Proseminar Harmonic Analysis: Mondays, 12-14, room S07.

Date	Title	Speaker
15th April	Introduction to Fourier Series	Sven Michel
22nd April	Convergence of Fourier series	Ben Trinh
29th April	Introduction to Hilbert spaces	Heidi Bullinger
6th May	Results on Hilbert spaces	Frederic Cüppers
13th May	The Fourier transform	Veronika Reimchen
27th May	The inversion formula and Plancherel's Theorem	Jonte Nagel
3rd June	An application of Fourier Analysis	Kanae Krüger
10th June	Finite abelian groups	Philipp Schmale
17th June	Metric spaces and topology	Konrad Rohlfs
24th June	Locally compact abelian groups	Gamze Ünal
1st July	Pontryagin Duality	Kai Dirschnabel
8th July	Integration on groups, Haar integration	Giacomo Gavelli
15th July	Plancherel's Theorem	Giacomo Gavelli

ABOUT THE SEMINAR

The seminar has two aims and three main topics. The first aim is to give an introduction to Fourier Analysis, and in this regard we will cover two of the three main topics, namely Fourier Series and the Fourier Transform. Fourier Series and the Fourier Transform are at the core of a lot of modern Analysis and applications, varying from abstract harmonic analysis to analytic number theory, differential equations, quantum mechanics, sound editing, computer imaging...

The second aim is to give an introduction to abstract Harmonic Analysis (the third main topic). We will see that both the incarnations of Fourier Analysis, Fourier Series and the Fourier Transform, are special case of a more general theory arising in the context of locally compact abelian groups.

When we talk about Fourier Series, we are basically trying to answer the question: when can a periodic function (a function which repeats itself after a fixed period of time) be written as a sum of simple waves? We will answer this question showing the L^2-completeness of the Fourier Series.

The Fourier Transform is another instance of dealing with the decomposition of a function into its constituent frequencies, but this time for non-periodic functions. We will deal with the problem of inversion of the Fourier Transform (how to "reconstruct" a function from its transform) and we will present the most important Plancherel Theorem, which states that the Fourier Transform can be defined for square integrable functions and that it is an isometry.

In the third and last part of the seminar, we will say what a Locally Compact Group is and introduce a new notion of Fourier Transform in this setting. The end goal is to show the general Plancherel Theorem and see that it is a simulatenous generalization of the completeness of Fourier Series and the Plancherel Theorem for the Fourier Transform, hence showing how abstract Harmonic Analysis is a generalization of Fourier Analysis.

About the structure of the seminar: we will have 13 talks of 90 minutes each, following the book "A first course in Harmonic Analysis", by Prof. Anton Deitmar ([1]). What it is exptected from you is that you present the assigned topic to the whole class when the moment comes.

For the preparation of the talks, I would like to meet all of you at least ones before you give your presentation, to know how you plan to structure it and, hopefully, be able to give useful advice. If you need to come more than once, you are more than welcome to do so and I will be glad to help you to the extent of my capabilities. When you plan to come to my office, please send me an e-mail in advance so that I can organize.

TALKS

1. Sections 1.1-1.3,1.6: **Introduction to Fourier Series**. Periodic functions, orthogonality of exponential functions, Fourier coefficients, Fourier series of a Riemann integral periodic function, Bessel inequality: estimate on the sum of the Fourier coefficients with L^2-norm of the function. Periodic functions seen on the unit circle (Maybe to do early in the talk).

2. Sections 1.4-1.5: **Convergence of Fourier series**. L^2-convergence and uniform convergence. Uniform convergence implies L^2-convergence. A technical Lemma to carry out computations. L^2-convergence of Fourier series for Riemann step functions. L^2-convergence for periodic Riemann integrable functions. Uniform convergence of Fourier series for periodic, continuous and piecewise continuously differentiable functions.

   We introduce Hilbert spaces, and this leads us to a more profound understanding of some results about Fourier series. Also, the theory of Hilbert spaces, in particular the concept of "completion", is needed to understand the Plancherel's Theorem, arguably the most important Theorem in Fourier Analysis. Hilbert spaces arise as the natural environment to do analysis guided by some intuitions from $2D$ geometry. In a Hilbert space we have exact analogs of the Pythagorean Theorem and parallelogram law. Moreover, in Hilbert spaces, it makes sense to talk about perpendicular projection onto a subspace. An element of a Hilbert space can be uniquely specified by its coordinates with respect to an orthonormal basis, in analogy with Cartesian coordinates in classical geometry.

3. Sections 2.1-2.2: **Introduction to Hilbert spaces**. Pre-Hilbert spaces. Cauchy-Schwarz inequality. isometries and examples. Cauchy sequences and Hilbert spaces. An important class of Hilbert spaces: the l^2-spaces.

4. Sections 2.3-2.4: **Results on Hilbert spaces**. Complete and hortonormal systems, hortonormal basis. Existence of an hortonormal basis in separable

Hilbert spaces. Coefficients of an element, isometry with $l^2(\mathbb{N})$. Completion Theorem. Orthonormal basis for $L^2(\mathbb{R}/\mathbb{Z})$, Fourier series revisited.

Given some additional technical tools, we define the Fourier Transform and show some properties. We deal with two problems. 1) What is the biggest domain of definition for the Fourier Transform? 2) Which functions can be completely reconstructed from their Fourier transform? We will give a partial answer to the first question and a complete answer to the second one via the inversion Formula and Plancherel's Theorem.

5. Sections 3.1-3.3: **The Fourier Transform**. Technical tools: dominated convergence Theorem, Monotone convergence Theorem, Convolution. Definition of the Fourier Transform and first properties. Fourier Transform of Schwarz (fast decaying) functions.

6. Sections 3.4-3.5: **The inversion formula and Plancherel's Theorem**. Approximation via convolution. Inversion formula. Completion of L^1_{bc} and Plancherel's Theorem.

7. Sections 3.6-3.7 + Appendix A: **An application of Fourier Analysis**. We bring together Fourier series and Fourier transform to obtain the Poisson Summation formula. We see an application of Fourier Analysis to analytic number theory, obtaining a functional expression for the Riemann Zeta function thanks to the aforementioned Poisson Summation Formula.

The second part of the proseminar is an introduction to abstract Harmonic Analysis. We see how Fourier Analysis is an istance of a more general theory arising in the context of locally compact abelian groups. The goal of this part is to show a general Plancherel Theorem, which simulatenously extends the Completeness of the Fourier series and the Plancherel's Theorem for the Fourier transform on the real line, hence showing that abstract Harmonic Analysis is a generalization of Fourier Analysis.

8. Sections 5.1-5.3: **Finite Abelian Groups**. The group of characters of a finite abelian group, the Pontryagin dual. Fourier transform on the Pontryagin dual. "Plancherel Theorem" for finite abelian groups. The transform of a convolution is the product of the transforms.

9. Section 6.1: **Metric spaces and Topology**. Metric spaces, continuity. Equivalent metrics and metrizable metric spaces. Dense subsets. Topological spaces. Closed and compact subsets. σ-compactness and compact exhaustions. Locally compact and σ-locally compact spaces.

10. Sections 6.2-6.3: **Locally compact abelian groups**. Cauchy sequences. Isometries and completeness. Completion of a metric space. Completion of $L^1_{bc}(\mathbb{R})$. Metrizable abelian groups and LCA groups. Separability of LCA groups. Absorbing exhaustions.

11. Sections 7.1-7.2: **Pontryagin duality**. The set of characters is a group. The dual is an LCA group. The Pontryagin dual of $\mathbb{R}$. Canonical isomorphism with the bidual (Pontryagin duality).

12. Sections 8.1-8.2: **Integration on groups, Haar integration**. A different description of the Riemann integral. Integrals on $C_c(G)$. Haar integral on a locally compact group (no proof). Inner product and $L^2(G)$. Examples of Haar integrals. Haar integral of a product. Weak Fubini's Theorem. (This could be split into two talks)

13. Sections 8.3-8.4: **Plancherel's Theorem**. Fourier transform on an LCA group. The transform of a convolution is the product of the transform. Plancherel's Theorem.

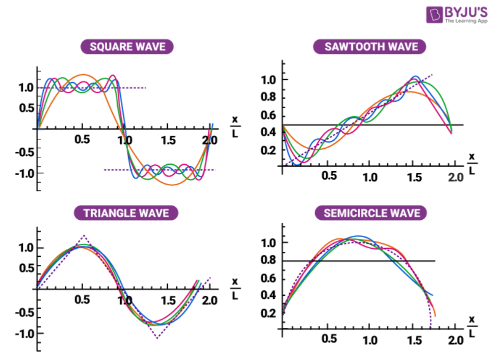

Figure 1: Approximation via Fourier Series

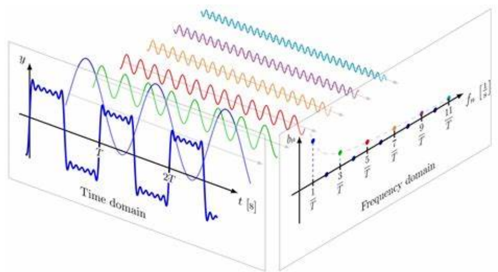

Figure 2: Decomposition in waves

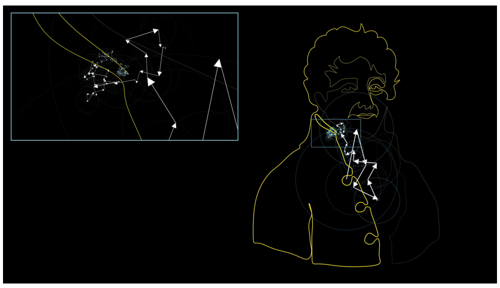

Figure 3: Drawing with Fourier series

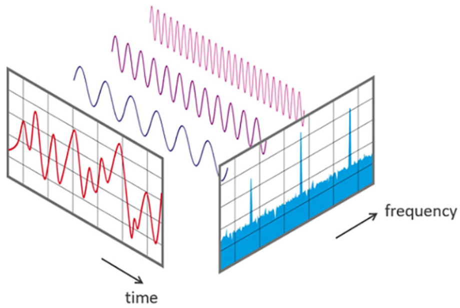

Figure 4: Fourier transform visualized

Source of the images is this page

Some nice videos:

- Fourier series
- Fourier transform

Bibliography

[1] Anton Deitmar. *A first course in harmonic analysis.* Springer, 2005.