A VIDEO COURSE ON COX RINGS

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This document provides a short introductory course to Cox rings via video clips, short notes and exercises. The video clips can be activated by clicking the respective starting buttons in the document. The prerequesites for this course are basic knowledge in algebraic geometry and toric varieties.

The idea is to survey basic concepts, principles and facts around Cox rings. The exercises may help to get deeper into the matter. For a more detailed study, we refer to [1].

The present course format grew out of the need of teaching without classroom during the Corona Time. Certainly, this course is a handmade product by a non-expert concerning multimedia techniques as well as the didactic aspects of e-learning. The author will highly appreciate any comments and suggestions.

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1. COX RINGS AND CHARACTERISTIC SPACES

Part 1-A. We recall the basic notions around divisors, introduce and discuss sheaves of divisorial algebras and finally define the Cox sheaf and the Cox ring in the case of a torsion free divisor class group.

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PART 1-A: SHORT NOTES

Reminder 1.1. Let X be a normal variety. The group of *Weil divisors* of X is the free abelian group generated by the prime divisors,

$$\mathrm{WDiv}(X) \ = \ \bigoplus_{D \text{ prime}} \mathbb{Z}D.$$

where a *prime divisor* is an irreducible closed subvariety $D \subseteq X$ of codimension one. Every $0 \neq f \in \mathbb{C}(X)^*$ defines a Weil divisor

$$\mathrm{div}(f) \ = \ \sum_{D \text{ prime}} \mathrm{ord}_D(f) \cdot D,$$

where $\mathrm{ord}_D(f)$ is the *order* of f along the prime divisor D. For any two $0 \neq f, f' \in \mathbb{C}(X)^*$, we have

$$\mathrm{div}(ff') \ = \ \mathrm{div}(f) + \mathrm{div}(f').$$

Reminder 1.2. Every $D \in \mathrm{WDiv}(X)$ gives rise to a sheaf $\mathcal{O}(D)$ of $\mathcal{O}$-modules: for any open $U \subseteq X$, one sets

$$\Gamma(U, \mathcal{O}(D)) \ := \ \{0\} \cup \{f \in \mathbb{C}(X)^*; \ (\mathrm{div}(f) + D)|_U \geq 0\},$$

where one defines $D|_U := D \cap U$ if D intersects U and $D|_U := 0$ otherwise for prime divisors D. We always have

$$f \in \Gamma(U, \mathcal{O}(D)), \ f' \in \Gamma(U, \mathcal{O}(D')) \ \Rightarrow \ ff' \in \Gamma(U, \mathcal{O}(D + D')).$$

Example 1.3. Consider the projective plane $\mathbb{P}_2$ and the Weil divisor $D := V(T_0)$ on $\mathbb{P}_2$. We have

$$\Gamma(\mathbb{P}_2, \mathcal{O}(D)) \ = \ \mathbb{C} \oplus \mathbb{C}\frac{T_1}{T_0} \oplus \mathbb{C}\frac{T_2}{T_0}.$$

Moreover, denoting by $\mathbb{C}[T_0, T_1, T_2]_k$ the vector space of all homogeneous polynomials of degree k, we have an isomorphism

$$\mathbb{C}[T_0, T_1, T_2]_k \ \to \ \Gamma(\mathbb{P}_2, \mathcal{O}(kD)), \qquad f \ \mapsto \ \frac{f(T_0, T_1, T_2)}{T_0^k}.$$

Reminder 1.4. Let K be an abelian group. A *K-graded $\mathbb{C}$-algebra* is a $\mathbb{C}$-algebra A coming with a direct sum decomposition

$$A \ = \ \bigoplus_{w \in K} A_w$$

such that for any two $w, w' \in K$ we have $A_w A_{w'} \subseteq A_{w+w'}$. The vector subspaces $A_w \subseteq A$ are the *homogeneous components* of A.

Construction 1.5. Let X be a normal variety and $K \subseteq \mathrm{WDiv}(X)$ a subgroup. The associated *sheaf of divisorial algebras* is the sheaf

$$\mathcal{S} \ := \ \bigoplus_{D \in K} \mathcal{S}_D, \qquad\qquad \mathcal{S}_D \ := \ \mathcal{O}(D)$$

of K-graded $\mathcal{O}$-algebras, where the multiplication stems from that in the field $\mathbb{C}(X)$ of rational functions:

$$\Gamma(U, \mathcal{S}_D) \times \Gamma(U, \mathcal{S}_{D'}) \ \to \ \Gamma(U, \mathcal{S}_{D+D'}), \qquad (f, f') \ \mapsto \ ff'.$$

Example 1.6. Consider again $\mathbb{P}_2$, the divisor $D = V(z_0)$ and the divisorial sheaf $\mathcal{S}$ given by the subgroup

$$K \ := \ \mathbb{Z}D \ \subseteq \ \mathrm{WDiv}(\mathbb{P}_2).$$

Then $K \cong \mathbb{Z}$ and we have an isomorphism $\mathbb{C}[T_0, T_1, T_2] \to \Gamma(\mathbb{P}_2, \mathcal{S})$ of $\mathbb{Z}$-graded algebras given component-wise by

$$\mathbb{C}[T_0, T_1, T_2]_k \ \to \ \Gamma(\mathbb{P}_2, \mathcal{S}_{kD}), \qquad f \ \mapsto \ \frac{f(T_0, T_1, T_2)}{T_0^k}.$$

Reminder 1.7. On a normal variety X, the assignment $f \mapsto \mathrm{div}(f)$ yields a homomorphism $\mathbb{C}(X)^* \to \mathrm{WDiv}(X)$. The image

$$\mathrm{PDiv}(X) \ \subseteq \ \mathrm{WDiv}(X)$$

is the subgroup of *principal divisors*. The *divisor class group* of X is the factor group

$$\mathrm{Cl}(X) \ := \ \mathrm{WDiv}(X) / \mathrm{PDiv}(X).$$

Theorem 1.8. *Let X be a normal variety and $K \subseteq \mathrm{WDiv}(X)$ a finitely generated subgroup providing an epimorphism*

$$K \ \to \ \mathrm{Cl}(X), \qquad D \ \mapsto \ [D].$$

Let $\mathcal{S}$ be the sheaf of divisorial algebras associated with K. Then the algebra of global sections $\Gamma(X, \mathcal{S})$ is a unique factorization domain.

Construction 1.9. Let X be a normal variety with $\mathrm{Cl}(X)$ finitely generated and torsion free. Then the canonical map

$$K \ \to \ \mathrm{Cl}(X), \qquad D \ \mapsto \ [D]$$

is an isomorphsim for a suitable subgroup $K \subseteq \mathrm{WDiv}(X)$. We define the *Cox sheaf* and the *Cox ring* of X as

$$\mathcal{R} \ := \ \bigoplus_{D \in K} \mathcal{O}(D), \qquad \mathcal{R}(X) \ := \ \bigoplus_{D \in K} \Gamma(X, \mathcal{O}(D)).$$

Remark 1.10. Construction 1.9 does not depend on the choice of $K \subseteq \mathrm{WDiv}(X)$. In particular, the Cox ring $\mathcal{R}(X)$ is unique up to isomorphy.

Example 1.11. The class of $D = V(T_0) \subseteq \mathbb{P}_2$ generates $\mathrm{Cl}(\mathbb{P}_2)$ and the Cox ring of $\mathbb{P}$ is the polynomial ring

$$\mathcal{R}(\mathbb{P}_2) \; = \; \bigoplus_{k\in\mathbb{Z}} \Gamma(\mathbb{P}_2, \mathcal{O}(kD)) \; \cong \; \bigoplus_{k\in\mathbb{Z}} \mathbb{C}[T_0,T_1,T_2]_k \; = \; \mathbb{C}[T_0,T_1,T_2].$$

Theorem 1.12. *Let X be a normal variety with finitely generated torsion free divisor class group. Then the Cox ring $\mathcal{R}(X)$ is a unique factorization domain.*

Part 1-A: Exercises

Exercise 1.13. Let $K \subseteq \mathrm{WDiv}(X)$ have $D_1,\ldots,D_l$ as a $\mathbb{Z}$-basis and $U \subseteq X$ be open subset such that

$$D_i \; = \; \mathrm{div}(f_i), \qquad i = 1,\ldots,l.$$

Show that, with $\deg(T_i) = D_i$ and $f_i^{-1} \in \Gamma(X, \mathcal{S}_{D_i})$, we have an isomorphism of K-graded algebras

$$\Gamma(U,\mathcal{O})[T_1^{\pm 1},\ldots,T_l^{\pm 1}] \to \Gamma(U,\mathcal{S}), \quad gT_1^{\nu_1}\cdots T_l^{\nu_l} \mapsto gf_1^{-\nu_1}\cdots f_l^{-\nu_l}.$$

Exercise 1.14. Let X be a normal variety and $\mathcal{S}$ the sheaf of divisorial algebras arising from a subgroup $K \subseteq \mathrm{WDiv}(X)$. Show that for the set $X_{\mathrm{reg}} \subseteq X$ of smooth points, we have

$$\Gamma(X_{\mathrm{reg}},\mathcal{S}) \; = \; \Gamma(X,\mathcal{S}).$$

Exercise 1.15. Prove Remark 1.10. *Hint:* Given two subgroups K and K' of $\mathrm{WDiv}(X)$ mapping isomorphically onto $\mathrm{Cl}(X)$, there are $\mathbb{Z}$-bases $D_1,\ldots,D_l$ of K and $D_1',\ldots,D_l'$ of K' with $D_i' = D_i + \mathrm{div}(f_i)$. Use this to relate the associated sheaves of divisorial algebras.

Part 1-B. We define Cox sheaf and Cox ring in the general case, look at the example case of a toric variety and discuss algebraic aspects of the Cox ring, in particular its divisibility properties.

Clip 1-B **Notes 1-B** **Exercises 1-B**

Part 1-B: Short Notes

Construction 1.16. Let X be a normal variety with $\Gamma(X,\mathcal{O}^*) = \mathbb{C}^*$ and $\mathrm{Cl}(X)$ finitely generated. Fix a subgroup $K \subseteq \mathrm{WDiv}(X)$ such that

$$c\colon K \; \to \; \mathrm{Cl}(X), \quad D \; \mapsto \; [D]$$

is surjective. Set $K^0 = \ker(c)$ and choose a group homomorphism $\chi\colon K^0 \to \mathbb{C}(X)^*$ with

$$\mathrm{div}(\chi(E)) \; = \; E \qquad \text{for all } E \in K^0.$$

Let $\mathcal{S}$ be the associated sheaf of divisorial algebras associated with K. Then we obtain a sheaf of ideals in of $\mathcal{S}$ by

$$\mathcal{I} \ = \ \mathcal{S}\langle 1-\chi(E),\ E \in K^0\rangle, \quad 1 \in \Gamma(X,\mathcal{S}_0), \quad \chi(E) \in \Gamma(X,\mathcal{S}_{-E}).$$

The *Cox sheaf* of X is the quotient sheaf $\mathcal{R} := \mathcal{S}/\mathcal{I}$ together with the $\mathrm{Cl}(X)$-grading

$$\mathcal{R} \ = \ \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{R}_{[D]}, \qquad\qquad \mathcal{R}_{[D]} \ := \ \pi\left(\bigoplus_{D' \in c^{-1}([D])} \mathcal{S}_{D'}\right).$$

where $\pi\colon \mathcal{S} \to \mathcal{R}$ denotes the projection. The *Cox ring* is the algebra of global sections

$$\mathcal{R}(X) \ := \ \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{R}_{[D]}(X), \qquad\qquad \mathcal{R}_{[D]}(X) \ := \ \Gamma(X,\mathcal{R}_{[D]}).$$

Remark 1.17. The assumption $\Gamma(X,\mathcal{O})^* = \mathbb{C}^*$ in Construction 1.16 ensures that the Cox sheaf $\mathcal{R}$ and the Cox ring $\mathcal{R}(X)$ are unique up to isomorphy.

Example 1.18. Let Z be a toric variety with acting torus $\mathbb{T}$ and base point z_0. With the invariant prime divisors $D_i \subseteq Z$, we have

$$Z \setminus \mathbb{T}\cdot z_0 \ = \ D_1 \cup \ldots \cup D_r$$

The divisor class group $\mathrm{Cl}(Z)$ is generated by the classes $[D_i]$. Assume $\Gamma(Z,\mathcal{O}^*) = \mathbb{C}^*$. Then the Cox ring of Z is given by

$$\mathcal{R}(Z) = \mathbb{C}[T_1,\ldots,T_r], \qquad \deg(T_i) = [D_i] \in \mathrm{Cl}(Z), \quad i = 1,\ldots,r.$$

More explicitly, let Z arise from a lattice fan (Σ, N) with primitive generators $v_1,\ldots,v_r \in N$ spanning $N_{\mathbb{Q}}$. Then

$$P\colon \mathbb{Z}^r \ \to \ N, \qquad e_i \ \mapsto \ v_i$$

defines a linear map and the divisor class group of Z is the cokernel of the dual map $P^*\colon M \to \mathbb{Z}^r$:

$$\mathrm{Cl}(Z) \ \cong \ \mathbb{Z}^r/P^*M.$$

Denote by $Q\colon \mathbb{Z}^r \to \mathbb{Z}^r/P^*M$ the projection. Then the Cox ring of the toric variety Z is given as

$$\mathcal{R}(Z) = \mathbb{C}[T_1,\ldots,T_r], \qquad \deg(T_i) = Q(e_i),\ i = 1,\ldots,r.$$

Remark 1.19. Consider the setting of Construction 1.16. For every divisor $D \in K$, we have an isomorphism of sheaves

$$\pi_{|\mathcal{S}_D}\colon \mathcal{S}_D \ \to \ \mathcal{R}_{[D]}.$$

Moreover, for every open set $U \subseteq X$, we have a canonical isomorphism of $\mathrm{Cl}(X)$-graded algebras

$$\Gamma(U,\mathcal{S})/\Gamma(U,\mathcal{I}) \ \cong \ \Gamma(U,\mathcal{S}/\mathcal{I}).$$

In particular, the Cox ring $\mathcal{R}(X)$ of X is the quotient of the algebra of global sections of $\mathcal{S}$:

$$\mathcal{R}(X) \;\cong\; \Gamma(X,\mathcal{S})/\Gamma(X,\mathcal{I}).$$

Construction 1.20. Let X be a normal variety with Cox ring $\mathcal{R}(X)$. For any homogeneous element $0 \neq f \in \mathcal{R}_{[D]}(X)$ we define the $[D]$-*divisor* and the $[D]$-*localization* as

$$\mathrm{div}_{[D]}(f) \;:=\; \mathrm{div}(\tilde{f}) + D, \qquad X_{[D],f} \;:=\; X \setminus \mathrm{supp}(\mathrm{div}(\tilde{f}) + D),$$

where, in the setting of Construction 1.16, we choose $D \in K$ and take $\tilde{f} \in \Gamma(X, \mathcal{S}_D)$ with $\pi(\tilde{f}) = f$. The $[D]$-divisor and the $[D]$-localization of f do not depend on the choices made.

Theorem 1.21. *The Cox ring $\mathcal{R}(X)$ is an integral normal ring. Moreover, one has the following statements on localization and units.*

(i) *For every non-zero homogeneous element $f \in \mathcal{R}_{[D]}(X)$, there is a canonical isomorphism*

$$\Gamma(X,\mathcal{R})_f \;\cong\; \Gamma(X_{[D],f},\mathcal{R}).$$

(ii) *Every homogeneous unit of $\mathcal{R}(X)$ is constant. If $\Gamma(X,\mathcal{O}) = \mathbb{C}$ holds, then we have $\mathcal{R}(X)^* = \mathbb{C}^*$.*

Definition 1.22. Consider a finitely generated abelian group K and a K-graded integral $\mathbb{C}$-algebra $R = \oplus_K R_w$.

(i) A non-zero non-unit $f \in R$ is *K-prime* if it is homogeneous and $f|gh$ with homogeneous $g, h \in R$ implies $f|g$ or $f|h$.

(ii) R is *K-factorial*, or *factorially graded*, if every homogeneous non-zero non-unit $f \in R$ is a product of K-primes.

Remark 1.23. For torsion free grading group and finitely generated $\mathbb{C}$-algebra, the concepts “factorially graded” and “factorial” coincide. As soon as there is torsion in the grading group, they may differ.

Proposition 1.24. *Let X be a normal variety with Cox ring $\mathcal{R}(X)$.*

(i) *An element $0 \neq f \in \Gamma(X,\mathcal{R}_{[D]})$ divides $0 \neq g \in \Gamma(X,\mathcal{R}_{[E]})$ if and only if $\mathrm{div}_{[D]}(f) \leq \mathrm{div}_{[E]}(g)$ holds.*

(ii) *An element $0 \neq f \in \Gamma(X,\mathcal{R}_{[D]})$ is $\mathrm{Cl}(X)$-prime if and only if the divisor $\mathrm{div}_{[D]}(f) \in \mathrm{WDiv}(X)$ is prime.*

Theorem 1.25. *Let X be a normal variety with Cox ring $\mathcal{R}(X)$.*

(i) *$\mathcal{R}(X)$ is $\mathrm{Cl}(X)$-factorial.*

(ii) *If $\mathrm{Cl}(X)$ is torsion free, then $\mathcal{R}(X)$ is a UFD.*

Part 1-B: Exercises

Exercise 1.26. Verify Example 1.18. *Hint:* Recall that the we have an isomorphism of groups

$$\mathbb{Z}^r \ \to \ \mathrm{WDiv}^{\mathbb{T}}(Z), \qquad a \ \mapsto \ D_a := a_1D_1 + \ldots + a_rD_r.$$

Now proceed as follows. Show that for any $a \in \mathbb{Z}^r$, we have an isomorphism of vector spaces

$$\psi_a \colon \Gamma(Z, \mathcal{O}(D_a)) \ \mapsto \ \mathbb{C}[T_1, \ldots, T_r]_{Q(a)} \qquad \chi^u \ \mapsto \ T^{P^*u+a},$$

where as usual $T^\mu = T_1^{\mu_1} \cdots T_r^{\mu_r}$. Show that the ψ_a, where $a \in \mathbb{Z}^r$ fit together to a homomorphism of $\mathbb{C}$-algebras

$$\psi \colon \Gamma(Z, \mathcal{S}) \ \to \ \mathbb{C}[T_1, \ldots, T_r]$$

The invariant principal divisors of Z are precisely the divisors of the character functions χ^u, where $u \in M$. This defines a homomorphism

$$\chi \colon \ \ker(Q) = P^*M \ \to \ \mathbb{C}(Z)^*, \qquad P^*u \ \mapsto \ \chi^u.$$

Show that $\ker(\psi)$ is generated by $1 - \chi^u$, where $u \in M$, and use Remark 1.19 to conclude

$$\mathcal{R}(Z) \ \cong \ \Gamma(Z, \mathcal{S})/\Gamma(Z, \mathcal{I}) \ \cong \ \mathbb{C}[T_1, \ldots, T_r].$$

Exercise 1.27. Verify the statements made in Remark 1.19. Show that the Cox sheaf and the Cox ring do not depend on the choices made in Construction 1.16.

Exercise 1.28. Show that the $[D]$-divisor does not depend on the choices made in Construction 1.20.

Exercise 1.29. Let X be a variety with Cox sheaf $\mathcal{R}$. Show that every affine open set $U \subseteq X$ is the $[D]$-localization of some $f \in \mathcal{R}_{[D]}(X)$. *Hint:* Use the fact that in any variety, the complement of a proper affine open subset is of pure codimension one.

Exercise 1.30. Show that the following $\mathbb{C}$-algebra does not admit unique factorization:

$$A \ := \ \mathbb{C}[T_1, T_2, T_3]/\langle T_1^2 + T_2^2 + T_3^2 \rangle$$

Show that A becomes factorially K-graded with $K = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ by setting

$$\deg(T_1) := (1, \bar{0}, \bar{0}), \quad \deg(T_2) := (1, \bar{1}, \bar{0}), \quad \deg(T_3) := (1, \bar{0}, \bar{1}).$$

Part 1-C. We introduce the characteristic space as the geometric realization of the Cox sheaf, consider the action of the characteristic quasitorus and discuss the geometry of the characteristic space.

Clip 1-C **Notes 1-C** **Exercises 1-C**

Part 1-C: Short Notes

Remark 1.31. Consider a variety X with Cox sheaf $\mathcal{R}$. Then $\mathcal{R}$ is *locally of finite type* if X is covered by an affine open $U \subseteq X$ such that the $\mathbb{C}$-algebra $\Gamma(U, \mathcal{R})$ is finitely generated.

(i) If $\mathcal{R}(X)$ is finitely generated, then $\mathcal{R}$ is locally of finite type.
(ii) If X is $\mathbb{Q}$-factorial, then $\mathcal{R}$ is locally of finite type.

Construction 1.32. Let X be a variety with Cox sheaf $\mathcal{R}$ locally of finite type. Cover X by open affine $X_i \subseteq X$ such that $R_i := \Gamma(X_i, \mathcal{R})$ is finitely generated. Consider the commutative diagrams

$$\begin{tikzcd}
R_i \arrow[r] & R_{ij} \arrow[r, equal] & R_{ji} & R_j \arrow[l] \\
(R_i)_0 \arrow[u] \arrow[r] & (R_{ij})_0 \arrow[u] \arrow[r, equal] & (R_{ji})_0 \arrow[u] & R(X_j)_0 \arrow[u] \arrow[l]
\end{tikzcd}$$

where $R_{ij} = \Gamma(X_{ij}, \mathcal{R})$ with $X_{ij} := X_i \cap X_j$ and the lower row lists the respective parts of $\mathrm{Cl}(X)$-degree zero. Passing to the spectra, we obtain gluing data

$$\begin{tikzcd}
\hat{X}_i & \hat{X}_{ij} \arrow[l] \arrow[r, equal] \arrow[d] & \hat{X}_{ji} \arrow[r] \arrow[d] & \hat{X}_j \arrow[d] \\
X_i \arrow[from=u] & X_{ij} \arrow[l] \arrow[r, equal] & X_{ji} \arrow[r] & X_j
\end{tikzcd}$$

Gluing yields a normal quasiaffine variety $\hat{X} = \mathrm{Spec}_X \, \mathcal{R}$, coming with an affine morphism $p\colon \hat{X} \to X$. We call $\hat{X}$ *the characteristic space* over X. Note that we have

$$p_*(\mathcal{O}_{\hat{X}}) \; = \; \mathcal{R}, \qquad \Gamma(\hat{X}, \mathcal{O}) \; = \; \Gamma(X, \mathcal{R}).$$

Remark 1.33. In Construction 1.32, all the algebras are graded by the finitely generated abelian group $\mathrm{Cl}(X)$ and the involved homomorphisms respect these gradings.

Reminder 1.34. A *quasitorus* is an algebraic group isomorphic to a direct product of a torus and a finite abelian group. We have a contravariant equivalence of categories:

$$\begin{array}{rcl}
\left\{ \begin{array}{c} \text{finitely generated} \\ \text{abelian groups} \end{array} \right\} & \longleftrightarrow & \{\text{quasitori}\} \\
K & \mapsto & \mathrm{Spec}\ \mathbb{C}[K], \\
\mathbb{X}(H) & \leftarrow\!\shortmid & H.
\end{array}$$

Reminder 1.35. Let a quasitorus H act on an affine variety X. We call $f \in \Gamma(X, \mathcal{O})$ *homogeneous* with respect to a character $\chi \in \mathbb{X}(H)$ if

$$f(h \cdot x) \; = \; \chi(h) f(x), \quad \text{for all } h \in H, \ x \in X.$$

Let $\Gamma(X,\mathcal{O})_\chi \subseteq \Gamma(X,\mathcal{O})$ denote the vector subspace of all χ-homogeneous functions. Then one has a grading

$$\Gamma(X,\mathcal{O}) \ = \ \bigoplus_{\chi\in\mathbb{X}(H)} \Gamma(X,\mathcal{O})_\chi.$$

Fact 1.36. The correspondence between affine algebra and affine varieties specializes to a contravariant equivalence of categories

$$\begin{array}{rcl} \{\text{graded affine algebras}\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{affine varieties with} \\ \text{quasitorus action} \end{array} \right\} \\ A & \mapsto & \operatorname{Spec} A, \\ \Gamma(X,\mathcal{O}) & \leftarrow\!\shortmid & X. \end{array}$$

Here we mean by a graded affine algebra an integral finitely generated $\mathbb{C}$-algebra graded by a finitely generated abelian group.

Definition 1.37. Let a quasitorus H act on a variety X. A morphism $p\colon X \to Y$ is called a *good quotient* if

(i) $p\colon X \to Y$ is affine and H-invariant,
(ii) the pullback $p^*\colon \mathcal{O}_Y \to (p_*\mathcal{O}_X)^H$ is an isomorphism.

A good quotient $p\colon X \to Y$ is called *geometric* if its fibers are precisely the orbits of the H-action.

Proposition 1.38. *Let a quasitorus H act on a variety X with good quotient $p\colon X \to Y$. Then p is surjective and for any $y \in Y$ one has:*

(i) *There is exactly one closed H-orbit $H \cdot x_y$ in the fiber $p^{-1}(y)$.*
(ii) *Every orbit $H \cdot x \subseteq p^{-1}(y)$ has $H \cdot x_y$ in its closure.*

Remark 1.39. Let X be a variety with characteristic space $p\colon \hat{X} \to X$. Then the *characteristic quasitorus*

$$H \ = \ \operatorname{Spec} \mathbb{C}[\operatorname{Cl}(X)]$$

acts on on the variety $\hat{X}$ and the morphism $p\colon \hat{X} \to X$ is a good quotient for this action.

Definition 1.40. Let X be a normal variety and let $x \in X$ be any point. Then we have the subgroup

$$\operatorname{PDiv}(X,x) \ \subseteq \ \operatorname{WDiv}(X)$$

of all Weil divisors being principal near x. The *local class group* of X at x is the factor group

$$\operatorname{Cl}(X,x) \ := \ \operatorname{WDiv}(X)/\operatorname{PDiv}(X,x).$$

Proposition 1.41. *Consider the characteristic space $p\colon \hat{X} \to X$ and the action of the characteristic quasitorus H on $\hat{X}$. Given $x \in X$, fix $\hat{x} \in p^{-1}(x)$ with closed H-orbit. Then*

$$\operatorname{Cl}(X,x) \ \cong \ \mathbb{X}(H_{\hat{x}}).$$

Corollary 1.42. *Consider the characteristic space* $p\colon \hat{X} \to X$ *and the action of the characteristic quasitorus* H *on* $\hat{X}$*.*

(i) *The action of* H_X *on* $\hat{X}$ *is free if and only if* X *is locally factorial.*
(ii) *The good quotient* $p\colon \hat{X} \to X$ *is geometric if and only if* X *is* $\mathbb{Q}$*-factorial.*

Reminder 1.43. A Weil divisor on a normal variety X is *Cartier* if it is locally principal. We write

$$\mathrm{CDiv}(X) \ \subseteq \ \mathrm{WDiv}(X)$$

for the subgroup of all Cartier divisors of X. The *Picard group* of X is the factor group

$$\mathrm{Pic}(X) \ = \ \mathrm{CDiv}(X)/\,\mathrm{PDiv}(X).$$

Corollary 1.44. *Consider the characteristic space* $p\colon \hat{X} \to X$ *and the action of the characteristic quasitorus* H *on* $\hat{X}$*. Let* $\hat{H} \subseteq H$ *be the subgroup generated by all isotropy groups* $H_{\hat{x}}$*, where* $\hat{x} \in \hat{X}$*. Then*

$$\mathrm{Pic}(X) \ \cong \ \mathbb{X}(H/\hat{H}).$$

Part 1-C: Exercises

Exercise 1.45. Prove Remark 1.31. *Hint:* For the first statement, use Theorem 1.21 (i). For the second one, use $\mathcal{R} = \mathcal{S}/\mathcal{I}$, Exercise 1.13 and [1, Cor. 1.1.2.6].

Exercise 1.46. Let $A = \oplus_K A_w$ be a K-graded affine algebra and suppose that we are given homogeneous generators

$$A \ = \ \mathbb{C}[f_1,\dots,f_r], \qquad f_i \in A_{w_i},\ i = 1,\dots,r.$$

Consider $X = \mathrm{Spec}\ A$ with the action of $H = \mathrm{Spec}\ \mathbb{C}[K]$. Convince yourself about the following. We have a closed embedding

$$X \ \to \ \mathbb{C}^r, \qquad x \ \mapsto \ (f_1(x),\dots,f_r(x)).$$

The image $X \subseteq \mathbb{C}^r$ is invariant under the diagonal action of $H = \mathrm{Spec}\ \mathbb{C}[K]$ given by

$$h \cdot z \ = \ (\chi^{w_1}(h)z_1,\dots,\chi^{w_r}(h)z_r),$$

where $\chi^{w_1} \in \mathbb{X}(H)$ is character given by $\mathbb{Z} \to K$, $a \mapsto aw$. For any $f \in A$ homogeneity is characterized by

$$f \in A_w \ \Leftrightarrow \ f(h \cdot x) = \chi^w(h)f(x) \text{ for all } h \in H,\ x \in X.$$

Exercise 1.47. Verify all the statements made in Remark 1.39.

Exercise 1.48. Prove Corollary 1.44 by using Proposition 1.41.

Exercise 1.49. Consider the characteristic space $p\colon \hat{X} \to X$ and the action of the characteristic quasitorus H on $\hat{X}$. Show that if there is an H-fixed point in $\hat{X}$, then the Picard group $\mathrm{Pic}(X)$ is trivial.

2. Varieties with finitely generated Cox ring

Part 2-A. We introduce the total coordinate space and the irrelevant ideal and we characterize coordinate spaces as well as characteristic spaces via properties of the characteristic quasitorus action.

Clip 2-A **Notes 2-A** **Exercises 2-A**

Part 2-A: Short Notes

Construction 2.1. Let X be a variety with finitely generated Cox ring $\mathcal{R}(X)$. The *total coordinate space* of X is the normal affine variety

$$\bar{X} \ := \ \mathrm{Spec}\ \mathcal{R}(X)$$

with the action of the characteristic quasitorus $H = \mathrm{Spec}\ \mathbb{C}[\mathrm{Cl}(X)]$ given by the $\mathrm{Cl}(X)$-grading of $\mathcal{R}(X)$. The isomorphisms

$$\Gamma(\mathcal{O}, \bar{X}) \ = \ \mathcal{R}(X) \ = \ \Gamma(\mathcal{R}, X) \ = \ \Gamma(\mathcal{O}, \hat{X})$$

of graded algebras yield an H-equivariant open embedding $\hat{X} \subseteq \bar{X}$ with complement $\bar{X} \setminus \hat{X}$ of codimension at least two in $\bar{X}$. Altogether:

$$\begin{tikzcd}
\mathrm{Spec}_X \mathcal{R} \arrow[r, equal] & \hat{X} \arrow[d, "p"', "/\!\!/ H"] \arrow[r, phantom, "\subseteq"] & \bar{X} \arrow[r, equal] & \mathrm{Spec}\ \mathcal{R}(X) \\
& X & &
\end{tikzcd}$$

Definition 2.2. In the setting of Construction 2.1, the *irrelevant ideal* of X is the vanishing ideal of the complement $\bar{X} \setminus \hat{X}$ in the Cox ring:

$$\mathcal{J}_{\mathrm{irr}}(X) \ := \ \{f \in \mathcal{R}(X);\ f|_{\bar{X} \setminus \hat{X}} = 0\} \ \subseteq \ \mathcal{O}(\bar{X}) \ = \ \mathcal{R}(X).$$

Proposition 2.3. *Situation as in Construction 2.1. The irrelevant ideal $\mathcal{J}_{\mathrm{irr}}(X) \subseteq \mathcal{R}(X)$ is homogeneous and we have the following.*

(i) *For any homogeneous element $f \in \mathcal{R}(X)$, the membership in the irrelevant ideal is characterized by*

$$f \in \mathcal{J}_{\mathrm{irr}}(X) \iff \bar{X}_f = \hat{X}_f \iff \hat{X}_f \text{ is affine.}$$

(ii) *Let $0 \neq f \in \mathcal{R}_{[D]}(X)$. If the $[D]$-localization $X_{[D],f}$ is affine, then we have $f \in \mathcal{J}_{\mathrm{irr}}(X)$.*

(iii) *Let $0 \neq f_i \in \mathcal{R}_{[D_i]}(X)$, where $1 \leq i \leq r$, be such that the sets $X_{[D_i],f_i}$ are affine and cover X. Then we have*

$$\mathcal{J}_{\mathrm{irr}}(X) \ = \ \sqrt{\langle f_1, \ldots, f_r \rangle}.$$

Example 2.4. Consider the toric variety Z arising from a non-degenerate lattice fan (Σ, N) and the linear map

$$P\colon \mathbb{Z}^r \ \to \ N, \qquad e_i \ \mapsto \ v_i,$$

defined by the primitive $v_1, \ldots, v_r$ of Σ. The divisor class group of Z and the characteristic quasitorus are given by

$$\mathrm{Cl}(Z) = \mathbb{Z}^r / P^* M, \qquad H = \operatorname{Spec} \mathbb{C}[\mathbb{Z}^r / P^* M],$$

where $P^*\colon M \to \mathbb{Z}^r$ denotes the dual map of P. With the projection $Q\colon \mathbb{Z}^r \to \mathbb{Z}^r / P^* M$, the Cox ring of Z is

$$\mathcal{R}(Z) = \mathbb{C}[T_1, \ldots, T_r], \qquad \deg(T_i) = Q(e_i).$$

Now, write $\delta^r := \mathbb{Q}^r_{\ge 0}$ for the positive orthant and define a subfan $\hat{\Sigma}$ of the fan $\bar{\Sigma}$ of faces of δ^r by

$$\hat{\Sigma} \ := \ \{\delta \preccurlyeq \delta^r; \ P(\delta) \subseteq \sigma \text{ for some } \sigma \in \Sigma\}.$$

Then $P\colon \mathbb{Z}^r \to N$ is a map of fans from $\hat{\Sigma}$ to Σ. The associated toric morphisms give us the picture from Construction 2.1:

$$\begin{tikzcd}
\operatorname{Spec}_Z \mathcal{R} \arrow[r, equal] & \hat{Z} \arrow[r, phantom, "\subseteq"] \arrow[d, "p"', "/\!\!/ H"] & \bar{Z} \arrow[r, equal] & \operatorname{Spec} \mathcal{R}(Z) \arrow[r, equal] & \mathbb{C}^r \\
& Z & & &
\end{tikzcd}$$

Observe that $H = \ker(p)$ acts on $\bar{Z}$ as a subgroup of $\mathbb{T}^r$. For the irrelevant ideal of Z, we obtain

$$\mathcal{J}_{\mathrm{irr}}(X) \ = \ \langle T^\sigma; \ \sigma \in \Sigma \rangle \ \subseteq \ \mathbb{C}[T_1, \ldots, T_r], \qquad T^\sigma \ = \ \prod_{v_i \notin \sigma} T_i.$$

Definition 2.5. Let a quasitorus H act on a variety W.

(i) $f \in \mathbb{C}(W)$ is *homogeneous* w.r.t. to $\chi \in \mathbb{X}(H)$ if f is defined and χ-homogeneous on a non-empty invariant open set.
(ii) W is *H-factorial* if it is normal and any H-invariant Weil divisor of W is the divisor of a homogeneous rational function.

Remark 2.6. Let W be a normal quasiaffine variety with an action of a quasitorus H. Then we have the grading

$$\Gamma(W, \mathcal{O}) \ = \ \bigoplus_{\mathbb{X}(H)} \Gamma(W, \mathcal{O})_\chi.$$

The variety W is H-factorial if and only if the algebra $\Gamma(W, \mathcal{O})$ is $\mathbb{X}(H)$-factorial.

Remark 2.7. Let X be a variety with finitely generated Cox ring $\mathcal{R}(X)$. Since $\mathcal{R}(X)$ is $\mathrm{Cl}(X)$-factorial, the total coordinate space $\bar{X}$ is H-factorial.

Definition 2.8. The action of a quasitorus H on a variety W is *strongly stable* if there is an open invariant subset $W' \subseteq W$ such that

(i) $W \setminus W'$ is of codimension at least two in W,
(ii) the group H acts freely on W',
(iii) for every $x \in W'$ the orbit $H \cdot x$ is closed in W.

Remark 2.9. Let X be a variety with finitely generated Cox ring $\mathcal{R}(X)$. Then the action of the characteristic quasitorus H on the total coordinate space $\bar{X}$ is strongly stable.

Theorem 2.10. *Let a quasitorus H act on a quasiaffine variety W with a good quotient $p\colon W \to X$. Assume that*

(i) *W has only constant invertible homogeneous global functions,*
(ii) *W is H-factorial,*
(iii) *the H-action is strongly stable.*

Then X is a normal variety with $\Gamma(X, \mathcal{O}^*) = \mathbb{C}^*$. Moreover, divisor class group, Cox sheaf, characteristic space of X are given by

$$\mathrm{Cl}(X) \;\cong\; \mathbb{X}(H), \qquad \mathcal{R} \cong p_*\mathcal{O}_W, \qquad p\colon W \to X.$$

Corollary 2.11. *Let H be a quasitorus, $\bar{W}$ an H-factorial affine variety with only constant invertible homogeneous global functions and $W \subseteq \bar{W}$ an open, H-invariant subset such that*

(i) *$\bar{W} \setminus W$ is of codimension at least two in $\bar{W}$,*
(ii) *the H-action on W is strongly stable,*
(iii) *the H-action on W has a good quotient $p\colon W \to X$.*

Then $p\colon W \to X$ is a characteristic space over X and $\bar{W}$ is a total coordinate space for X.

Remark 2.12. Building on Corollary 2.11, it will be our task for the remaining unit to provide systematic constructions of varieties with finitely generated Cox ring.

Part 2-A: Exercises

Exercise 2.13. Verify the details of the toric case as discussed in Example 2.4.

Exercise 2.14. Verify the statements of Remark 2.9. *Hint:* Use Proposition 1.41.

Exercise 2.15. Consider the quadric $X = V(T_0^2 + \ldots + T_n^2)$ in $\mathbb{P}_n$. Use Corollary 2.11 to concude $\mathrm{Cl}(X) = \mathbb{Z}$ and

$$\mathcal{R}(X) = \mathbb{C}[T_0, \ldots, T_n]/\langle T_0^2 + \ldots + T_n^2\rangle, \quad \deg(T_i) = 1,\ i = 0, \ldots, n.$$

Part 2-B. Construction of GIT-quotients, the GIT-fan, good quotients with a quotient space embeddable into a toric variety.

Clip 2-B **Notes 2-B** **Exercises 2-B**

Part 2-B: Short Notes

Reminder 2.16. Consider the action of a quasitorus H on a variety X.

(i) If X is affine, thus $X = \text{Spec } A$ with a K-graded algebra A, then one has the *algebraic quotient*

$$\pi\colon X \ \to \ Y = \text{Spec } \Gamma(X, \mathcal{O})^H = \text{Spec } A_0.$$

(ii) A *good quotient* is an affine, H-invariant morphism $\pi\colon X \to Y$ such that $\pi^*\colon \mathcal{O}_Y \to \pi_*(\mathcal{O}_X)^H$ is an isomorphism.

Example 2.17. For any $a, b \in \mathbb{Z}$, we have the diagonal action of $\mathbb{C}^*$ on the affine plane $\mathbb{C}^2$ given by

$$t \cdot z \ = \ (t^a z_1, t^b z_2).$$

In the *elliptic case*, the origin is an attractive fixed point. For instance, for $a = b = 1$, we have two good quotients:

$$\mathbb{C}^2 \ \to \ \{\text{pt}\}, \qquad\qquad \mathbb{C}^2 \setminus \{0\} \ \to \ \mathbb{P}_1, \quad z \ \mapsto [z].$$

The first one is the algebraic quotient, the second one is a geometric quotient, meaning that the fibers are the orbits.

In the *parabolic case*, we have curve consisting of fixed points. For instance, for $a = 0$ and $b = 1$, we have two algebraic quotients:

$$\mathbb{C}^2 \ \to \ \mathbb{C}, \quad z \mapsto z_1, \qquad \mathbb{C}^2 \setminus V(T_2) \ \to \ \mathbb{C}, \quad z \mapsto z_1.$$

In the *hyperbolic case*, the general orbit is closed. We have three good quotients, the latter two of them geometric:

$$\mathbb{C}^2 \ \to \ \mathbb{C}, \quad z \mapsto z_1 z_2, \qquad \begin{array}{l} \mathbb{C}^2 \setminus V(T_1) \ \to \ \mathbb{C}, \quad z \mapsto z_1 z_2, \\ \mathbb{C}^2 \setminus V(T_1) \ \to \ \mathbb{C}, \quad z \mapsto z_1 z_2. \end{array}$$

Construction 2.18. Let the quasitorus H act on the affine variety X. The set of *semistable points* associated with $\chi \in \mathbb{X}(H)$ is

$$X^{ss}(\chi) \ := \ \{x \in X;\ f(x) \neq 0 \text{ for } f \in \Gamma(X, \mathcal{O})_{n\chi},\ n > 0\}$$

The subset $X^{ss}(\chi) \subseteq X$ is open and H-invariant. Given $n > 0$ and $f \in \Gamma(X, \mathcal{O})_{n\chi}$, we have the algebraic quotient

$$\pi_f\colon X_f \ \to \ U_f, \qquad U_f \ = \ \text{Spec } \Gamma(X_f, \mathcal{O})^H.$$

By definition, these sets X_f cover $X^{ss}(\chi)$. Provided $X^{ss}(\chi) \neq \emptyset$, the above algebraic quotients yield gluing data

$$\begin{tikzcd}
X_f \arrow[d, "\pi_f"'] & \supseteq & X_{fg} \arrow[d] & \subseteq & X_g \arrow[d, "\pi_g"] \\
U_f & & U_{fg} \arrow[ll] \arrow[rr] & & U_g
\end{tikzcd}$$

The result of the gluing process is a good quotient $\pi\colon X^{ss}(\chi) \to Y(\chi)$ for the H-action on $X^{ss}(\chi)$, the *GIT-quotient* associated with χ.

Remark 2.19. The quotient space $Y(\chi)$ equals the homogeneous spectrum $\mathrm{Proj}(A(\chi))$ of the finitely generated $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-algebra

$$A(\chi) \;:=\; \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \Gamma(X, \mathcal{O})_{n\chi} \;\subseteq\; \Gamma(X, \mathcal{O}).$$

Thus, the quotient $X^{ss}(\chi) \to Y(\chi)$ is *projective* in the sense that the morphism $Y(\chi) \to Y(0)$ is projective, where $Y(0) = \mathrm{Spec}\ \Gamma(X, \mathcal{O})^H$.

Construction 2.20. Let the quasitorus H act on the affine variety X. We have the following cones in the rational vector space $\mathbb{X}_{\mathbb{Q}}(H)$:

$$\begin{array}{rcl}
\omega_X & := & \mathrm{cone}(\chi \in \mathbb{X}(G);\ \Gamma(X, \mathcal{O})_\chi \neq \{0\}), \\
\omega_x & := & \mathrm{cone}(\chi \in \mathbb{X}(G);\ f(x) \neq 0 \text{ for a } f \in \Gamma(X, \mathcal{O})_\chi), \\
\lambda_\chi & := & \bigcap\limits_{\substack{x \in X, \\ \chi \in \omega_x}} \omega_x,
\end{array}$$

the *weight cone* ω_X of X, the *orbit cone* ω_x of a point $x \in X$ and the *GIT-cone* λ_χ of a character $\chi \in \omega_X$.

Remark 2.21. The weight cone, the orbit cones and the GIT-cones are all convex and polyhedral. Moreover, the following sets are finite

$$\Omega_x \;:=\; \{\omega_x;\ x \in X\}, \qquad \Lambda(X) \;:=\; \{\lambda_\chi;\ \chi \in \omega_X\}.$$

Theorem 2.22. *Let the quasitorus H act on an affine variety X. Then $\Lambda(X)$ is a quasifan with support $\omega_X \subseteq \mathbb{X}_{\mathbb{Q}}(H)$. Moreover,*

$$\lambda_{\chi_1} \;\preccurlyeq\; \lambda_{\chi_2} \quad\Longleftrightarrow\quad X^{ss}(\chi_1) \;\supseteq\; X^{ss}(\chi_2)$$

holds for any two characters $\chi_1, \chi_2 \in \omega_X$. If X is H-factorial, then we have a bijection

$$\begin{array}{rcl}
\Lambda(X) & \longrightarrow & \left\{ \begin{array}{l} H\text{-invariant open } X' \subseteq X \text{ with} \\ \text{projective good quotient } X' \to Y' \end{array} \right\} \\
\lambda & \mapsto & X^{ss}(\chi), \text{ where } \chi \in \lambda^\circ.
\end{array}$$

Remark 2.23. A variety has the *A_2-property* if any two of its points admit a common affine neighborhood. A normal variety has the A_2-property if and only if it admits a closed embedding into a normal toric variety.

Definition 2.24. Let the quasitorus H act on an affine variety X. A *bunch of orbit cones* is a set $\emptyset \neq \Phi \subseteq \Omega_X$ of orbit cones such that

(i) given $\omega_1, \omega_2 \in \Phi$, one has $\omega_1^\circ \cap \omega_2^\circ \neq \emptyset$,
(ii) given $\omega \in \Phi$ and $\omega_0 \in \Omega_X$ with $\omega^\circ \subseteq \omega_0^\circ$, one has $\omega_0 \in \Phi$.

A *maximal bunch of orbit cones* is a bunch of orbit cones which cannot be enlarged by adding further orbit cones.

Construction 2.25. Let the quasitorus H act on an affine variety X. For any bunch of orbit cones $\Phi \subseteq \Omega_X$, we set

$$X(\Phi) \ := \ \{x \in X;\ \omega \preccurlyeq \omega_x \text{ for some } \omega \in \Phi\}.$$

Then $X(\Phi) \subseteq X$ is an $(H,2)$*-set* in the sense that it is open, H-invariant and there is a good quotient

$$X(\Phi) \ \to \ Y(\Phi)$$

for the action of H on $X(\Phi)$ and the quotient variety $X(\Phi)$ has the A_2-property.

Definition 2.26. Let the quasitorus H act on an affine variety X. An $(H,2)$*-maximal set* is an $(H,2)$-set $U \subseteq X$ such that given any $(H,2)$-set $U_1 \subseteq X$ with good quotient $\pi_1 \colon U_1 \to Y_1$ and an open $V \subseteq Y_1$ with $U = \pi_1^{-1}(V)$, we have $U = U_1$.

Theorem 2.27. *Let the quasitorus H act on an affine variety X and assume that X is H-factorial. Then we have a bijection*

$$\left\{ \begin{array}{l} \textit{maximal bunches of} \\ \textit{orbit cones in } \Omega_X \end{array} \right\} \ \longrightarrow \ \left\{ \begin{array}{l} \textit{(H,2)-maximal} \\ \textit{subsets of } X \end{array} \right\}$$
$$\Phi \ \mapsto \ X(\Phi).$$

PART 2-B: EXERCISES

Exercise 2.28. Verify all the statements made in Example 2.17. Compute the GIT-fan $\Lambda(\mathbb{C}^2)$ for each of the discussed cases and retrieve the quotients presented there as GIT-quotients.

Exercise 2.29. Let the quasitorus H act on an affine variety X. Convince yourself about the following.

(i) The algebra $\Gamma(X, \mathcal{O})$ is generated by finitely many χ_i-homogeneous functions f_i, say $i = 1, \ldots, r$.
(ii) Every orbit cone ω_x, where $x \in X$ is generated by some of the characters $\chi_1, \ldots, \chi_r$.
(iii) The weight cone ω_X is generated by $\chi_1, \ldots, \chi_r$ and there is a non-empty open $u \subseteq X$ with $\omega_X = \omega_x$ for all $x \in U$.

Use these observations to prove all the statements made in Remark 2.21.

Exercise 2.30. Let the quasitorus H act on an affine variety X. Given GIT-cones $\lambda_{\chi_1} \preccurlyeq \lambda_{\chi_2}$, show that we have a commutative diagram

$$\begin{tikzcd}
X^{ss}(\chi_2) \arrow[r, phantom, "\subseteq"] \arrow[d] & X^{ss}(\chi_1) \arrow[d] \\
Y(\chi_2) \arrow[r] & Y(\chi_1)
\end{tikzcd}$$

where the induced morphism $Y(\chi_2) \to Y(\chi_2)$ is projective. *Hint:* Use that $Y(\chi_i) \to Y(0)$ is projective.

Part 2-C. We discuss linear Gale duality, introduce concept of a bunched ring, perform the construction of the associated variety and figure out the bunched rings producing projective varieties.

Clip 2-C **Notes 2-C** **Exercises 2-C**

Part 2-C: Short Notes

Remark 2.31. Consider a non-degenerate lattice fan (Σ, N) with primitive generators $v_1, \ldots, v_r$. Then we have exact sequences

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] & \mathbb{Z}^r \arrow[r, "P\colon e_i \mapsto v_i"] & N & \\
 & & \delta \succcurlyeq \hat{\sigma} \arrow[d] & \sigma \arrow[l] & \\
 & Q(\hat{\sigma}^*) & \hat{\sigma}^* \preccurlyeq \gamma \arrow[l] & & \\
0 & K \arrow[l] & \mathbb{Z}^r \arrow[l, "Q"'] & M \arrow[l, "P^*"'] & 0 \arrow[l]
\end{tikzcd}$$

and a process, also called *linear Gale duality*, associating with any cone $\sigma \in \Sigma$ the cone $Q(\hat{\sigma}^*) \subseteq K_{\mathbb{Q}}$ via

$$\operatorname{cone}(e_i;\ v_i \in \sigma) = \hat{\sigma} \ \mapsto \ \hat{\sigma}^* = \operatorname{cone}(e_j;\ v_j \notin \sigma).$$

Here, $\delta = \mathbb{Q}^r_{\geq 0}$ and $\gamma = \mathbb{Q}^r_{\geq 0}$ are regarded as dual to each other and the displayed assignment is the face correspondence.

Remark 2.32. Let the toric variety Z arise from a non-degenerate lattice fan (Σ, N). Divisor class group and Cox ring of Z are given by

$$\operatorname{Cl}(Z) = K, \qquad \mathcal{R}(Z) = \mathbb{C}[T_1, \ldots, T_r], \qquad \deg(T_i) = Q(e_i).$$

The characteristic quasitorus $H = \operatorname{Spec} \mathbb{C}[K]$ acts via the grading on the total coordinate space $\bar{Z} = \mathbb{C}^r$. We observe:

(i) the algebra $\mathcal{R}(Z)$ is K-factorial and $T_1, \ldots, T_r$ form a system of pairwise non-associated K-prime generators,

(ii) the K-grading of $\mathcal{R}(Z)$ is almost free, that means that any $r-1$ of $\deg(T_1), \ldots, \deg(T_r)$ generate K as a group,

(iii) the collection $\Phi = \{Q(\hat{\sigma}^*);\ \sigma \in \Sigma\}$ is a bunch of orbit cones of the H-action on $\bar{Z}$, containing all $Q(\varrho_i^*)$, where $i = 1, \ldots, r$.

The toric variety Z is an open subset of $Z(\Phi) = \bar{Z}(\Phi) /\!\!/ H$. If Z is suitably maximal, for instance affine or complete, then $Z = Z(\Phi)$ holds.

Definition 2.33. Let R be a normal, K-factorial $\mathbb{C}$-algebra with a system $\mathfrak{F} = (f_1, \ldots, f_r)$ of pairwise non-associated K-prime generators.

(i) Set $w_i := \deg(f_i) \in K$. The K-grading of R is *almost free*, if any $r-1$ of $w_1, \ldots, w_r$ generate the group K.

(ii) The *projected cone* assigned to $\mathfrak{F}$ is $(\mathbb{Z}^r \xrightarrow{Q} K, \gamma)$, where the homomorphism Q and the cone γ are

$$Q\colon \mathbb{Z}^r \to K, \quad e_i \to \deg(f_i), \quad \gamma = \mathrm{cone}(e_1, \ldots, e_r) \subseteq E_{\mathbb{Q}}.$$

(iii) An $\mathfrak{F}$*-face* is a face $\gamma_0 \preccurlyeq \gamma$ admitting a point $\bar{x} \in \bar{X} := \operatorname{Spec} R$ such that for every $i = 1, \ldots, r$ we have

$$f_i(\bar{x}) \neq 0 \ \Leftrightarrow\ e_i \in \gamma_0.$$

(iv) Set $\Omega_{\mathfrak{F}} = \{Q(\gamma_0);\ \gamma_0 \preccurlyeq \gamma\ \mathfrak{F}\text{-face}\}$. By an $\mathfrak{F}$*-bunch* we mean a nonempty subset $\Phi \subseteq \Omega_{\mathfrak{F}}$ such that
  (a) given $\omega_1, \omega_2 \in \Phi$, one has $\omega_1^\circ \cap \omega_2^\circ \neq \emptyset$,
  (b) given $\omega \in \Phi$ and $\omega_0 \in \Omega_{\mathfrak{F}}$ with $\omega^\circ \subseteq \omega_0^\circ$, one has $\omega_0 \in \Phi$.

(v) We say that an $\mathfrak{F}$-bunch Φ is *true* if for every facet $\gamma_0 \prec \gamma$ the image $Q(\gamma_0)$ belongs to Φ.

Definition 2.34. A *bunched ring* $(R, \mathfrak{F}, \Phi)$ consists of a normal, almost free K-factorial affine $\mathbb{C}$-algebra R with $R^*_\times = \mathbb{C}^*$, a system $\mathfrak{F}$ of pairwise non-associated K-prime generators and a true $\mathfrak{F}$-bunch Φ.

Construction 2.35. Let $(R, \mathfrak{F}, \Phi)$ be a bunched ring. Then the quasitorus $H = \operatorname{Spec} \mathbb{C}[K]$ acts on $\bar{X} = \operatorname{Spec} R$ and we obtain

$$\begin{tikzcd}
\bar{X}(\Phi) \arrow[r, phantom, "="] & \hat{X} \arrow[d, "p"', "/\!\!/ H"] \arrow[r, phantom, "\subseteq"] & \bar{X} \arrow[r, phantom, "="] & \operatorname{Spec} R \\
X(R, \mathfrak{F}, \Phi) \arrow[r, phantom, "="] & X & &
\end{tikzcd}$$

where the open set $\hat{X} \subseteq \bar{X}$ and the good quotient $p\colon \hat{X} \to X$ arise from Φ, regarded as a bunch of orbit cones of the H-action.

Remark 2.36. For a bunched polynomial ring, Construction 2.35 delivers a toric variety; the defining fan is given by linear Gale duality as

$$\Sigma \ = \ \{P(\delta_0);\ \delta_0 \preccurlyeq \delta,\ Q(\delta_0^*) \in \Phi\}.$$

Theorem 2.37. *Let X arise from a bunched ring $(R, \mathfrak{F}, \Phi)$. Then X is a normal A_2-variety with*

$$\dim(X) \ = \ \dim(R) - \dim(K_{\mathbb{Q}}), \qquad \Gamma(X, \mathcal{O}^*) = \mathbb{C}^*,$$

Moreover, the divisor class group, the Cox ring and the characteristic space of X are given by

$$\mathrm{Cl}(X) \ = \ K, \qquad \mathcal{R}(X) \ = \ R, \qquad p\colon \hat{X} \ \to \ X.$$

Remark 2.38. An A_2*-maximal variety* is an A_2-variety X such that for any open embedding $X \subseteq X'$ into an A_2-variety X' one has $X = X'$ or $X' \setminus X$ is of codimension one in X'. Note that

(i) every affine variety is A_2-maximal,
(ii) every projective variety is A_2-maximal,
(iii) every complete A_2-variety is A_2-maximal.

Theorem 2.39. *Every A_2-maximal variety with finitely generated Cox ring arises from a bunched ring.*

Remark 2.40. Let R be a K-factorial $\mathbb{C}$-algebra and $\mathfrak{F} = (f_1, \dots, f_r)$ a system of pairwise non-associated K-prime generators. Let

$$w \ \in \ Q(\gamma_1)^\circ \cap \ldots \cap Q(\gamma_r)^\circ \ \subseteq \ K_{\mathbb{Q}},$$

for $w \in K$, where $\gamma_1, \dots, \gamma_r \prec \gamma$ denote the facets of the orthant $\gamma = \mathbb{Q}^r_{\geq 0}$. Then we obtain a true $\mathfrak{F}$-bunch

$$\Phi(w) \ := \ \{\omega \in \Omega_{\mathfrak{F}}; \ w \in \omega^\circ\},$$

which is as well a bunch of orbit cones for the action of $H = \mathrm{Spec}\ \mathbb{C}[K]$ on $\bar{X} = \mathrm{Spec}\ R$. The associated open set of is a set of semistable points:

$$\hat{X} \ := \ \bar{X}(\Phi(w)) \ = \ \bar{X}^{ss}(w) \ \subseteq \ \bar{X}.$$

In particular, $X = \hat{X} /\!\!/ H$ is projective over Spec R_0. In case of a bunched ring $(R, \mathfrak{F}, \Phi(w))$, we adopt the short notation

$$(R, \mathfrak{F}, w) \ := \ (R, \mathfrak{F}, \Phi(w)).$$

The bunched rings $(R, \mathfrak{F}, w)$ deliver precisely the varieties with finitely generated Cox ring for which

$$X \ \to \ \mathrm{Spec}\ \Gamma(X, \mathcal{O}(X))$$

is a projective morphism; varieties with the latter property are also called *semiprojective.*

Part 2-C: Exercises

Exercise 2.41. Verify the observations made in Remark 2.32 (i), (ii) and (iii). *Hint:* Look at [1, Lemmas 2.1.4.1 and 2.2.3.2].

Exercise 2.42. Elaborate the details of Remark 2.36, where for a bunched polynomial ring $R = \mathbb{C}[T_1, \ldots, T_r]$ we tacitly assume $\mathfrak{F}$ to be $(T_1, \ldots, T_r)$. *Hint:* Look at [1, Lemmas 2.1.4.1 and 2.2.3.2].

Exercise 2.43. Verify the statements (i), (ii) and (iii) made in Remark 2.38.

Exercise 2.44. Verify Remark 2.40 in the case that R is a polynomial ring and the variables T_i are K-homogeneous. Moreover, show that $w \in K$ is the class of an ample divisor on the toric variety $X(R, \mathfrak{F}, w)$.

Exercise 2.45. Elaborate the following. We obtain an almost free K-factorial algebra R by setting

$$R \ := \ \mathbb{C}[T_1, \ldots, T_5]/\langle T_1T_2 + T_3^2 + T_4T_5 \rangle, \qquad K \ := \ \mathbb{Z}^2$$

and defining $\deg(T_i)$ to be the i-th column of the matrix

$$Q \ := \ \begin{bmatrix} 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 2 \end{bmatrix}.$$

Moreover, $T_1, \ldots, T_5$ induce a system $\mathfrak{F} = (f_1, \ldots, f_5)$ of pairwise non-associated K-prime generators. The $\mathfrak{F}$-faces are

$$\{0\}, \ \gamma_1, \ \gamma_2, \ \gamma_4, \ \gamma_5, \ \gamma_{1,4}, \ \gamma_{1,5}, \ \gamma_{2,4}, \ \gamma_{2,5}, \ \gamma_{1,2,3}, \ \gamma_{3,4,5},$$

$$\gamma_{1,2,3,4}, \ \gamma_{1,2,3,5}, \ \gamma_{1,2,4,5}, \ \gamma_{1,3,4,5}, \ \gamma_{2,3,4,5}, \ \gamma_{1,2,3,4,5},$$

where we set $\gamma_{i_1,\ldots,i_k} := \operatorname{cone}(e_{i_1}, \ldots, e_{i_k})$. We obtain a bunched ring $(R, \mathfrak{F}, w_3)$ with $w_3 = \deg(T_3)$. In

$$\bar{X} \ = \ V(T_1T_2 + T_3^2 + T_4T_5) \ \subseteq \ \mathbb{C}^5,$$

the open set $\hat{X} = \bar{X}(\Phi(w_3))$ equals $\bar{X}^{ss}(w_3)$ and $\hat{X}$ is the union of the four affine open subsets:

$$\hat{X} \ = \ \bar{X}_{f_1f_4} \cup \bar{X}_{f_2f_5} \cup \bar{X}_{f_1f_2f_3} \cup \bar{X}_{f_3f_4f_5}.$$

The resulting variety $X = X(R, \mathfrak{F}, w_3)$ is a projective surface with divisor class group and Cox ring given as

$$\operatorname{Cl}(X) \ = \ K, \qquad \mathcal{R}(X) \ = \ R.$$

In fact, the methods presented later yield that X is a $\mathbb{Q}$-factorial Gorenstein del Pezzo $\mathbb{C}^*$-surface with one singularity, of type A_2.

3. Geometry via defining data

Part 3-A. We construct the minimal ambient toric variety of the variety arising from a bunched ring, look at the induced stratification, describe the Picard group and discuss singularities and smooth points.

Clip 3-A **Notes 3-A** **Exercises 3-A**

Part 3-A: Short Notes

Remark 3.1. For any bunched ring $(R, \mathfrak{F}, \Phi)$, the associated projected cone $(\mathbb{Z}^r \xrightarrow{Q} K, \gamma)$ gives rise to a linear Gale duality scheme:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] & \mathbb{Z}^r \arrow[r, "P"] & N & \\
& & \mathrm{faces}(\delta) \supseteq \hat{\Phi}^* \arrow[r] & \Phi^* & \\
& \Phi \arrow[r] & \hat{\Phi} \subseteq \mathrm{faces}(\gamma) \arrow[u] & & \\
0 & K \arrow[l] & \mathbb{Z}^r \arrow[l, "Q"'] & M \arrow[l, "P^*"'] & 0 \arrow[l]
\end{tikzcd}$$

where $P^*\colon M \to \mathbb{Z}^r$ denotes the inclusion of $\ker(Q)$ into $\mathbb{Z}^r$ and $P\colon \mathbb{Z}^r \to N$ the dual map. Moreover,

$$\Phi^* \ = \ \{P(\gamma_0^*);\ \gamma_0 \preccurlyeq \gamma,\ Q(\gamma_0) \in \Phi\}.$$

Any $\sigma \in \Phi^*$ is a cone in $N_{\mathbb{Q}}$ generated by some $v_i := P(e_i)$. Any two cones $\sigma, \sigma' \in \Phi^*$ intersect in a common face. Set

$$\Sigma \ := \ \Sigma(\Phi) \ := \ \bigcup_{\sigma \in \Phi^*} \mathrm{faces}(\sigma).$$

Then Σ is a non-degenerate fan in the lattice N and $v_1, \ldots, v_r$ are the primitive generators of Σ.

Remark 3.2. Let $(R, \mathfrak{F}, \Phi)$ be a bunched ring. Then $\mathfrak{F} = (f_1, \ldots, f_r)$ provides us with the H-equivariant closed emdedding

$$\bar{X} \ \to \ \mathbb{C}^r, \qquad \bar{x} \ \mapsto \ (f_1(\bar{x}), \ldots, f_r(\bar{x}))$$

where $\bar{X}$ is the total coordinate space of $X = X(R, \mathfrak{F}, \Phi)$ and the characteristic quasitorus H acts on $\mathbb{C}^r$ via

$$h \cdot z \ = \ (\chi^{w_1}(h)z_1, \ldots, \chi^{w_r}(h)z_r), \qquad w_i \ = \ \deg(f_i).$$

Consider the map $P\colon \mathbb{Z}^r \to N$ dual to the inclusion of the kernel of Q. Then we have fans in N and $\mathbb{Z}^r$:

$$\Sigma \ = \ \Sigma(\Phi), \qquad \hat{\Sigma} \ = \ \{\delta_0 \preccurlyeq \delta;\ P(\delta_0) \in \sigma \text{ for some } \sigma \in \Sigma\}.$$

The morphism $\hat{Z} \to Z$ of the associated toric varieties defined by P is the characteristic space; it fits into the commutative diagram

$$\begin{tikzcd}
 & & \bar{X} \arrow[r] & \bar{Z} & = & \mathbb{C}^r \\
\bar{X}(\Phi) \arrow[d] & = & \hat{X} \arrow[r] \arrow[u] \arrow[d] & \hat{Z} \arrow[u] \arrow[d] & & \\
X(R, \mathfrak{F}, \Phi) & = & X \arrow[r] & Z & &
\end{tikzcd}$$

All horizontal arrows are closed embeddings, we call Z the *minimal ambient toric variety* of X and $X \to Z$ the *minimal toric embedding.*

Remark 3.3. Consider the variety X arising from a bunched ring $(R, \mathfrak{F}, \Phi)$ and the minimal ambient toric variety Z. Then we have

$$\mathrm{Cl}(X) \ = \ K \ = \ \mathrm{Cl}(Z), \qquad \mathcal{R}(X) \ = \ \mathcal{R}(Z)/I(\bar{X}).$$

Remark 3.4. Consider the variety X arising from a bunched ring $(R, \mathfrak{F}, \Phi)$ and the minimal ambient toric variety Z. For $\sigma \in \Sigma$, set

$$X(\sigma) \ := \ X \cap \mathbb{T}_N \cdot z_\sigma \ \subseteq \ X.$$

Each $X(\sigma) \subseteq X$ is locally closed. With the face $\hat{\sigma} = \mathrm{cone}(e_i;\ v_i \in \sigma)$ of the orthant $\delta = \mathbb{Q}^r_{\geq 0}$, we have

$$X(\sigma) \neq \emptyset \ \Leftrightarrow \ \hat{\sigma}^* \preccurlyeq \gamma \text{ is an } \mathfrak{F}\text{-face.}$$

If one of these conditions is satisfied, we call σ an *X-cone.* Denoting by $\Sigma_X \subseteq \Sigma$ the set of all X-cones, we have

$$X \ = \ \bigsqcup_{\sigma \in \Sigma_X} X(\sigma).$$

Proposition 3.5. *Consider $X = X(R, \mathfrak{F}, \Phi)$ and its minimal ambient toric variety Z. Given $\sigma \in \Sigma$ and $x \in X(\sigma)$, we have*

$$\mathrm{Cl}(X, x) \ = \ K/Q(\mathrm{lin}(\hat{\sigma}^*) \cap \mathbb{Z}^r) \ = \ \mathrm{Cl}(Z, x).$$

Proposition 3.6. *Consider $X = X(R, \mathfrak{F}, \Phi)$ and its minimal ambient toric variety Z. Then, in $\mathrm{Cl}(X) = K = \mathrm{Cl}(Z)$, we have*

$$\mathrm{Pic}(X) \ = \ \bigcap_{\sigma \in \Sigma} Q(\mathrm{lin}(\hat{\sigma}^*) \cap \mathbb{Z}^r) \ = \ \mathrm{Pic}(Z).$$

Proposition 3.7. *Let X arise from a bunched ring $(R, \mathfrak{F}, \Phi)$. For every X-cone $\sigma \in \Sigma$, the following statements are equivalent:*

(i) *$X(\sigma)$ contains a factorial point of X,*
(ii) *every $x \in X(\sigma)$ is a factorial point of X,*
(iii) *we have $Q(\mathrm{lin}(\hat{\sigma}^*) \cap \mathbb{Z}^r) = K$,*
(iv) *σ is a regular cone in N.*

In particular, the variety X is locally factorial if and only if its minimal ambient toric variety Z is smooth.

Proposition 3.8. *Let X arise from a bunched ring $(R, \mathfrak{F}, \Phi)$. For every X-cone $\sigma \in \Sigma$, the following statements are equivalent:*

(i) *$X(\sigma)$ contains a $\mathbb{Q}$-factorial point of X,*
(ii) *every $x \in X(\sigma)$ is a $\mathbb{Q}$-factorial point of X,*
(iii) *the cone $Q(\hat{\sigma}^*)$ is of full dimension in $K_{\mathbb{Q}}$,*
(iv) *the cone $\sigma \subseteq N_{\mathbb{Q}}$ is simplicial.*

In particular, the variety X is $\mathbb{Q}$-factorial if and only if its minimal ambient toric variety Z is $\mathbb{Q}$-factorial.

Remark 3.9. Consider $X = X(R, \mathfrak{F}, \Phi)$, its minimal ambient toric variety Z and $x \in X$. We have the commutative diagam

$$\begin{tikzcd}
p^{-1}(x) \arrow[r, phantom, "\subseteq"] \arrow[d] & \hat{X} \arrow[r] \arrow[d] & \hat{Z} \arrow[r, phantom, "\subseteq"] \arrow[d] & \mathbb{C}^r \\
x \arrow[r, phantom, "\in"] & X \arrow[r] & Z &
\end{tikzcd}$$

Moreover, $x \in X(\sigma)$ holds for some $\sigma \in \Sigma$. For $\hat{\sigma} = \operatorname{cone}(e_i;\ v_i \in \sigma)$, the corresponding toric orbit in $\mathbb{C}^r$ is

$$\mathbb{T}^r \cdot z_{\hat{\sigma}} \ = \ \{z \in \mathbb{C}^r;\ z_i = 0 \Leftrightarrow v_i \in \sigma\} \ \subseteq \ \mathbb{C}^r.$$

This leads to an explicit characterization: for any point $\hat{x} \in p^{-1}(x)$, the following statements are equivalent:

(i) the orbit $H \cdot \hat{x}$ is closed in $\hat{X}$,
(ii) we have $\hat{x} \in \mathbb{T}^r \cdot z_{\hat{\sigma}}$.

Proposition 3.10. *Let X arise from a bunched ring $(R, \mathfrak{F}, \Phi)$. Given $\sigma \in \Sigma$ and $x \in X(\sigma)$, the following statements are equivalent:*

(i) *the point $x \in X$ is smooth,*
(ii) *we have $Q(\operatorname{lin}(\hat{\sigma}^*) \cap \mathbb{Z}^r) = K$ and some $\hat{x} \in p^{-1}(x) \cap \mathbb{T}^r \cdot z_{\hat{\sigma}}$ is smooth in $\bar{X}$,*
(iii) *the cone σ in N is regular and some $\hat{x} \in p^{-1}(x) \cap \mathbb{T}^r \cdot z_{\hat{\sigma}}$ is smooth in $\bar{X}$.*

In particular, the variety X is smooth if and only if $\hat{X}$ and the minimal ambient toric variety Z are smooth.

PART 3-A: EXERCISES

Exercise 3.11. Elaborate the details of Remark 3.1 *Hint:* Look at [1, Lemmas 2.1.4.1 and 2.2.3.2].

Exercise 3.12. Consider the setting of Remark 3.9. Show that the intersection $p^{-1}(x) \cap \mathbb{T}^r \cdot z_{\hat{\sigma}}$ equals the unique closed H-orbit of $p^{-1}(x)$. For any $\hat{x} = (z_1, \ldots, z_r)$ from $p^{-1}(x) \cap \mathbb{T}^r \cdot z_{\hat{\sigma}}$, we call $x = [z_1, \ldots, z_r]$ a presentation of $x \in X$ in *Cox coordinates.*

Exercise 3.13. Consider the surface X defined by the bunched ring $(R, \mathfrak{F}, \Phi)$ from Exercise 2.45. We have $r = 5$ and $N = \mathbb{Z}^2$ and the linear map $\mathbb{Z}^5 \to \mathbb{Z}^2$ dual to $\ker(Q) \subseteq \mathbb{Z}^5$ is given by the matrix

$$P = \begin{bmatrix} -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 1 & 1 \\ -1 & 0 & 1 & -1 & 0 \end{bmatrix}$$

Determine the fan Σ of the minimal ambient toric variety Z of X. Show that X is $\mathbb{Q}$-factorial. Determine all local class groups, the Picard group and the singularities of X .

Part 3-B. For the variety arising of a bunched ring, we determine base loci of divisors and the cones of effective, movable, semiample and ample divisor classes. Moreover, we discuss the Mori equivalence.

Clip 3-B **Notes 3-B** **Exercises 3-B**

Part 3-B: Short Notes

Reminder 3.14. The *base locus* and the *stable base locus* of a Weil divisor D on a normal variety X are:

$$\mathrm{Bs}(D) \ := \ \bigcap_{f \in S_D} \mathrm{supp}(\mathrm{div}(f) + D), \qquad \mathbf{B}(D) \ := \ \bigcap_{k \in \mathbb{Z}_{\geq 1}} \mathrm{Bs}(kD),$$

where we denote by S_D the set of all functions $f \in \mathbb{C}(X)^*$ such that $\mathrm{div}(f) + D > 0$ holds.

Proposition 3.15. *Consider a variety* $X = (R, \mathfrak{F}, \Phi)$ *and its minimal ambient toric variety* Z*. Given any* $D \in \mathrm{WDiv}(X)$*, set*

$$w \ := \ [D] \ \in \ \mathrm{Cl}(X) \ = \ K.$$

Moreover, let $\Sigma_X \subseteq \Sigma$ *be the set of* X*-cones. Then the base locus and the stable base locus of* D *are*

$$\mathrm{Bs}(D) \ := \ \bigcup_{\substack{\sigma \in \Sigma_X \\ w \notin Q(\hat{\sigma}^* \cap \mathbb{Z}^r)}} X(\sigma), \qquad \mathbf{B}(D) \ := \ \bigcup_{\substack{\sigma \in \Sigma_X \\ w \notin Q(\hat{\sigma}^*)}} X(\sigma).$$

Reminder 3.16. Let X be any normal variety. Then a Weil divisor D on X is called

(i) *effective* if D admits non-zero sections,
(ii) *movable* if $\mathrm{Bs}(D) \subseteq X$ is of codimension at least two.
(iii) *semiample* if D has empty stable base locus,

(iv) *ample* if X is covered by affine open sets of the form

$$X_f \;=\; X \setminus \operatorname{supp}(\operatorname{div}(f) + kD)$$

with $k \in \mathbb{Z}_{\ge 1}$ and a non-zero section $f \in \Gamma(X, \mathcal{O}(kD))$.

Proposition 3.17. *Consider a variety $X = (R, \mathfrak{F}, \Phi)$ and its minimal ambient toric variety Z. The cones generated by the effective, movable, semiample and ample divisor classes of X in $K_{\mathbb{Q}} = \operatorname{Cl}_{\mathbb{Q}}(X)$ are*

$$\operatorname{Eff}(X) \;=\; Q(\gamma), \qquad\qquad \operatorname{Mov}(X) \;=\; \bigcap_{i=1}^{r} \operatorname{cone}(Q(e_j);\ j \ne i),$$

$$\operatorname{SAmple}(X) \;=\; \bigcap_{\sigma \in \Sigma_X} Q(\hat{\sigma}^*), \qquad\qquad \operatorname{Ample}(X) \;=\; \bigcap_{\sigma \in \Sigma_X} Q(\hat{\sigma}^*)^\circ.$$

Remark 3.18. Let R be a K-factorial $\mathbb{C}$-algebra and $\mathfrak{F} = (f_1, \dots, f_r)$ a system of pairwise non-associated K-prime generators. Then

$$\Phi(w) = \{\omega \in \Omega_{\mathfrak{F}};\ w \in \omega^\circ\}, \quad w \in K, \quad w \in \bigcap_{i=1}^{r} \operatorname{cone}(Q(e_j);\ j \ne i)^\circ$$

provides us with true $\mathfrak{F}$-bunches. For the action of $H = \operatorname{Spec} \mathbb{C}[K]$ on $\bar{X} = \operatorname{Spec} R$, we have

$$\hat{X} \;:=\; \bar{X}(\Phi(w)) \;=\; \bar{X}^{ss}(w).$$

Now assume $R^*_\times = \mathbb{C}^*$ and that the K-grading is almost free. Then $(R, \mathcal{F}, w) = (R, \mathcal{F}, \Phi(w))$ is a bunched ring and

$$X := X(R, \mathcal{F}, w) = \bar{X}^{ss}(w) /\!\!/ H, \qquad \operatorname{SAmple}(X) = \lambda(w) \in \Lambda(\bar{X}).$$

That means in particular that the GIT-fan $\Lambda(\bar{X})$ stores the semiample cones of all semiprojective varieties with Cox ring R.

Remark 3.19. Let $X = (R, \mathfrak{F}, \Phi)$ be complete. Every Weil divisor D on X defines a positively graded sheaf

$$\mathcal{S}^+(D) \;:=\; \bigoplus_{n \in \mathbb{Z}_{\ge 0}} \mathcal{S}^+_n(D), \qquad\qquad \mathcal{S}^+_n(D) \;:=\; \mathcal{O}_X(nD).$$

As $R = \mathcal{R}(X)$ is finitely generated, also $\Gamma(X, \mathcal{S}^+(D))$ is so. Thus, we obtain a rational map

$$\varphi(D)\colon X \dashrightarrow X(D), \qquad X(D) \;:=\; \operatorname{Proj}(\Gamma(X, \mathcal{S}^+(D))).$$

For suitable n and a vector space basis $f_0, \dots, f_m$ of $\Gamma(X, \mathcal{O}(nD))$ we obtain $X(D)$ as the closure of the image of the rational map

$$X \dashrightarrow \mathbb{P}_m, \qquad x \;\mapsto\; [f_0(x), \dots, f_m(x)].$$

Remark 3.20. Let $X = X(R, \mathfrak{F}, \Phi)$ be complete and $D \in \mathrm{WDiv}(X)$. Set $w := [D]$ and $U := \bar{X} \setminus V(f_1 \cdots f_r)$. Then we have a commutative diagram

$$\begin{tikzcd}
\hat{X} \arrow[d, "/\!\!/H"', "p"] & \supseteq & U \arrow[d] & \subseteq & \bar{X}^{ss}(w) \arrow[d, "/\!\!/H"] \\
X \arrow[rrrrd, dashed, bend right, "\varphi(D)"'] & \supseteq & U/H \arrow[rr] & & \bar{X}^{ss}(w)/\!\!/H \arrow[d, equal] \\
 & & & & X(D)
\end{tikzcd}$$

The area of definition of $\varphi(D)\colon X \dashrightarrow X(D)$ equals $X \setminus \mathbf{B}(D)$. Moreover, $\mathbf{B}(D)$ equals the $p(\hat{X} \setminus \bar{X}^{ss}(w))$.

Definition 3.21. A birational map $X \dashrightarrow X'$ of varieties is a *small quasimodification* if it defines an isomorphism $V \to V'$ of open sets $V \subseteq X$ and $V' \subseteq X'$ with complements of codimension at least two.

Remark 3.22. Let $X = X(R, \mathfrak{F}, \Phi)$ be complete and $D \in \mathrm{WDiv}(X)$. The rational map $\varphi(D)\colon X \to X(D)$ is

(i) birational if and only if $[D] \in \mathrm{Eff}(X)^\circ$ holds.
(ii) a small quasimodification if and only if $[D] \in \mathrm{Mov}(X)^\circ$ holds.
(iii) a morphism if and only if $[D] \in \mathrm{SAmple}(X)$ holds.
(iv) an isomorphism if and only if $[D] \in \mathrm{Ample}(X)$ holds.

Definition 3.23. Two Weil divisors D, D' on $X = X(R, \mathfrak{F}, \Phi)$a normal are called *Mori equivalent*, if $\mathbf{B}(D) = \mathbf{B}(D')$ and there is a commutative diagram

$$\begin{tikzcd}
 & X \arrow[dl, dashed, "\varphi(D)"'] \arrow[dr, dashed, "\varphi(D')"] & \\
X(D) \arrow[rr, leftrightarrow, "\cong"'] & & X(D')
\end{tikzcd}$$

Proposition 3.24. *Let $X = X(R, \mathfrak{F}, \Phi)$ be complete and consider the GIT-fan $\Lambda(\bar{X})$ of the action of $H = \mathrm{Spec}\ \mathbb{C}[K]$ on $\bar{X} = \mathrm{Spec}\ R$. For any two $D, D' \in \mathrm{WDiv}(X)$, the following statements are equivalent.*

(i) *The divisors D and D' are Mori equivalent.*
(ii) *One has $[D], [D'] \in \lambda^\circ$ for some GIT-cone $\lambda \in \Lambda(\bar{X})$.*

Part 3-B: Exercises

Exercise 3.25. Let X be a variety with finitely generated Cox ring $\mathcal{R}(X)$ and $D \in \mathrm{WDiv}(X)$. Show the following:

$$\mathrm{Bs}(D) \ := \bigcap_{0 \neq f \in \mathcal{R}_{[D]}(X)} \mathrm{supp}(\mathrm{div}_{[D]}(f)).$$

Exercise 3.26. Let $X = X(R, \mathfrak{F}, \Phi)$ with minimal ambient toric variety Z. Let $D \in \mathrm{WDiv}(X)$ and $E \in \mathrm{WDiv}(Z)$ with $[D] = [E]$ in $\mathrm{Cl}(X) = \mathrm{Cl}(Z)$. Show that $\mathrm{Bs}(D)$ equals $\mathrm{Bs}(E) \cap X$.

Exercise 3.27. Show that a variety $X = X(R, \mathfrak{F}, w)$ is projective if and only if the cone $\mathrm{Eff}(X)$ is pointed.

Exercise 3.28. Make the definition of $\varphi(D)$ of Remark 3.19 explicit: Take an open $U \subseteq X$ such that D is principal on U and look at the $\mathbb{Z}_{\geq 0}$-graded inclusion of $\Gamma(X, \mathcal{S}^+(D))$ in $\Gamma(U, \mathcal{S}^+(D))$.

Exercise 3.29 (Further Reading)**.** The decomposition of $\mathrm{Mov}(X)$ into finitely many polyhedral cones $\mathrm{SAmple}(X')$ as seen in Remark 3.18 characterizes finite generation of the Cox ring of a complete variety. This was originally proven in [4]; see also [1, Thm. 4.3.3.1]. The projective varieties with finitely generated Cox ring are also called *Mori dream spaces.*

Exercise 3.30 (Further Reading)**.** Let X be a normal complete surface with finitely generated divisor class group. Then X has finitely generated Cox ring if and only if $\mathrm{Mov}(X)$ is polyhedral and coincides with $\mathrm{SAmple}(X)$. Moreover, if one of the statements holds, then X is $\mathbb{Q}$-factorial and projective. See [1, Thm. 4.3.3.5].

Part 3-C. We introduce intrinsic quadrics, provide a systematic construction of all intrinsic quadrics and survey results on smooth intrinsic quadrics of low Picard number.

Clip 3-C **Notes 3-C** **Exercises 3-C**

PART 3-C: SHORT NOTES

Definition 3.31. An *intrinsic quadric* is a normal projective variety X with Cox ring defined by a single purely quadratic relation,

$$\mathcal{R}(X) \;\cong\; \mathbb{C}[T_1, \ldots, T_s]/\langle g \rangle, \qquad g \;=\; \sum_{1 \leq i \leq j \leq s} a_{ij} T_i T_j,$$

where $g \neq 0$ and the variables $T_1, \ldots, T_s$ define $\mathrm{Cl}(X)$-homogeneous elements in the Cox ring $\mathcal{R}(X)$.

Remark 3.32. A normal projective variety with finitely generated divisor class group is toric if and only if its Cox ring is a polynomial ring. Thus, among all Mori dream spaces, the intrinsic quadrics cautiously extend the class of toric varieties.

Example 3.33. Consider the quadratic polynomial $g = T_1T_2 + T_3^2 + T_4^2$ and the factor algebra

$$R \; = \; \mathbb{C}[T_1, \dots, T_4]/\langle g \rangle$$

Let $K = \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. Define a K-grading on $\mathbb{C}[T_1, \dots, T_4]$ by taking as $\deg(T_i)$ the i-th column of the *degree matrix*

$$Q \; = \; \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ \bar{1} & \bar{3} & \bar{2} & \bar{0} \end{array} \right].$$

As g is K-homogeneous, we have an induced K-grading on R, which turns out to be factorial. We even obtain a bunched ring $(R, \mathfrak{F}, w)$ with

$$\mathfrak{F} \; = \; (T_1, \dots, T_4), \qquad w \; = \; (1, \bar{0}) \; \in \; K.$$

The associated $X = X(R, \mathfrak{F}, w)$ is an intrinsic quadric. The generator matrix of the fan Σ of the minimal ambient toric variety Z is

$$P \; = \; \left[\begin{array}{rrrr} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ 0 & -2 & 1 & 1 \end{array} \right]$$

Observe that the upper two rows of P store the exponents of g and that we have $\deg(T_3) \neq \deg(T_4)$.

Construction 3.34. Let $q, t, n, m \in \mathbb{Z}_{\geq 0}$ with q even, $3 \leq q/2 + t$ and $q + t = n$, write $\mathbb{C}[T, S] = \mathbb{C}[T_1, \dots, T_n, S_1, \dots, T_m]$ and consider

$$g_{q,t} \; := \; T_1T_2 + \ldots + T_{q-1}T_q + T_{q+1}^2 + \ldots + T_{q+t}^2 \; \in \; \mathbb{C}[T, S].$$

We install a grading on the factor algebra $\mathbb{C}[T, S]/\langle g_{q,t} \rangle$. First we store the exponents of $g_{q,t}$ into a $r \times n$ matrix P_0, where $r = q/2 + t - 1$:

$$P_0 \; := \; \left[\begin{array}{ccc} -l_1 & l_2 & 0 \\ \vdots & & \ddots \\ -l_1 & 0 & l_r \end{array} \right], \qquad l_i \; = \; \begin{cases} (1, 1), & i \leq \frac{q}{2}, \\ (2), & i > \frac{q}{2}. \end{cases}$$

Next we build an $(r{+}s) \times (n{+}m)$ integral block matrix P with primitive pairwise distinct columns that generate $\mathbb{Q}^{r+s}$ as a cone:

$$P \; := \; \left[\begin{array}{cc} P_0 & 0 \\ d & d' \end{array} \right]$$

where the $s \times n$ block d and the $s \times m$ block d' can be choosen subject to the above conditions. With $K := \mathbb{Z}^{n+m}/P^*\mathbb{Z}^{r+s}$ and

$$\deg(T_i) \; := \; Q(e_i), \qquad \deg(S_j) \; := \; Q(e_{n+j})$$

we define a K-grading on $\mathbb{C}[T, S]$, where $Q \colon \mathbb{Z}^{n+m} \to K$ is the projection. As $g_{q,t}$ is K-homogeneous, we have an induced K-grading on

$$R(P) \; := \; \mathbb{C}[T, S]/\langle g_{q,t} \rangle.$$

We speak of $R(P)$ as a *standard K-graded quadratic algebra* if $\deg(T_i)$ are pairwise distinct for $i = q + 1, \dots, q + t$.

Remark 3.35. For any standard K-graded quadratic algebra $R(P)$, the grading is almost free and factorial. Moreover, $R(P)$ is a unique factorization domain if and only if $n \geq 5$. For $n = 3, 4$, we obtain K-factorial $R(P)$ that don't admit unique factorization; see Example 3.33.

Construction 3.36. Consider a standard K-graded quadratic algebra $R = R(P)$. As a system of K-prime generators for R fix

$$\mathfrak{F} \;=\; (T_1, \ldots, T_n, S_1, \ldots, S_m).$$

Then we obtain a bunched ring $(R, \mathfrak{F}, w)$ by choosing any $w \in K$ from the relative interior of the moving cone:

$$w \;\in\; \mathrm{Mov}(R)^\circ \;=\; \bigcap_{\substack{\gamma_0 \prec \gamma \\ \text{facet}}} Q(\gamma_0)^\circ, \qquad\qquad \gamma \;=\; \mathbb{Q}^{n+m}_{\geq 0}.$$

The variety $X = X(P, w)$ associated with the bunched ring $(R, \mathfrak{F}, w)$ is a *standard intrinsic quadric.* We have the commutative diagram

$$\begin{tikzcd}
V(g_{q,t}) \arrow[r, phantom, "="] & \bar{X} \arrow[r, phantom, "\subseteq"] & \bar{Z} \arrow[r, phantom, "="] & \mathbb{C}^{n+m} \\
\bar{X}^{ss}(w) \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, phantom, "="] \arrow[d, "/\!\!/H"'] & \hat{X} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r] \arrow[d, "/\!\!/H"'] & \hat{Z} \arrow[u, phantom, sloped, "\subseteq"] \arrow[d, "/\!\!/H"] & \\
X(P,w) \arrow[r, phantom, "="] & X \arrow[r] & Z &
\end{tikzcd}$$

where $H = \mathrm{Spec}\ \mathbb{C}[K]$ is the characteristic quasitorus and $X \to Z$ the minimal toric embedding. Moreover, we have

$$\dim(X) + 1 = \dim(Z) = n + m - \dim(K_{\mathbb{Q}}), \quad \mathrm{Cl}(X) = K = \mathrm{Cl}(Z),$$

the maps $\hat{X} \to X$ and $\hat{Z} \to Z$ are characteristic spaces and the Cox ring of X is the standard K-graded quadratic algebra

$$\mathcal{R}(X) \;=\; R(P) \;=\; \mathbb{C}[T_1, \ldots, T_n, S_1, \ldots, S_m]/\langle g_{q,t} \rangle.$$

The primitive generators of the fan Σ of the minimal ambient toric variety Z of X are precisely the columns of P.

Proposition 3.37. *Every intrinsic quadric is isomorphic to a standard intrinsic quadric.*

Proposition 3.38. *Let X be a smooth intrinsic quadric of Picard number one. Then X is isomorphic to the quadric $V(T_0^2 + \ldots + T_n^2) \subseteq \mathbb{P}_n$.*

Example 3.39. For $n \geq 6$, consider the unique factororization domain $R := \mathbb{C}[T_1, \ldots, T_n]/\langle g \rangle$, where

$$g \;=\; T_1 T_2 + \ldots + T_{n-1} T_n.$$

Set $K = \mathbb{Z}^2$. Then R we obtain an almost free K-grading on R by defining the degree of T_i to be the i-th column of

$$Q = \begin{bmatrix} 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 0 & 1 \end{bmatrix}$$

Setting $\mathfrak{F} = (T_1, \ldots, T_n)$ and $w = (1,1)$, gives a bunched ring $(R, \mathfrak{F}, w)$. For the associated variety X and its minimal toric embedding, we have

$$X(1, n/2-2, 1) \ = \ X \ \to \ Z \ \subseteq \ \mathbb{P}_{\frac{n}{2}-1} \times \mathbb{P}_{\frac{n}{2}-1}.$$

where $X(1, n/2-2, 1)$ is the flag variety of type $(1, n/2-2, 1)$ and Z is an open toric subvariety of $\mathbb{P}_{n/2-1} \times \mathbb{P}_{n/2-1}$.

Remark 3.40. An intrinsic quadric X is called *full* if $X \cong X(P, w)$ with $t = 0$.

(i) Every smooth full intrinsic quadric X of Picard number $\rho(X) = 2$ is isomorphic to a flag variety $X(1, k, 1)$.
(ii) The smooth full intrinsic quadrics X of Picard number $\rho(X) = 3$ are explicitly described in [2, Thm. 1.3].
(iii) Any full Fano intrinsic quadric X satsfies $\rho(X) \le 3$. If $\rho(X) = 3$, then X is $\mathbb{Q}$-fatorial but not smooth.

Remark 3.41. The smooth intrinsic quadrics of Picard number $\rho(X) = 2$ are described in [2, Thm 1.1].

Part 3-C: Exercises

Exercise 3.42. Show that an intrinsic quadric is full if and only if its total coordinate space has precisely one singular point.

Exercise 3.43. Give an example of a bunched ring that defines a singular variety having smooth mininmal ambient toric variety. *Hint:* Look at $V(T_0T_1 + T_2T_3 + T_4^2) \subseteq \mathbb{P}_5$.

Exercise 3.44. Consider the setting of Construction 3.34. Prove the following statements:

(i) For all $q+1 \le i < j \le q+t$, the vector $2e_i - 2e_j$ lies in the $\mathbb{Z}$-linear row space of the matrix P.
(ii) $R(P)$ is a standard K-graded quadratic algebra if and only if the $\mathbb{Z}$-linear row space of P contains none of the $e_i - e_j$, where $q+1 \le i < j \le q+t$.

Exercise 3.45. Show that the K-grading of any standard $R(P)$ arising from Construction 3.34 is almost free and factorial. *Hint:* Consult [3, Constructions 6.3 and 6.13].

Exercise 3.46. Convince yourself about the following. An intrinsic quadric has a non-UFD as Cox ring if and only if it is isomorphic to standard intrinsic quadric with (q, t) being one of $(0, 4)$, $(2, 2)$, $(0, 3)$. Give concrete examples for each of the cases.

Exercise 3.47. Prove Proposition 3.37. *Hint:* Use [2, Proposition 2.1].

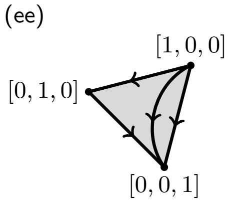
4. Rational $\mathbb{C}^*$-surfaces

Part 4-A. We give first examples of $\mathbb{C}^*$-surfaces, discuss the three types of fixed points as well as source and sink, look at $\mathbb{C}^*$-actions on toric surfaces and at a non-toric example.

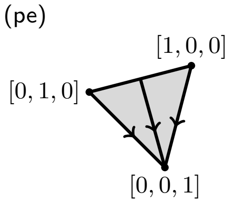
Clip 4-A **Notes 4-A** **Exercises 4-A**

Part 4-A: Short Notes

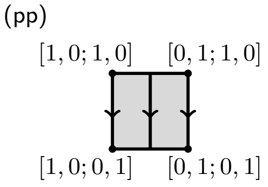
Example 4.1. Consider the following $\mathbb{C}^*$-actions (ee) and (pe) on the projective plane $\mathbb{P}_2$ and (pp) on $\mathbb{P}_1 \times \mathbb{P}_1$:



$t \cdot [z] = [z_0, tz_1, t^2 z_2]$



$t \cdot [z] = [z_0, z_1, tz_2]$



$t \cdot [z; w] = [z_0, z_1; w_0, tw_1]$

The closure of any orbit $\mathbb{C}^* \cdot x$ is obtained by adding $\lim_{t \to 0} t \cdot x$ and $\lim_{t \to \infty} t \cdot x$. The arrows indicate where $t \cdot x$ tends to for $t \to \infty$.

Remark 4.2. Let X be a normal surface with a non-trivial $\mathbb{C}^*$-action. There are only three possible types of fixed points:

(i) an *elliptic* fixed point lies in the closure of infinitely many non-trivial $\mathbb{C}^*$-orbits,
(ii) a *parabolic* fixed point lies in the closure of precisely one non-trivial $\mathbb{C}^*$-orbit,
(iii) a *hyperbolic* fixed point lies in the closure of precisely two non-trivial $\mathbb{C}^*$-orbits.

If X is projective, then there are irreducible components F^+ and F^- of the fixed point set admitting open neighborhoods U^+ and U^- with

$$\lim_{t \to 0} t \cdot x \in F^+ \text{ for all } x \in U^+, \qquad \lim_{t \to \infty} t \cdot x \in F^- \text{ for all } x \in U^-.$$

One calls F^+ the *source* and F^- the *sink.* Each of them consists either of an elliptic fixed point or it is a smooth curve of parabolic fixed points. Apart from F^+ and F^-, we find at most hyperbolic fixed points.

Remark 4.3. Consider a toric surface Z arsising from a fan Σ in $\mathbb{Z}^2$. Every $v \in \mathbb{Z}^2$ defines a one-parameter subgroup

$$\lambda_v \colon \mathbb{C}^* \to \mathbb{T}^2, \qquad t \mapsto (t^{v_1}, t^{v_2})$$

and thus a $\mathbb{C}^*$-action on Z via $t * z := \lambda_v(t) \cdot z$. For $v \neq 0$, a point z_σ is a $\mathbb{C}^*$-fixed point if and only if $v \in \operatorname{lin}(\sigma)$. More specifically,

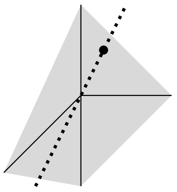
$$\begin{array}{lcl} z_\sigma \text{ is elliptic} & \Leftrightarrow & \dim(\sigma) = 2,\ \mathbb{Q}v \cap \sigma^\circ \neq \emptyset, \\ z_\sigma \text{ is parabolic} & \Leftrightarrow & \{0\} \neq \mathbb{Q}v \cap \sigma \preccurlyeq \sigma, \\ z_\sigma \text{ is hyperbolic} & \Leftrightarrow & \dim(\sigma) = 2,\ \mathbb{Q}v \cap \sigma = \{0\}. \end{array}$$

Moreover, if z_σ is elliptic or parabolic, then it belongs to the source (to the sink) if and only if $v \in \sigma$ holds ($-v \in \sigma$ holds).

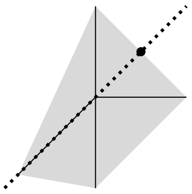
Example 4.4. The first Hirzebruch surface is the toric surface Z given by the complete fan Σ in $\mathbb{Z}^2$ with the primitive generators

$$(1,0), \qquad (0,1), \qquad (-1,-1), \qquad (0,-1).$$

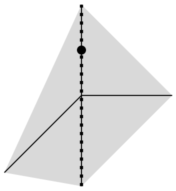
Choosing the vector $v \in \mathbb{Z}^2$ as below we obtain a $\mathbb{C}^*$-action on Z of source/sink type (ee), (ep) and (pp):



$v = (1,2)$



$v = (1,1)$



$v = (0,1)$

Example 4.5. Consider the diagonal action of $H := \mathbb{C}^*$ on $\mathbb{C}^4$ and the H-invariant hypersurface $\bar{X} \subseteq \mathbb{C}^4$ given by

$$t \cdot z \ = \ (tz_1, t^3 z_2, t^2 z_3, t^3 z_4), \qquad \bar{X} \ = \ V(T_1^3 T_2 + T_3^3 + T_4^2).$$

Then with $W_4 = \mathbb{C}^4 \setminus \{0\}$ and $\hat{X} = \bar{X} \setminus \{0\}$, we have a commutative diagram where the horizontal arrows are closed embeddings:

$$\begin{tikzcd}
\bar{X} \arrow[r, phantom, "\subseteq"] & \mathbb{C}^4 \\
\hat{X} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r] \arrow[d, "p", "/H"'] & W_4 \arrow[u, phantom, sloped, "\subseteq"] \arrow[d, "p"', "/H"] \\
X \arrow[r] & \mathbb{P}_{1,3,2,3}
\end{tikzcd}$$

In particular, $X = \hat{X}/H$ is a surface in the weighted projective space $\mathbb{P}_{1,3,2,3}$. Observe that on X we have the prime divisors

$$D_i \ = \ V(T_i) \cap X, \qquad i \ = \ 1, \ldots, 4.$$

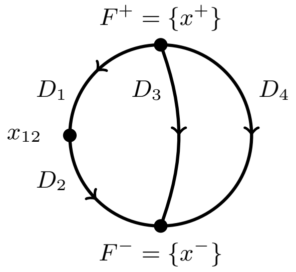
We install a $\mathbb{C}^*$-action on X. First note that we obtain a well defined $\mathbb{C}^*$-action on $\mathbb{P}_{1,3,2,3}$ by

$$s \star z \ = \ [sz_1, s^{-3} z_2, z_3, z_4].$$

By construction, X is invariant and thus comes with the induced $\mathbb{C}^*$-action. The fixed points are

$$x^+ \ = \ [0,1,0,0], \qquad x^- \ = \ [1,0,0,0], \qquad x_{12} \ = \ [0,0,-1,1],$$

where x^+ and x^- are elliptic and x_{12} is hyperbolic. Moreover, we have the following picture



Each of the prime divisors $D_i \subseteq X$ is the closure of a non-trivial orbit $\mathbb{C}^* \cdot x_i$. For the orders of the corresponding isotropy groups, we have

$$|\mathbb{C}^*_{x_1}| = 3, \qquad |\mathbb{C}^*_{x_2}| = 1, \qquad |\mathbb{C}^*_{x_3}| = 3, \qquad |\mathbb{C}^*_{x_4}| = 2.$$

Example 4.6 (continued)**.** The surface X constructed in Example 4.5 arises from the bunched ring $(R, \mathfrak{F}, w)$, where R is the $\mathbb{Z}$-graded algebra given in terms of generators and relations and degree matrix by

$$R \ = \ \mathbb{C}[T_1, \ldots, T_4]/\langle T_1^3T_2 + T_3^3 + T_4^2 \rangle, \qquad Q \ = \ \left[\begin{array}{cccc} 1 & 3 & 2 & 3 \end{array}\right],$$

we take the canonical generator system $\mathfrak{F} = (T_1, \ldots, T_4)$ and $w = 1$ in $K = \mathbb{Z}$. The ambient weighted projective space $\mathbb{P}_{1,3,2,3}$ of X arises as a toric variety from the fan Σ in $\mathbb{Z}^3$ with generator matrix

$$P \ = \ [v_1, v_2, v_3, v_4] \ = \ \left[\begin{array}{cccc} -3 & -1 & 3 & 0 \\ -3 & -1 & 0 & 2 \\ -2 & -1 & 1 & 1 \end{array}\right]$$

and the maximal cones $\sigma_i = \operatorname{cone}(v_j;\ j \neq i)$. The minimal ambient toric variety of X is the open toric subvariety Z of $\mathbb{P}_{1,3,2,3}$ given by the subfan of Σ with the maximal cones

$$\sigma^+ = \operatorname{cone}(v_1, v_3, v_4), \quad \sigma^- = \operatorname{cone}(v_2, v_3, v_4), \quad \tau_{12} = \operatorname{cone}(v_1, v_2).$$

The corresponding toric orbits of Z host the $\mathbb{C}^*$-fixed points of X. Note that the orders of the isotropy groups of $D_1, \ldots, D_4$ are reflected in the exponents of the defining relation and in the upper two rows of P.

PART 4-A: EXERCISES

Exercise 4.7. Consider a variety X with a $\mathbb{C}^*$-action and let $x \in X$. Give a precise definition of (existence of) the limits

$$x_0 \ := \ \lim_{t \to 0} t \cdot x \ \in \ X, \qquad\qquad x_\infty \ := \ \lim_{t \to \infty} t \cdot x \ \in \ X.$$

Hint: Limits exist if and only if the orbit map $t \mapsto t \cdot x$ extends to a morphism $\mathbb{C} \to X$ or $\mathbb{P}_1 \to X$.

Exercise 4.8. Let X be a normal affine surface with a non-trivial $\mathbb{C}^*$-action and consider the good quotient

$$p\colon X \ \to \ Y := \operatorname{Spec} \Gamma(X, \mathcal{O})^{\mathbb{C}^*}.$$

Show that p maps the fixed point set isomorphically onto a closed subvariety of Y. Moreover show that we are in exactly one of the following three situations:

(i) Y is a point. Then we have $X^{\mathbb{C}^*} = \{x\}$ with an elliptic fixed point $x \in X$ and precisely one of the following holds:

$$\lim_{t \to 0} t \cdot x' = x \text{ for all } x' \in X, \qquad \lim_{t \to \infty} t \cdot x' = x \text{ for all } x' \in X.$$

(ii) Y is a smooth curve and X has infinitely many fixed points. Then $X^{\mathbb{C}^*} = F$ with a curve $F \subseteq X$ of parabolic fixed points and precisely one of the following holds:

$$\lim_{t \to 0} t \cdot x' \in F \text{ for all } x' \in X, \qquad \lim_{t \to \infty} t \cdot x' \in F \text{ for all } x' \in X.$$

(iii) Y is a smooth curve and X has at most finitely many fixed points. Then every fixed point of X is hyperbolic.

Exercise 4.9. Let X be a normal projective $\mathbb{C}^*$-surface. Then [5] guarantees that X is covered by $\mathbb{C}^*$-invariant open affine subsets. Show that for any $x \in X$ with non-trivial $\mathbb{C}^*$-orbit, the limits x_0 and x_∞ differ from each other. Moreover, use Exercise 4.8 to prove Remark 4.2.

Exercise 4.10. Verify all the statements made in Example 4.1. Moreover, prove the following.

(i) In the case (ee), we have three fixed points: the two elliptic ones $[1,0,0]$ and $[0,0,1]$ and the hyperbolic $[0,1,0]$. Moreover,

$$F^+ \ = \ \{[1,0,0]\}, \qquad F^- \ = \ \{[0,0,1]\}.$$

(ii) In the case (pe), the curve $V(z_2)$ consists of parabolic fixed points and there is the elliptic fixed point $[0,0,1]$. Moreover,

$$F^+ \ = \ V(z_2), \qquad F^- \ = \ \{[0,0,1]\}.$$

(iii) In the case (pp), the curves $V(w_0)$ and $V(w_1)$ consist of parabolic fixed points. Moreover,

$$F^+ \ = \ V(w_1), \qquad F^- \ = \ V(w_0).$$

Exercise 4.11. Prove all the statements made in Remark 4.3.

Exercise 4.12. Work out the details of Examples 4.5 and 4.5. Show that the constellation of fixed points and isotropy group orders cannot be realized by a one-parameter subgroup action on a toric surface.

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