

Glue together the prescribed boundary geodesics. This yields a closed hyperbolic surface $S_t(N)$.



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- The sequence $(S_n)_{n\in\mathbb{N}}$ is Benjamini–Schramm convergent.

Applications: sneak peek

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- L^2-Betti numbers: asymptotics of $\frac{b_k(X_n)}{\mathrm{vol}(X_n)}$.

Thank you for your attention!