A VIDEO COURSE ON TORIC VARIETIES

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This document provides a short introduction to toric varieties. The idea is to present basic concepts, principles and facts of the theory of toric varieties and, in parallel, to train the explict working with toric varieties. In particular, the intention is by no means to give a comprehensive treatment. For the more detailed study, we refer to the excellent existing introductory texts, as for instance [1–3, 5].

The prerequesites for this course are basic knowledge in algebraic geometry. The course comprises video clips, short notes and exercises. The video clips can be activated by clicking the respective starting buttons in the present document. The exercises are recommended for immediate elaboration right after each video clip. The aim of the short notes is to support the videos and the working with the exercises.

The present course format grew out of the need of teaching without classroom during the Corona Time. Certainly, this course is a handmade product by a non-expert concerning multimedia techniques as well as the didactic aspects of e-learning. The author will highly appreciate any comments and suggestions.

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1. From toric varieties to lattice fans

Part 1-A. We discuss the projective plane as a basic example of a toric variety, present the general definition of toric varieties and indicate how to obtain examples of affine and projective toric varieties.

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Part 1-A: Short Notes

Example 1.1. Consider the *complex projective plane* $\mathbb{P}_2$. The elements of $\mathbb{P}_2$ are the lines through $0 \in \mathbb{C}^3$. In *homogeneous coordinates*,

$$[z] \;=\; [z_0, z_1, z_2] \;\in\; \mathbb{P}_2$$

denotes the line through 0 and a non-zero vector $z \in \mathbb{C}^3$. The projective plane is a *toric variety:* the *torus* $\mathbb{T}^2 = \mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathbb{P}_2$ via

$$\mathbb{T}^2 \times \mathbb{P}_2 \;\to\; \mathbb{P}_2, \qquad (t, [z]) \;\mapsto\; t \cdot [z] \;=\; [z_0, t_1 z_1, t_2 z_2],$$

we have an open orbit

$$\begin{aligned} \mathbb{T}^2 \cdot [1,1,1] \;&=\; \{[1, t_1, t_2];\; t = (t_1, t_2) \in \mathbb{T}^2\} \\ &=\; \mathbb{P}_2 \setminus (V(z_0) \cup V(z_1) \cup V(z_2)) \end{aligned}$$

and, even more, the orbit map $\mathbb{T}^2 \to \mathbb{P}_2$, $t \mapsto\; t \cdot [1,1,1]$ through the point $[1,1,1]$ is an open embedding.

Definition 1.2. The *standard n-torus* is the n-fold direct product $\mathbb{T}^n := (\mathbb{C}^*)^n$. A *torus* is an algebraic group $\mathbb{T}$ isomorphic to some $\mathbb{T}^n$.

Definition 1.3. A *toric variety* is an irreducible variety X with a morphical *torus action* $\mathbb{T} \times X \to X$ and a *base point* $x_0 \in X$ such that the *orbit map* $\mathbb{T} \to X$, $t \mapsto t \cdot x_0$ is an open embedding.

Remark 1.4. Our *short notation* for a toric variety X with acting torus $\mathbb{T}$ and base point x_0 is $(X, \mathbb{T}, x_0)$.

Example 1.5. Set $\mathbb{1}_n := (1, \ldots, 1) \in \mathbb{C}^n$. We turn $\mathbb{C}^n$ and $\mathbb{P}_n$ into toric varieties via the *standard toric structures:*

(i) $(\mathbb{C}^n, \mathbb{T}^n, \mathbb{1}_n)$ with $t \cdot z = (t_1 z_1, \ldots, t_n z_n)$,

(ii) $(\mathbb{P}_n, \mathbb{T}^n, [\mathbb{1}_{n+1}])$ with $t \cdot [z] = [z_0, t_1 z_1, \ldots, t_n z_z]$.

Remark 1.6. Let $\varphi \colon \mathbb{T} \to \mathbb{T}^n$ be a monomorphism of tori.

(i) Using the standard toric structure on $\mathbb{C}^n$, we obtain an affine toric variety $(X, \mathbb{T}, \mathbb{1}_n)$ by

$$X \;:=\; \overline{\varphi(\mathbb{T}) \cdot \mathbb{1}_n} \;\subseteq\; \mathbb{C}^n, \qquad t \cdot z \;:=\; \varphi(t) \cdot z.$$

(ii) Using the standard toric structure on $\mathbb{P}^n$, we obtain a projective toric variety $(X, \mathbb{T}, [\mathbb{1}_{n+1}])$ by

$$X \;:=\; \overline{\varphi(\mathbb{T}) \cdot [\mathbb{1}_{n+1}]} \;\subseteq\; \mathbb{P}_n, \qquad t \cdot [z] \;:=\; \varphi(t) \cdot [z].$$

Example 1.7. We apply Remark 1.6 (i) to the monomorphism of tori

$$\varphi\colon \mathbb{T}^2 \ \to \ \mathbb{T}^3, \qquad (t_1,t_2) \ \mapsto \ \left(\frac{t_1}{t_2}, t_1t_2, t_1\right)$$

and obtain an affine toric surface $(X, \mathbb{T}, \mathbb{1}_3)$, where

$$X \ = \ \overline{\varphi(\mathbb{T}^2)} \ = \ V(z_1z_2 - z_3^2) \ \subseteq \ \mathbb{C}^3.$$

Example 1.8. We apply Remark 1.6 (i) to the monomorphism of tori

$$\varphi\colon \mathbb{T}^3 \ \to \ \mathbb{T}^4, \qquad (t_1,t_2,t_3) \ \mapsto \ \left(t_1, t_2, t_3, \frac{t_1t_2}{t_3}\right)$$

and obtain an affine toric threefold $(X, \mathbb{T}, \mathbb{1}_4)$, where

$$X \ = \ \overline{\varphi(\mathbb{T}^3)} \ = \ V(z_1z_2 - z_3z_4) \ \subseteq \ \mathbb{C}^3.$$

Fact 1.9. Remark 1.6 (i) delivers (up to isomorphy) all affine toric varieties and via Remark 1.6 (ii), we obtain in particular (up to isomorphy) all normal projective toric varieties.

Part 1-A: Exercises

Exercise 1.10. Elaborate the details of Example 1.1: The $\mathbb{T}^2$-orbit through $[1,1,1]$ is open in $\mathbb{P}_2$ and the orbit map $t \mapsto \ t \cdot [1,1,1]$ defines an isomorphism of varieties from $\mathbb{T}^2$ onto $\mathbb{T}^2 \cdot [1,1,1]$.

Exercise 1.11. Show that $(\mathbb{C}^n, \mathbb{T}^n, \mathbb{1}_n)$ and $(\mathbb{P}^n, \mathbb{T}^n, [\mathbb{1}_{n+1}])$ from Example 1.5 satisfy the definition of a toric variety.

Exercise 1.12. Verify the presentations $X = V(z_1z_2 - z_3^2)$ from Example 1.7 and $X = V(z_1z_2 - z_3z_4)$ from Example 1.8.

Exercise 1.13. Show that $(X, \mathbb{T}, \mathbb{1}_n)$ and $(X, \mathbb{T}, \mathbb{1}_{[n+1]})$ defined in Remark 1.6 are indeed toric varieties. *Hint:* Use the fact that $\varphi(\mathbb{T}) \subseteq \mathbb{T}^n$ is a closed subtorus and φ defines an isomorphism onto its image.

Part 1-B. We provide a little background on convex polyhedral cones, introduce the notions of lattice cones and lattice fans and discuss an example.

Clip 1-B **Notes 1-B** **Exercises 1-B**

Part 1-B: Short Notes

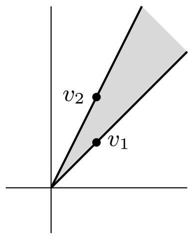
Definition 1.14. A *lattice* is a $\mathbb{Z}$-module $N \cong \mathbb{Z}^n$. The rational vector space associated with a lattice $N \cong \mathbb{Z}^n$ is $N_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} N \cong \mathbb{Q}^n$.

Definition 1.15. A (convex, polyhedral) *cone* in a finite-dimensional rational vector space V is a subset of the form

$$\sigma \;=\; \mathrm{cone}(v_1,\ldots,v_r) \;:=\; \left\{\sum \alpha_i v_i;\; \alpha_i \in \mathbb{Q}_{\ge 0}\right\} \;\subseteq\; V,$$

where $v_1,\ldots,v_r \in V$. If $V = N_{\mathbb{Q}}$ with a lattice N, then we call the pair (σ, N) a *lattice cone* or say that σ is a *cone in* N.

Example 1.16. Set $v_1 = (1,1)$ and $v_2 = (1,2)$. Then the cone in $\mathbb{Z}^2$ generated by v_1 and v_2 looks as follows:



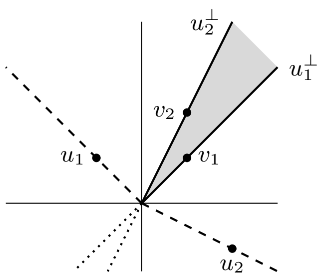
Fact 1.17. A subset σ of a finite-dimensional rational vector space V is a cone if and only if

$$\sigma \;=\; \mathrm{posloc}(u_1,\ldots,u_s) \;:=\; \{v \in V;\; u_1(v) \ge 0,\ldots,u_s(v) \ge 0\}$$

holds with linear forms $u_1,\ldots,u_s \in U := \mathrm{Hom}(V,\mathbb{Q})$. In this case, the linear forms $u_1,\ldots,u_s$ generate the *dual cone* of σ:

$$\sigma^\vee \;:=\; \{u \in U;\; u|_\sigma \ge 0\} \;=\; \mathrm{cone}(u_1,\ldots,u_s) \;\subseteq\; U.$$

Example 1.18. Consider again the cone σ in $\mathbb{Z}^2$ generated by the vectors $v_1 = (1,1)$ and $v_2 = (1,2)$.



Identifying $\mathbb{Q}^2$ with its dual via the standard scalar product, we can represent the dual cone $\sigma^\vee$ of σ as

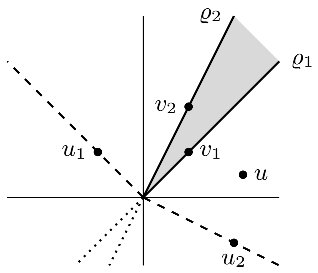
$$\sigma^\vee \;=\; \mathrm{cone}(u_1,u_2), \qquad u_1 \;=\; (-1,1), \quad u_2 \;=\; (2,-1).$$

Definition 1.19. A *face* of a cone $\sigma \subseteq V$ is a subset $\tau \subseteq \sigma$ cut out by a linear form, that means that there is a $u \in \sigma^\vee$ with

$$\tau = u^\perp \cap \sigma = \{v \in \sigma;\; u(v) = 0\}.$$

We write $\tau \preccurlyeq \sigma$ if $\tau \subseteq \sigma$ is a face of the cone $\sigma \subseteq V$. A cone $\sigma \subseteq V$ is called *pointed*, if $\{0\} \preccurlyeq \sigma$ holds.

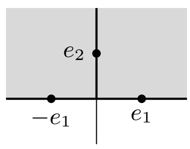
Example 1.20. Consider once more the cone σ in $\mathbb{Z}^2$ generated by the vectors $v_1 = (1,1)$ and $v_2 = (1,2)$.



Let u be any point in the relative interior of $\sigma^\vee$. Then the faces of σ are given by

$$\sigma = 0^\perp \cap \sigma \preccurlyeq \sigma, \qquad \{0\} = u^\perp \cap \sigma \preccurlyeq \sigma,$$
$$\varrho_1 = u_2^\perp \cap \sigma \preccurlyeq \sigma, \qquad \varrho_2 = u_2^\perp \cap \sigma \preccurlyeq \sigma.$$

Example 1.21. Let $e_1 = (1,0)$ and $e_2 = (0,1)$. Then $\mathrm{cone}(e_1, e_2, -e_1)$ is not pointed.

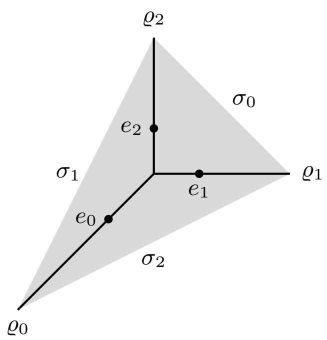


Definition 1.22. A *fan* in a finite dimensional vector space V is a finite set Σ of pointed cones in V such that

(i) for each $\sigma \in \Sigma$, also every $\tau \preccurlyeq \sigma$ belongs to Σ,
(ii) for any two $\sigma, \sigma' \in \Sigma$, we have $\sigma \cap \sigma' \preccurlyeq \sigma, \sigma'$.

If $V = N_{\mathbb{Q}}$ holds with a lattice N, then we call the pair (Σ, N) a *lattice fan* or refer to Σ as a *fan in* N.

Example 1.23. Let $e_1 = (1,0)$ and $e_2 = (0,1)$ be the canonical basis vectors in $\mathbb{Z}^2$ and set $e_0 = -e_1 - e_2$. The fan Σ in $\mathbb{Z}^2$ drawn below consists of the three maximal cones $\sigma_i = \mathrm{cone}(e_j;\ j \neq i)$, the three rays $\varrho_i = \mathrm{cone}(e_i)$ and the zero cone $\{0\}$.



Part 1-B: Exercises

Exercise 1.24. Determine the faces of the cones σ_i of the fan Σ from Example 1.23. Express them explicitly as $u_i^\perp \cap \sigma_i$ with linear forms u_i on $\mathbb{Q}^2$ satisfying $u_i|_{\sigma_i} \geq 0$.

Exercise 1.25. Determine all faces of $\mathrm{cone}(v_1, v_2, v_3, v_4) \subseteq \mathbb{Q}^3$, where the vectors v_i are given by

$$v_1 = (1,0,0), \quad v_2 = (0,1,0), \quad v_3 = (0,1,1), \quad v_4 = (1,0,1).$$

Exercise 1.26. Consider a cone $\sigma = \mathrm{cone}(v_1, \ldots, v_r) \subseteq V$. Show that every face $\tau \preccurlyeq \sigma$ is a cone in V generated by some of the v_i. In particular, observe that σ has only finitely many faces.

Exercise 1.27. Let $\sigma \subseteq V$ be a pointed cone in a finite dimensional rational vector space V. Show that the set

$$\mathcal{F}(\sigma) \ = \ \{\tau;\ \tau \preccurlyeq \sigma\}$$

is a fan in V. *Hint:* given $\tau_i = u_i^\perp \cap \sigma$ with $u_i \in \sigma^\vee$, where $i = 1, 2$, look at $u_1 + u_2$.

Exercise 1.28. Show that a cone is pointed if and only if it contains no line through the origin.

Part 1-C. We show how to assign via one parameter subgroups and their limits to any toric variety a lattice fan, discuss an example and rigorously define the notions behind this process.

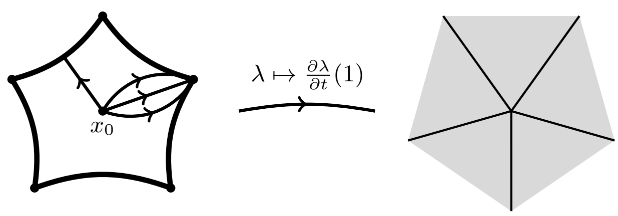
Clip 1-C **Notes 1-C** **Exercises 1-C**

Part 1-C: Short Notes

Remark 1.29. The idea for assigning a lattice fan to a given toric variety $(X, \mathbb{T}, x_0)$ is to look at one parameter subgroups (1-psg) of the acting torus $\mathbb{T}$ and their limits in X:

$$\lambda\colon \mathbb{C}^* \ \to \ \mathbb{T}, \qquad\qquad \lim_{t\to 0} \lambda(t)\cdot x_0 \ \in \ X.$$

It turns out that there are only finitely many such limit points. The directions at x_0 of the 1-psg sharing the same limit point generate the interiors of the cones of the desired fan.



Example 1.30. Consider the toric variety $(\mathbb{P}_2, \mathbb{T}^2, [1,1,1])$. The 1-psg of the torus $\mathbb{T}^2$ are given as

$$\lambda_v(t) \;=\; (t^{v_1}, t^{v_2}), \qquad v \;\in\; \mathbb{Z}^2.$$

We look for the possible limits of such one parameter subgroups in $\mathbb{P}_2$, that means for

$$\lim_{t\to 0} \lambda_v(t) \cdot [1,1,1] \;=\; \lim_{t\to 0} [1, t^{v_1}, t^{v_2}] \;\in\; \mathbb{P}_2.$$

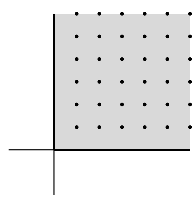
For instance, for all $v \in \mathbb{Z}^2$ with $v_1 > 0$ and $v_2 > 0$, the associated λ_v share the same limit

$$\lim_{t\to 0} \lambda_v(t) \cdot [1,1,1] \;=\; [1,0,0].$$

Conversely, for any λ_v with limit $[1,0,0]$, we have $v_1, v_2 > 0$. Observe that the direction of λ_v at x_0 is given by

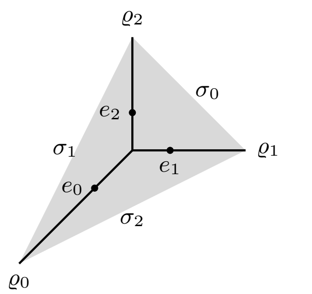
$$\frac{\partial \lambda_v}{\partial t}(1) \;=\; (v_1, v_2) \;=\; v.$$

Thus, the directions of 1-psg $\lambda_v\colon \mathbb{C}^* \to \mathbb{T}$ with limit $x = [1,0,0]$ are located in the positive quadrant:



The resulting closed cone is $\sigma(x) = \mathrm{cone}(e_1, e_2)$. Set $e_0 := -e_1 - e_2$. Here are all 1-psg, limits and associated cones for $(\mathbb{P}_2, \mathbb{T}^2, [1,1,1])$:

λ_v	limit x	$\sigma(x)$
$0 < v_1, v_2$	$[1,0,0]$	σ_0
$v_1 < 0, v_2$	$[0,1,0]$	σ_1
$v_2 < 0, v_1$	$[0,0,1]$	σ_2
$v_1 = v_2 < 0$	$[0,1,1]$	ϱ_0
$v_1 > 0,\ v_2 = 0$	$[1,0,1]$	ϱ_1
$v_1 = 0,\ v_2 > 0$	$[1,1,0]$	ϱ_2
$v_1 = v_2 = 0$	$[1,1,1]$	$\{0\}$



Definition 1.31. A *one parameter subgroup (1-psg)* of a torus $\mathbb{T}$ is a homomorphism $\lambda\colon \mathbb{C}^* \to \mathbb{T}$ of algebraic groups.

Remark 1.32. The set $\Lambda(\mathbb{T})$ of all 1-psg of a torus $\mathbb{T}$ becomes a lattice by defining the addition as pointwise multiplication:

$$(\lambda + \lambda')(t) \;:=\; \lambda(t)\lambda'(t).$$

In order to see that this is indeed a lattice, it suffices to look at $\mathbb{T} = \mathbb{T}^n$. There we have mutually inverse isomorphisms

$$\begin{array}{ccc} \mathbb{Z}^n & \longleftrightarrow & \Lambda(\mathbb{T}^n) \\ v & \mapsto & [\lambda_v \colon t \mapsto (t^{v_1}, \ldots, t^{v_n})], \\ \dfrac{\partial \lambda}{\partial t}(1) & \mapsfrom & \lambda. \end{array}$$

Definition 1.33. A 1-psg $\lambda \colon \mathbb{C}^* \to \mathbb{T}$ *converges* in $(X, \mathbb{T}, x_0)$ if the map $t \mapsto \lambda(t) \cdot x_0$ extends to a morphism $\bar{\lambda} \colon \mathbb{C} \to X$. In this case,

$$\lim(\lambda) \ := \ \lim_{t \to 0} \lambda(t) \cdot x_0 \ = \ \bar{\lambda}(0) \ \in \ X$$

is the *limit* of the 1-psg λ in X. The *convergency cone* of such a limit point $x \in X$ is the lattice cone $\sigma(x)$ in $\Lambda(\mathbb{T})$ given by

$$\sigma(x) \ := \ \overline{\mathrm{cone}(\lambda \in \Lambda(\mathbb{T});\ \lim(\lambda) = x)} \ \subseteq \ \Lambda_{\mathbb{Q}}(\mathbb{T}).$$

Theorem 1.34. *Let $(X, \mathbb{T}, x_0)$ be a normal toric variety. Then the set $\Sigma(X)$ of all convergency cones of X is a fan in the lattice $\Lambda(\mathbb{T})$.*

Remark 1.35. Let $(X, \mathbb{T}, x_0)$ be an affine toric variety. Then there is a unique closed $\mathbb{T}$-orbit in X. This orbit is of the form $\mathbb{T} \cdot x$ with a limit point $x \in X$ and the fan $\Sigma(X)$ equals the fan of faces of the convergency cone $\sigma(X) := \sigma(x)$.

Part 1-C: Exercises

Exercise 1.36. Verify the table of limit points and associated convergency cones given in Example 1.30.

Exercise 1.37. Consider the toric variety $(\mathbb{C}^n, \mathbb{T}^n, [\mathbb{1}_{n+1}])$. Show that the fan $\Sigma(\mathbb{C}^n)$ in $\Lambda(\mathbb{T}^n) = \mathbb{Z}^n$ is the fan of faces of the positive orthant $\mathrm{cone}(e_1, \ldots, e_n)$, where $e_1, \ldots, e_n \in \mathbb{Z}^n$ are the canonical basis vectors.

Exercise 1.38. Consider the toric variety $(\mathbb{P}_n, \mathbb{T}^n, [\mathbb{1}_{n+1}])$. Show that the fan $\Sigma(\mathbb{P}_n)$ in $\Lambda(\mathbb{T}^n) = \mathbb{Z}^n$ has the maximal cones

$$\sigma_i \ := \ \mathrm{cone}(e_j;\ j \neq i), \qquad i = 0, \ldots, r,$$

where we denote by $e_1, \ldots, e_n \in \mathbb{Z}^n$ the canonical basis vectors and we define $e_0 = -e_1 - \ldots - e_n$.

Exercise 1.39. Determine the fans $\Sigma(X)$ for the affine toric varieties $(X, \mathbb{T}^2, (1,1,1))$ from Example 1.7 and $(X, \mathbb{T}^3, (1,1,1,1))$ from Example 1.8.

Exercise 1.40. Show that $\mathbb{Z}^n \leftrightarrow \Lambda(\mathbb{T}^n)$ from Remark 1.32 are indeed isomorphisms of $\mathbb{Z}$-modules. *Hint:* Every 1-psg $\mathbb{C}^* \to \mathbb{C}^*$ is of the form $t \mapsto t^a$ with an $a \in \mathbb{Z}$.

2. Morphisms here and there

Part 2-A. We discuss homomorphisms of tori and lattice homomorphisms, intoduce toric morphisms and maps of fans and formulate the categorical equivalence between normal toric varieties and lattice fans.

Clip 2-A **Notes 2-A** **Exercises 2-A**

Part 2-A: Short Notes

Definition 2.1. A *toric morphism* between toric varieties $(X, \mathbb{T}, x_0)$ and $(X', \mathbb{T}', x_0')$ is a pair $(\varphi, \tilde{\varphi})$, where $\varphi\colon X \to X'$ is a morphism of varieties and $\tilde{\varphi}\colon \mathbb{T} \to \mathbb{T}'$ is a homomorphism of tori such that

$$\varphi(x_0) \;=\; x_0', \qquad \varphi(t \cdot x) \;=\; \tilde{\varphi}(t) \cdot \varphi(x) \text{ for all } t \in \mathbb{T},\ x \in X.$$

Example 2.2. We obtain a toric morphism $(\varphi, \tilde{\varphi})$ from $(\mathbb{C}^2, \mathbb{T}^2, \mathbb{1}_2)$ to $(\mathbb{C}^2, \mathbb{T}^2, \mathbb{1}_2)$ by defining φ and $\tilde{\varphi}$ as

$$\varphi\colon \mathbb{C}^2 \to \mathbb{C}^2,\ (z_1, z_2) \mapsto (z_1, z_1 z_2), \quad \tilde{\varphi}\colon \mathbb{T}^2 \to \mathbb{T}^2,\ (t_1, t_2) \mapsto (t_1, t_1 t_2).$$

Both, φ and $\tilde{\varphi}$ are monomial maps. The exponent vectors of $\tilde{\varphi}$ are the rows of the Jacobian $J_{\tilde{\varphi}}$ at $\mathbb{1}_2 \in \mathbb{T}^2$:

$$J_{\tilde{\varphi}}(\mathbb{1}_2) \;=\; \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Remark 2.3. With any matrix $A = (a_{ij}) \in \mathrm{Mat}(m, n; \mathbb{Z})$, we associate a homomorphism of standard tori by

$$\varphi_A\colon \mathbb{T}^n \;\to\; \mathbb{T}^m, \quad t \;\mapsto\; (t^{A_{1*}}, \ldots, t^{A_{m*}}), \quad t^{A_{i*}} \;:=\; t_1^{a_{i1}} \cdots t_n^{a_{in}}.$$

So, φ_A is the monomial map having the rows of A as its exponent vectors. We have

$$\varphi_{AB} \;=\; \varphi_A \circ \varphi_B$$

whenever the integral matrices A and B fit together. Moreover, there are mutually inverse bijections

$$\begin{array}{rcl} \mathrm{Hom}(\mathbb{T}^n, \mathbb{T}^m) & \longleftrightarrow & \mathrm{Mat}(m, n; \mathbb{Z}) \\ \varphi & \mapsto & J_\varphi(\mathbb{1}_n), \\ \varphi_A & \leftarrow\!\shortmid & A. \end{array}$$

Example 2.4. The toric morphisms from $(\mathbb{C}^n, \mathbb{T}^n, \mathbb{1}_n)$ to $(\mathbb{C}^m, \mathbb{T}^m, \mathbb{1}_m)$ are precisely the pairs $(\varphi, \tilde{\varphi})$, where $\varphi = \tilde{\varphi} = \varphi_A$ holds with a matrix $A \in \mathrm{Mat}(m, n; \mathbb{Z})$ having only non-negative entries.

Example 2.5. We define two different toric structures on $\mathbb{C}$. Take $1 \in \mathbb{C}$ as the base point in both cases and define $\mathbb{C}^*$-actions on $\mathbb{C}$ via

$$t \cdot z \;=\; tz, \qquad\qquad t * z \;=\; t^{-1} z.$$

So, the first structure is the standard one and the second is different. We obtain a toric isomorphism $(\varphi, \tilde{\varphi})$ between the two structures by

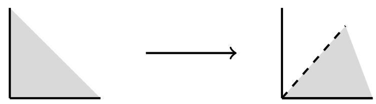
$$\varphi(z) \ = \ z, \qquad\qquad \tilde{\varphi}(t) \ = \ t^{-1}.$$

Definition 2.6. A *map of lattice fans* from (Σ, N) to (Σ', N') is a homomorphism $F\colon N \to N'$ such that for every cone $\sigma \in \Sigma$, there is a cone $\sigma' \in \Sigma$ with $F(\sigma) \subseteq \sigma'$.

Example 2.7. Consider the lattice cone $\sigma = \operatorname{cone}(e_1, e_2)$ in $\mathbb{Z}^2$ and let $\Sigma = \Sigma'$ be the fan of faces of σ. Then

$$F\colon \mathbb{Z}^2 \ \to \ \mathbb{Z}^2, \qquad (v_1, v_2) \ \mapsto \ (v_1 + v_2, v_2)$$

sends σ onto $\operatorname{cone}(e_1, e_1 + e_2) \subseteq \sigma$. In particular, F is a map of lattice fans from $(\Sigma, \mathbb{Z}^2)$ to $(\Sigma', \mathbb{Z}^2)$.



Remark 2.8. Let $\varphi\colon \mathbb{T} \to \mathbb{T}'$ be a homomorphism of tori. Then we have an induced *push forward* homomorphism

$$\varphi_*\colon \Lambda(\mathbb{T}) \ \to \ \Lambda(\mathbb{T}'), \qquad \lambda \ \mapsto \ \varphi \circ \lambda.$$

The pushforward of 1-psg is compatible with composition in the sense that it satisfies $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$.

Remark 2.9. Let $(\varphi, \tilde{\varphi})$ be a toric morphism from $(X, \mathbb{T}, x_0)$ to $(X', \mathbb{T}', x_0')$. Then $\tilde{\varphi}_*$ maps convergent 1-psg to convergent 1-psg and

$$\varphi\left(\lim_{t \to 0} \lambda(t) \cdot x_0\right) \ = \ \lim_{t \to 0} \tilde{\varphi}_* \lambda(t) \cdot x_0'.$$

Moreover, for any limit point $x \in X$, the image $\varphi(x) \in X'$ is a limit point and we have $\tilde{\varphi}_*(\sigma(x)) \subseteq \sigma(\varphi(x))$.

Theorem 2.10. *We have a covariant equivalence of categories:*

$$\begin{array}{rcl} \{\textit{normal toric varieties}\} & \longleftrightarrow & \{\textit{lattice fans}\} \\ (X, \mathbb{T}, x_0) & \mapsto & (\Sigma(X), \Lambda(\mathbb{T})) \\ (\varphi, \tilde{\varphi}) & \mapsto & \tilde{\varphi}_* \end{array}$$

Part 2-A: Exercises

Exercise 2.11. Prove the statements made in Remark 2.3. *Hint:* In order to see that every homomorphism $\varphi\colon \mathbb{T}^n \to \mathbb{T}^m$ is of the form $\varphi = \varphi_A$, look at the component functions $\varphi_1, \ldots, \varphi_m$ of φ and observe

$$\varphi_j(t) \ = \ \varphi_j(t_1, 1, \ldots, 1) \cdots \varphi_j(1, \ldots, 1, t_n).$$

Exercise 2.12. Show that for any homomorphism $\varphi\colon \mathbb{T}^n \to \mathbb{T}^m$ of standard tori, we have a commutative diagram

$$\begin{array}{ccc} \Lambda(\mathbb{T}^n) & \xrightarrow{\varphi_*} & \Lambda(\mathbb{T}^m) \\ {\scriptstyle \lambda\mapsto\frac{\partial\lambda}{\partial t}(\mathbb{1}_n)}\Big\downarrow\cong & & \cong\Big\downarrow{\scriptstyle \lambda\mapsto\frac{\partial\lambda}{\partial t}(\mathbb{1}_m)} \\ \mathbb{Z}^n & \xrightarrow[v\mapsto J_\varphi(\mathbb{1}_n)\cdot v]{} & \mathbb{Z}^m \end{array}$$

and the inverse isomorphism to $\lambda \mapsto \partial\lambda/\partial t(\mathbb{1}_n)$ is given by $v \mapsto \lambda_v$, where $\lambda_v(t) = (t^{v_1}, \ldots, t^{v_1})$.

Exercise 2.13. Verify the statements made in Remark 2.9. *Hint:* Look at Definition 1.33.

Exercise 2.14. Consider the cones $o := \{0\}$ and $\varrho := \mathbb{Q}_{\geq 0}$ in $\mathbb{Z}$. Show that, up to isomorphism, the only lattice fans in $\mathbb{Z}$ are

$$(\{o\}, \mathbb{Z}), \qquad (\{o, \varrho\}, \mathbb{Z}), \qquad (\{-\varrho, o, \varrho\}, \mathbb{Z}).$$

Use Theorem 2.10 to conclude that, up to isomorphism, the only one-dimensional normal toric varieties are

$$(\mathbb{C}^*, \mathbb{C}^*, 1), \qquad (\mathbb{C}, \mathbb{C}^*, 1), \qquad (\mathbb{P}_1, \mathbb{C}^*, [1, 1]).$$

Exercise 2.15. Show that every morphism $\mathbb{C}^n \to \mathbb{C}^m$ of algebraic varieties is a composition of a toric morphism $\mathbb{C}^n \to \mathbb{C}^l$ and a linear map $\mathbb{C}^l \to \mathbb{C}^m$.

Exercise 2.16. Prove that the functor from toric varieties to lattice fans is faithful, that means, injective on morphism sets. Proceed as follows:

(i) Show that for any two tori $\mathbb{T}$ and $\mathbb{T}'$, we have a bijection of homomorphism sets

$$\mathrm{Hom}(\mathbb{T}, \mathbb{T}') \ \mapsto\ \mathrm{Hom}(\Lambda(\mathbb{T}), \Lambda(\mathbb{T}')), \quad \varphi \ \mapsto\ \varphi_*.$$

(ii) Show that every toric morphism $(\varphi, \tilde{\varphi})$ is determined by the accompanying homomorphism $\tilde{\varphi}$.

Part 2-B. We consider the tautogical map onto the projective plane and the Veronese embedding of the projective line into the projective space and discuss them as toric morphisms and as maps of lattice fans.

Clip 2-B **Notes 2-B** **Exercises 2-B**

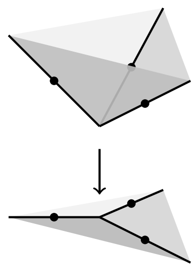
Part 2-B: Short Notes

Remark 2.17. Set $W_3 := \mathbb{C}^3 \setminus \{0\}$ and consider the tautological map sending $z \in W_3$ to the line $[z] \in \mathbb{P}_2$ through 0 and z:

$$p\colon W_3 \ \to\ \mathbb{P}_2, \qquad z = (z_0, z_1, z_2) \ \mapsto\ [z] = [z_0, z_1, z_2].$$

Then p can be regarded as a toric morphism and thus we can determine the associated map of fans.

$$\begin{array}{c} \mathbb{C}^3 \setminus \{0\} \\ {\scriptstyle p}\big\downarrow \\ \mathbb{P}_2 \end{array}$$



Reminder 2.18. We recall the standard toric structures on W_3 and $\mathbb{P}_2$. The action of $\mathbb{T}^3$ on W_3 and the action of $\mathbb{T}^2$ on $\mathbb{P}_2$ are given by

$$t \cdot z \ = \ (t_0 z_0, t_1 z_1, t_2 z_2), \qquad t \cdot [z] \ = \ [z_0, t_1 z_1, t_2 z_2].$$

Moreover, the base point in W_3 is $(1, 1, 1)$ and the base point in the projective plane $\mathbb{P}_2$ is $[1, 1, 1]$.

Remark 2.19. We turn the tautological map $p\colon W_3 \to \mathbb{P}_2$ into a toric morphism. First, note that the base points are respected:

$$p(1, 1, 1) \ = \ [1, 1, 1].$$

We look for the accompanying homomorphism $\tilde{p}$ of tori for p. For any torus element $t \in \mathbb{T}^3$, we have

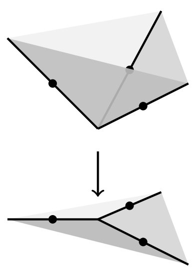
$$p(t \cdot (1, 1, 1)) \ = \ [t_0, t_1, t_2] \ = \ \left[1, \frac{t_1}{t_0}, \frac{t_2}{t_0}\right] \ = \ \left(\frac{t_1}{t_0}, \frac{t_2}{t_0}\right) \cdot p(1, 1, 1).$$

This determines $\tilde{p}$ and, altogether, we arrive at a toric morphism $(p, \tilde{p})$ from $(W_3, \mathbb{T}^3, (1, 1, 1))$ to $(\mathbb{P}_2, \mathbb{T}^2, [1, 1, 1])$, where

$$\begin{aligned} p\colon W_3 \ &\to\ \mathbb{P}^2, & (z_0, z_1, z_2) \ &\mapsto\ [z_0, z_1, z_2], \\ \tilde{p}\colon \mathbb{T}^3 \ &\to\ \mathbb{T}^2, & (t_0, t_1, t_2) \ &\mapsto\ \left(\frac{t_1}{t_0}, \frac{t_2}{t_0}\right). \end{aligned}$$

Remark 2.20. We look at the picture of fans. The fan of convergency cones $\hat{\Sigma} := \Sigma(W_3)$ lives in the lattice $\Lambda(\mathbb{T}^3) = \mathbb{Z}^3$ and consists of all proper faces of the positive orthant $\mathbb{Q}^3_{\ge 0}$.

$$\begin{array}{c} \hat{\Sigma} = \Sigma(W_3) \\ {\scriptstyle P}\big\downarrow \\ \Sigma = \Sigma(\mathbb{P}_2) \end{array}$$



Moerover, the fan $\Sigma := \Sigma(\mathbb{P}_2)$ lives in $\Lambda(\mathbb{T}^2) = \mathbb{Z}^2$ and has the three maximal cones $\sigma_i = \mathrm{cone}(e_j;\ j \neq i)$, where $j = 0, 1, 2$ and we set

$$e_0 \ := \ (-1,-1), \qquad e_1 \ := \ (1,0), \qquad e_2 \ := \ (0,1).$$

Under the identification $\Lambda(\mathbb{T}^n) = \mathbb{Z}^n$, the pushforward homomorphism $\tilde{p}_*\colon \Lambda(\mathbb{T}^3) \to \Lambda(\mathbb{T}^2)$ corresponds to (the linear map given by) the matrix

$$P \ = \ J_{\tilde{p}}(1,1,1) \ = \ \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \ \in \ \mathrm{Mat}(2,3;\mathbb{Z}).$$

Summarizing, $P\colon \mathbb{Z}^3 \to \mathbb{Z}^2$ is the map of lattice fans $(\hat{\Sigma}, \mathbb{Z}^3)$ and $(\Sigma, \mathbb{Z}^2)$ corresponding to the toric morphism $(p, \tilde{p})$ from $(W_3, \mathbb{T}^3, (1,1,1))$ to $(\mathbb{P}_2, \mathbb{T}^2, [1,1,1])$.

Example 2.21. The *Veronese embedding* of the projective line into the projective plane is given by

$$\iota\colon \mathbb{P}_1 \ \to \ \mathbb{P}_2, \qquad [z_0, z_1] \ \mapsto \ [z_0^2, z_0 z_1, z_1^2].$$

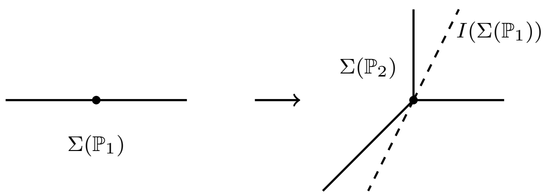
We obtain a toric morphism $(\iota, \tilde{\iota})$ from $(\mathbb{P}_1, \mathbb{T}^1, [1,1])$ to $(\mathbb{P}_2, \mathbb{T}^2, [1,1,1])$ by adding the homomorphism

$$\tilde{\iota}\colon \mathbb{T}^1 \ \to \ \mathbb{T}^2, \qquad t_1 \ \mapsto \ (t_1, t_1^2).$$

We go for the corresponding map of the lattice fans $\Sigma(\mathbb{P}_1)$ in $\Lambda(\mathbb{T}^1) = \mathbb{Z}$ and $\Sigma(\mathbb{P}_2)$ in $\Lambda(\mathbb{T}^2) = \mathbb{Z}^2$. For the push forward of 1-psg we have

$$\tilde{\iota}_* \ = \ I \ = \ J_{\tilde{\iota}}(1) \ = \ \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

identifying $\Lambda(\mathbb{T}^n)$ with $\mathbb{Z}^n$. Thus, the map of lattice fans associated with the toric Veronese embedding can be visualized as



Remark 2.22. The Veronese embedding and the tautological maps $W_{n+1} \to \mathbb{P}_n$, where $n = 1, 2$, fit into a commutative diagram

$$\begin{array}{ccc} W_2 & \xrightarrow[\varphi]{(z_0,z_1)\mapsto(z_0^2,z_0z_1,z_1^2)} & W_3 \\ \downarrow & & \downarrow \\ \mathbb{P}_1 & \xrightarrow[{[z_0,z_1]\mapsto[z_0^2,z_0z_1,z_1^2]}]{\iota} & \mathbb{P}_2 \end{array}$$

All involved maps are in fact toric morphisms. The acompanying homomorphism of tori associated with the lifting $\varphi\colon W_2 \to W_3$ is

$$\tilde{\varphi}\colon \mathbb{T}^2 \ \to \mathbb{T}^3, \qquad (t_0, t_1) \ \mapsto \ (t_0^2, t_0 t_1, t_1^2).$$

We consider the affine cone $X \subseteq \mathbb{C}^3$ over the image $\iota(\mathbb{P}_1) \subseteq \mathbb{P}_2$. It is explicitly given as

$$X \;=\; \overline{\varphi(W_2)} \;=\; V(z_0z_2 - z_1^2) \;\subseteq\; \mathbb{C}^3$$

and comes as a toric variety $(X, \varphi(\mathbb{T}^2), (1,1,1))$. We ask for the corresponding lattice fan. First,

$$\tilde{\varphi}_* \;=\; F \;=\; J_{\tilde{\varphi}}(1,1) \;=\; \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix},$$

gives us the push forward of 1-psg. The fan $\Sigma(X)$ in $\Lambda(\varphi(\mathbb{T}^2))$ turns out to be the fan of faces of the lattice cone

$$(\mathbb{Q}^3_{\ge 0} \cap F(\mathbb{Q}^2),\, \mathbb{Z}^3 \cap F(\mathbb{Q}^2)).$$

Part 2-B: Exercises

Exercise 2.23. Set $W_{n+1} := \mathbb{C}^{n+1} \setminus \{0\}$. Convince yourself about the following. We have a toric morphism $(p, \tilde{p})$ from $(W_{n+1}, \mathbb{T}^{n+1}, \mathbb{1}_{n+1})$ to $(\mathbb{P}_n, \mathbb{T}^n, [\mathbb{1}_{n+1}])$, where

$$p\colon W_{n+1} \to \mathbb{P}_n, \quad z \mapsto [z], \qquad \tilde{p}\colon \mathbb{T}^{n+1} \to \mathbb{T}^n, \quad t \mapsto \left(\frac{t_1}{t_0}, \dots, \frac{t_n}{t_0}\right).$$

The fan $\hat{\Sigma} = \Sigma(W_{n+1})$ in $\Lambda(\mathbb{T}^{n+1}) = \mathbb{Z}^{n+1}$ consists of all proper faces of the orthant $\mathbb{Q}^{n+1}_{\ge 0}$. In terms of the canonical basis vectors $e_i \in \mathbb{Z}^n$, the fan $\Sigma = \Sigma(\mathbb{P}_n)$ in $\Lambda(\mathbb{T}^n) = \mathbb{Z}^n$ has the maximal cones

$$\sigma_i = \operatorname{cone}(e_j;\ j \neq i), \quad i = 0, \dots, n, \quad e_0 = -e_1 \dots - e_n.$$

The map of lattice fans $(\hat{\Sigma}, \mathbb{Z}^{n+1})$ and $(\Sigma, \mathbb{Z}^n)$ corresponding to the toric morphism $(p, \tilde{p})$ is the linear map $\mathbb{Z}^{n+1} \to \mathbb{Z}^n$ given by the $n \times (n+1)$ matrix

$$P \;=\; [e_0, e_1, \dots, e_n] \;=\; \begin{bmatrix} -1 & 1 & & 0 \\ \vdots & & \ddots & \\ -1 & 0 & & 1 \end{bmatrix}.$$

Exercise 2.24. Let V be a finite dimensional rational vector space and $V_0 \subseteq V$ a vector subspace. Prove the following.

(i) If $\sigma \subseteq V$ is a cone, then $\sigma \cap V_0 \subseteq V_0$ is a cone.

(ii) If Σ is a fan in V, then $\{\sigma \cap V_0;\ \sigma \in \Sigma\}$ is a fan in V_0.

Exercise 2.25. Let $(\varphi, \tilde{\varphi})$ be a toric morphism from $(X, \mathbb{T}, x_0)$ to $(X', \mathbb{T}', x_0')$. Prove the following.

(i) The image $\mathbb{S} := \tilde{\varphi}(\mathbb{T}) \subseteq \mathbb{T}$ is a closed subtorus and its lattice of 1-psg satisfies

$$\Lambda(\mathbb{S}) \;=\; \Lambda(\mathbb{T}') \cap \tilde{\varphi}_*(\Lambda(\mathbb{T}))_{\mathbb{Q}} \;\subseteq\; \Lambda(\mathbb{T}').$$

(ii) With $Y := \overline{\varphi(X)} \subseteq X'$ and $y_0 := x'_0 \in X$, we obtain a toric variety $(Y, \mathbb{S}, y_0)$.
(iii) If X' is normal, then fan of convergency cones of the toric variety $(Y, \mathbb{S}, y_0)$ is given by

$$\Sigma(Y) \ = \ \{\sigma \cap \varphi_*(\Lambda(\mathbb{T}))_{\mathbb{Q}};\ \sigma \in \Sigma(X)\}.$$

Note: The assumption of X' being normal in (iii) is just to ensure that its convergency cones form a fan.

Exercise 2.26. Consider $(\mathbb{Q}^3_{\ge 0} \cap F(\mathbb{Q}^2), \mathbb{Z}^3 \cap F(\mathbb{Q}^2))$ from Remark 2.22. Show that there is a lattice isomorphism $G\colon \mathbb{Z}^3 \cap F(\mathbb{Q}^2) \to \mathbb{Z}^2$ with

$$G(\mathbb{Q}^3_{\ge 0} \cap F(\mathbb{Q}^2)) \ = \ \operatorname{cone}((1,0),\ (1,2)) \ \subseteq \ \mathbb{Q}^2.$$

Part 2-C. We introduce products of toric varieties and products of lattice fans and take a look at the Segre embedding of $\mathbb{P}_1 \times \mathbb{P}_1$ into $\mathbb{P}_3$ and the blowing up of the origin in the affine plane.

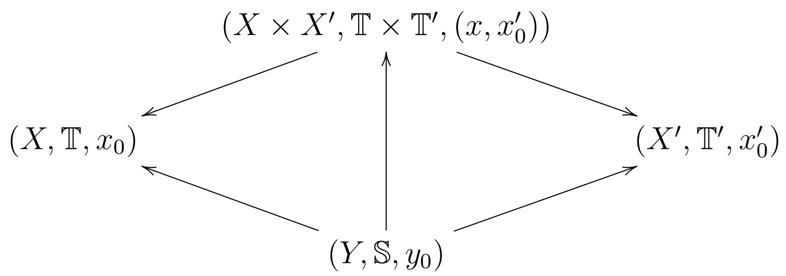
Clip 2-C **Notes 2-C** **Exercises 2-C**

Part 2-C: Short Notes

Remark 2.27. Consider toric varieties $(X, \mathbb{T}, x_0)$ and $(X', \mathbb{T}', x'_0)$. Then we obtain toric product variety

$$(X \times X', \mathbb{T} \times \mathbb{T}', (x, x'_0))$$

Moreover, the two toric projection morphisms given by $(\mathrm{pr}_X, \mathrm{pr}_{\mathbb{T}})$ and $(\mathrm{pr}_{X'}, \mathrm{pr}_{\mathbb{T}'})$ fit into a commutative diagram



with a unique upwards toric morphism, whenever we are given a toric variety $(Y, \mathbb{S}, y_0)$ and toric morphisms to $(X, \mathbb{T}, x_0)$ and $(X', \mathbb{T}', x'_0)$.

Remark 2.28. Let (Σ, N) and (Σ', N') be lattice fans. Then, abusing notation, we obtain a fan $\Sigma \times \Sigma'$ in $N \times N'$ by

$$\Sigma \times \Sigma' \ := \ \{\sigma \times \sigma';\ \sigma \in \Sigma,\ \sigma' \in \Sigma'\}$$

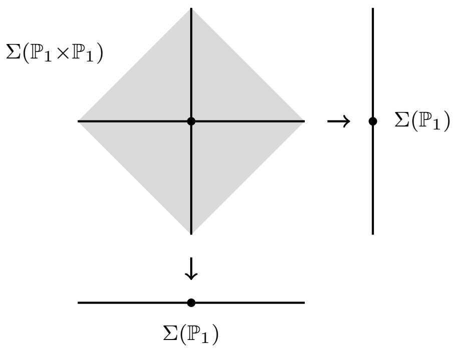
Together with the projections to pr_N and $\mathrm{pr}_{N'}$ this gives a categorical product of lattice fans.

Remark 2.29. By the categorical equivalence of normal toric varieties, the convergency fan of a product of two toric varieties is the product of their convergency fans.

Example 2.30. The product of the projective line $(\mathbb{P}_1, \mathbb{T}^1, [1,1])$ with itself is the toric variety

$$(\mathbb{P}_1 \times \mathbb{P}_1, \mathbb{T}^1 \times \mathbb{T}^1, ([1,1],[1,1])).$$

The associated convergency fan is the product of the convergency fans, lives in $\mathbb{Z}^2$ and, together with the projections, looks as follows:



Example 2.31. The *Segre embedding* of the product $\mathbb{P}_1 \times \mathbb{P}_1$ into the projective space $\mathbb{P}_3$ is given as

$$\iota\colon \mathbb{P}_1 \times \mathbb{P}_1 \;\to\; \mathbb{P}_3, \qquad ([z_0,z_1],[w_0,w_1]) \;\mapsto\; [z_0w_0, z_0w_1, z_1w_0, z_1w_1].$$

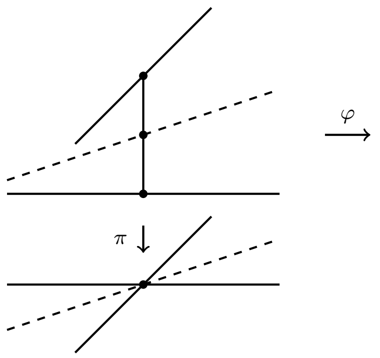
The Segre embedding defines a toric morphism $(\iota, \tilde{\iota})$ with the accompanying homomorphism

$$\tilde{\iota}\colon \mathbb{T}^1 \times \mathbb{T}^1 \;\to\; \mathbb{T}^2, \qquad (t_1,t_2) \;\mapsto\; (t_2, t_1, t_1t_2).$$

Reminder 2.32. The *blowing up* of the affine plane $\mathbb{C}^2$ at the origin $0 \in \mathbb{C}^2$ is the subvariety

$$\mathrm{bl}(2) \;=\; \{((x,y),[z,w]);\; xw = yz\} \;\subseteq\; \mathbb{C}^2 \times \mathbb{P}_1.$$

The projections from the product $\mathbb{C}^2 \times \mathbb{P}_1$ onto its factors induce morphisms $\pi\colon\ \mathrm{bl}(2) \to \mathbb{C}^2$ and $\varphi\colon\ \mathrm{bl}(2) \to \mathbb{P}_1$.



Blowing up $0 \in \mathbb{C}^2$ turns $\mathbb{C}^2$, being the union of all lines through 0, into a disjoint disjoint union of these lines, which are separated by storing their slopes in the $\mathbb{P}_1$-coordinate.

Remark 2.33. The blowing up bl(2) becomes a toric variety with base point $((1,1),[1,1])$ by installing the $\mathbb{T}^2$-action given by

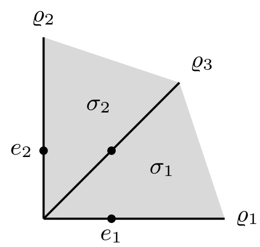
$$(t,s)\cdot((x,y),[z,w]) \;=\; ((tx,sy),[tz,sw]).$$

We ask for the fan of convergency cones of bl(2). That means to look at the possible limits of the form

$$\lim_{t\to 0}\lambda_v(t)\cdot((1,1),[1,1]) \;=\; \lim_{t\to 0}((t^{v_1},t^{v_2}),[t^{v_1},t^{v_2}]).$$

It turns out that there are six possible limit points. The following table lists them together with their convergency cones.

λ_v	limit x	$\sigma(x)$
$v_1 > v_2 > 0$	$((0,0),[0,1])$	σ_1
$v_2 > v_1 > 0$	$((0,0),[1,0])$	σ_2
$v_2 = 0,\ v_1 > 0$	$((0,1),[1,0])$	ϱ_1
$v_1 = 0,\ v_2 > 0$	$((1,0),[0,1])$	ϱ_2
$v_1 = v_2 > 0$	$((0,0),[1,1])$	ϱ_3
$v_1 = v_2 = 0$	$((1,1),[1,1])$	$\{0\}$



PART 2-C: EXERCISES

Exercise 2.34. Verify the details of Remarks 2.27, 2.28 and 2.29. Prove Remark 2.29 directly, without using the categorical equivalence.

Exercise 2.35. Observe that for the Segre embedding we have commutative diagrams

$$\begin{array}{ccc} W_2\times W_2 & \xrightarrow{(z_0,z_1,w_0,w_1)\mapsto(z_0w_0,z_0w_1,z_1w_0,z_1w_1)} & W_4 \\ \downarrow & & \downarrow \\ \mathbb{P}_1\times\mathbb{P}_1 & \xrightarrow[\iota_{\mathrm{Segre}}]{} & \mathbb{P}_3 \end{array}$$

where all involved maps are in fact toric morphisms. Determine the associated maps of convergency fans.

Exercise 2.36. Check the details of Remark 2.33. Generalize Remark 2.33 to the blowing up bl(n) of $0 \in \mathbb{C}^n$, which is defined as

$$\mathrm{bl}(n) \;=\; \{(z,[w]);\ z_iw_j = z_jw_i\} \;\subseteq\; \mathbb{C}^n\times\mathbb{P}_{n-1}.$$

Show that the fan of convergency cones of bl(n) is the fan in $\mathbb{Z}^n$ with the maximal cones

$$\sigma_j \;=\; \mathrm{cone}(v, e_i;\ i\neq j), \quad j = 1,\dots,n, \quad v = e_1+\dots+e_n.$$

Exercise 2.37. Show that $\pi\colon\ \mathrm{bl}(2) \to \mathbb{C}^2$ and $\pi\colon\ \mathrm{bl}(2) \to \mathbb{P}_1$ become toric morphisms by taking the accompanying homomorphisms

$$\tilde{\pi}\colon \mathbb{T}^2 \to \mathbb{T}^2,\ (t,s) \mapsto (t,s), \qquad \tilde{\varphi}\colon \mathbb{T}^2 \to \mathbb{T}^1,\ (t,s) \mapsto \frac{s}{t}$$

and the standard toric structures on $\mathbb{C}^2$ and $\mathbb{P}_1$. Determine the associated maps of fans.

Exercise 2.38. Consider the toric inclusion morphism $\mathrm{bl}(2) \to \mathbb{C}^2 \times \mathbb{P}_1$ and determine the images of the cones of $\Sigma(\mathrm{bl}(2))$ inside those of $\Sigma(\mathbb{C}^2 \times \mathbb{P}_1)$ under the corresponding map of fans.

3. From lattice fans to toric varieties

Part 3-A. We introduce characters of a torus, discuss the duality of characters and one parameter subgroups and present in detail the covariant equivalence between lattices and tori.

Clip 3-A **Notes 3-A** **Exercises 3-A**

Part 3-A: Short Notes

Definition 3.1. A *character* of a torus $\mathbb{T}$ is a homomorphism of algebraic groups $\chi\colon \mathbb{T} \to \mathbb{C}^*$.

Remark 3.2. The set $\mathbb{X}(\mathbb{T})$ of all characters of a torus $\mathbb{T}$ becomes a lattice by defining the addition as pointwise multiplication:

$$(\chi + \chi')(t) \;=\; \chi(t)\chi'(t).$$

As for the 1-psg, we see that this is indeed a lattice by looking at $\mathbb{T} = \mathbb{T}^n$. There we have mutually inverse isomorphisms

$$\begin{array}{rcl} \mathbb{Z}^n & \longleftrightarrow & \mathbb{X}(\mathbb{T}^n) \\ u & \mapsto & [\chi^u\colon t \mapsto t_1^{u_1} \cdots t_n^{u_n}], \\ \mathrm{grad}_\chi(\mathbb{1}_n) & \leftarrow\!\shortmid & \chi. \end{array}$$

Remark 3.3. The standard torus $\mathbb{T}^n$ is the localization $\mathbb{C}^n_{z_1\cdots z_n}$. Thus, its algebra of functions is the Laurent monomial algebra, given as

$$\mathcal{O}(\mathbb{T}^n) \;=\; \mathbb{C}[z_1, \ldots, z_n]_{z_1\cdots z_n} \;=\; \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \;=\; \bigoplus_{u\in\mathbb{Z}^n} \mathbb{C}\chi^u,$$

where the Laurent monomials $\chi^u \in \mathbb{X}(\mathbb{T}^n)$ equal the characters defined just before. Consequently, for any torus $\mathbb{T}$, we have

$$\mathcal{O}(\mathbb{T}) \;=\; \bigoplus_{\chi\in\mathbb{X}(\mathbb{T})} \mathbb{C}\chi.$$

Remark 3.4. Let $\mathbb{T}$ be a torus. Then the unit element $\mathbb{1} \in \mathbb{T}$ is determined by $\chi(\mathbb{1}) = 1$ for all $\chi \in \mathbb{X}(\mathbb{T})$. Moreover, the comorphism of the multiplication map $\mathbb{T} \times \mathbb{T} \to \mathbb{T}$ is given as

$$\mathcal{O}(\mathbb{T}) \;\to\; \mathcal{O}(\mathbb{T}) \otimes \mathcal{O}(\mathbb{T}), \qquad \chi \;\mapsto\; \chi \otimes \chi.$$

Remark 3.5. The characters and 1-psg of a standard torus come together in the following commutative diagram

$$\begin{array}{ccc} \mathbb{X}(\mathbb{T}^n) \times \Lambda(\mathbb{T}^n) & \xrightarrow{(\chi,\lambda)\mapsto \frac{\partial \chi\circ\lambda}{t}(1)} & \mathbb{Z} \\ {\scriptstyle (u,v)\mapsto(\chi^u,\lambda_v)}\uparrow & & \Vert \\ \mathbb{Z}^n \times \mathbb{Z}^n & \xrightarrow[(u,v)\mapsto\langle u,v\rangle]{} & \mathbb{Z} \end{array}$$

where $\langle u, v\rangle = u_1v_1 + \ldots + u_nv_n$ is the standard bilinear form. In particular, $\mathbb{X}(\mathbb{T}^n)$ and $\Lambda(\mathbb{T}^n)$ are mutually dual lattices. More generally,

$$\mathbb{X}(\mathbb{T}) \times \Lambda(\mathbb{T}) \ \to \ \mathbb{Z}, \qquad (\chi, \lambda) \ \mapsto \ \langle \chi, \lambda \rangle := \frac{\partial\, \chi \circ \lambda}{t}(1)$$

is a bilinear pairing turning $\mathbb{X}(\mathbb{T})$ and $\Lambda(\mathbb{T})$ into mutually dual lattices for any torus $\mathbb{T}$.

Construction 3.6. Let N be a lattice and set $M := \mathrm{Hom}(N, \mathbb{Z})$. The group algebra associated with M is defined by

$$\mathbb{C}[M] \ := \ \bigoplus_{u \in M} \mathbb{C}\chi^u, \qquad \chi^u\chi^{u'} \ := \ \chi^{u+u'}.$$

Note that $\mathbb{C}[M]$ is isomorphic to a Laurent monomial algebra. We obtain a torus by passing to the spectrum

$$\mathbb{T}_N \ := \ \mathrm{Spec}\ \mathbb{C}[M],$$

where the unit element $\mathbb{1}_N \in \mathbb{T}_N$ is determined by $\chi^u(\mathbb{1}_N) = 1$ for all $u \in M$ and the multiplication is given by its comorphism

$$\mathbb{C}[M] \ \to \ \mathbb{C}[M] \otimes \mathbb{C}[M], \qquad \chi^u \ \mapsto \ \chi^u \otimes \chi^u.$$

Remark 3.7. Let $F\colon N \to N'$ be a lattice homomorphism. The dual homomorphism $F^*\colon M' \to M$ yields a homomorphism

$$\psi_F\colon \mathbb{C}[M'] \ \to \ \mathbb{C}[M], \qquad \chi^{u'} \ \mapsto \ \chi^{F^*(u')} = \chi^{u' \circ F}.$$

Passing to the spectra, gives us a morphism $\varphi_F\colon \mathbb{T}_N \to \mathbb{T}_{N'}$, which turns out to be a homomorphism of algebraic groups.

Remark 3.8. Let N be a lattice. Then each $v \in N$ defines a lattice homomorphism

$$F_v\colon \mathbb{Z} \ \to \ N, \qquad k \ \mapsto \ kv.$$

The associated homomorphism $\lambda_v := \varphi_{F_v}\colon \mathbb{C}^* \to \mathbb{T}_N$ is a 1-psg. Moreover, we have an isomorphism of lattices

$$N \ \to \ \Lambda(\mathbb{T}_N), \qquad v \ \mapsto \ \lambda_v.$$

Theorem 3.9. *We have covariant functors being essentially inverse to each other:*

$$\{\textit{lattices}\} \ \longleftrightarrow \ \{\textit{tori}\}$$

$$\begin{array}{rcl} N & \mapsto & \mathbb{T}_N \\ F & \mapsto & \varphi_F \\ & & \\ \Lambda(\mathbb{T}) & \leftarrow\!\shortmid & \mathbb{T} \\ \varphi_* & \leftarrow\!\shortmid & \varphi \end{array}$$

Part 3-A: Exercises

Exercise 3.10. Verify all the statements made on the torus $\mathbb{T}$ in Remarks 3.3 and 3.4. *Hint:* Use $\mathbb{T} \cong \mathbb{T}^n$.

Exercise 3.11. Verify all statements made in Construction 3.6 and Remark 3.7.

Exercise 3.12. Show that $N \to \Lambda(\mathbb{T}_N)$, $v \mapsto \lambda_v$ from Remark 3.8 is an isomorphism of lattices.

Part 3-B. We discuss the monoids, monoid algebras and their spectra associated with lattice cones and present in detail the covariant equivalence between pointed lattice cones and normal affine toric varieties.

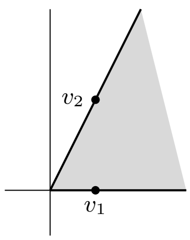
Clip 3-B **Notes 3-B** **Exercises 3-B**

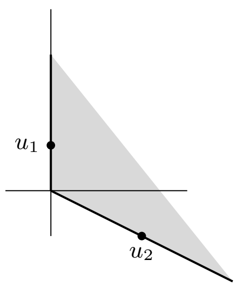
Part 3-B: Short Notes

Definition 3.13. By the *dual* of a lattice cone (σ, N), we mean the lattice cone $(\sigma^\vee, M)$, where

$$M \ = \ \mathrm{Hom}(N, \mathbb{Z}), \qquad \sigma^\vee \ = \ \{u \in M_{\mathbb{Q}};\ u|_\sigma \ge 0\} \ \subseteq \ M_{\mathbb{Q}}$$

Example 3.14. We regard the lattices $\mathbb{Z}^2$ and $\mathbb{Z}^2$ as dual to each other via the standard scalar product.

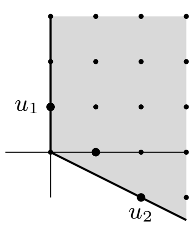




We see the lattice cone $(\sigma, \mathbb{Z}^2)$ generated by $v_1 = (1, 0)$ and $v_2 = (1, 2)$ and its dual $(\sigma^\vee, \mathbb{Z}^2)$ generated by $u_1 = (0, 1)$ and $u_2 = (2, -1)$.

Definition 3.15. Let (ω, M) be a lattice cone. The associated *cone monoid* is the additive submonoid $\omega \cap M \subseteq M$.

Example 3.16. Consider the lattice cone ω in $\mathbb{Z}^2$ generated by the vectors $u_1 = (0, 1)$ and $u_2 = (2, -1)$ and the cone monoid $\omega \cap \mathbb{Z}^2$.



Then $\omega \cap \mathbb{Z}^2$ is generated as a monoid by u_1, u_2 and $(1,0)$. Moreover, $\omega \cap \mathbb{Z}^2$ generates $\mathbb{Z}^2$ as a group.

Fact 3.17. Let (ω, M) be a lattice cone. Then the cone monoid $\omega \cap M$ is finitely generated. If ω is of full dimension in $M_{\mathbb{Q}}$, then $\omega \cap M$ generates M as a group.

Definition 3.18. Let (ω, M) be a lattice cone. The associated *monoid algebra* is defined by

$$\mathbb{C}[\omega \cap M] \ := \ \bigoplus_{u \in \omega \cap M} \mathbb{C}\chi^u, \qquad \chi^u \chi^{u'} \ := \ \chi^{u+u'}.$$

Fact 3.19. Let (ω, M) be a lattice cone. Then $\mathbb{C}[\omega \cap M]$ is an integral, normal, finitely generated subalgebra of $\mathbb{C}[M]$. If ω is of full dimension in $M_{\mathbb{Q}}$, then $\mathbb{C}[\omega \cap M]$ and $\mathbb{C}[M]$ have the same quotient field.

Construction 3.20. Let (σ, N) be a pointed lattice cone. Then the dual lattice cone $(\sigma^\vee, M)$ is of full dimension. Consider

$$\mathbb{T}_N \ = \ \mathrm{Spec}\ \mathbb{C}[M], \qquad\qquad X_\sigma \ = \ \mathrm{Spec}\, \mathbb{C}[\sigma^\vee \cap M]$$

and the point $x_0 \in X_\sigma$ defined by $\chi^u(x_0) = 1$ for all $u \in \sigma^\vee \cap M$. Then

$$\mathbb{C}[\sigma^\vee \cap M] \ \to \ \mathbb{C}[M] \otimes \mathbb{C}[\sigma^\vee \cap M], \qquad\qquad \chi^u \ \mapsto \ \chi^u \otimes \chi^u$$

is the comorphism of an action $\mathbb{T}_N \times X_\sigma \to X_\sigma$ and $(X_\sigma, \mathbb{T}_N, x_0)$ is a normal affine toric variety.

Example 3.21. Consider $\sigma = \mathrm{cone}((1,0),\ (1,2))$ in $\mathbb{Z}^2$. Then we have generators for the cone monoid of the dual cone:

$$\sigma^\vee \cap \mathbb{Z}^2 \ = \ \langle u_1, u_2, u_3 \rangle, \qquad u_1 = (2,-1),\ u_2 = (0,1),\ u_3 = (1,0).$$

Thus, $\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2]$ is generated by the elements $\chi^{u_1}, \chi^{u_2}, \chi^{u_3}$ and we obtain an epimorphism

$$\mathbb{C}[z_1, z_2, z_3] \ \to \ \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2], \qquad z_i \ \mapsto \ \chi^{u_i}.$$

This in turn gives rise to a closed embedding of the associated affine varieties:

$$X_\sigma \ \to \ \mathbb{C}^3, \qquad x \ \mapsto \ (\chi^{u_1}(x),\, \chi^{u_2}(x),\, \chi^{u_3}(x))\,.$$

Moreover, on the embedded $X_\sigma \subseteq \mathbb{C}^3$, the action of the torus $\mathbb{T}^2$ is given by

$$t \cdot z \ = \ \left(\frac{t_1^2}{t_2} z_1,\ t_2 z_2,\ t_1 z_3 \right).$$

Fact 3.22. Let $(\sigma, \mathbb{Z}^n)$ be a pointed lattice cone and suppose that we know generators for the monoid associated with the dual cone:

$$\sigma^\vee \cap \mathbb{Z}^n \ = \ \mathbb{Z}_{\ge 0} u_1 + \ldots + \mathbb{Z}_{\ge 0} u_r \ \subseteq \ \mathbb{Z}^n.$$

Then the variety $X_\sigma = \mathrm{Spec}\ \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n]$ can be concretely realized via the closed embedding

$$X_\sigma \ \to \ \mathbb{C}^r, \qquad x \ \mapsto \ (\chi^{u_1}(x), \ldots, \chi^{u_r}(x))\,.$$

Here, the base point x_0 is mapped to $\mathbb{1}_r$ and the action of $\mathbb{T}^r$ on X is explicitly given by

$$t \cdot z \;=\; (\chi^{u_1}(t)z_1, \ldots, \chi^{u_r}(t)z_r)\,.$$

Definition 3.23. A *map of lattice cones* from (σ, N) to (σ', N') is a homomorphism $F\colon N \to N'$ with $F(\sigma) \subseteq \sigma'$.

Remark 3.24. Let F be a map of pointed lattice cones (σ, N) and (σ', N'). Then the dual map $F^*\colon M' \to M$ satisfies $F^*((\sigma')^\vee) \subseteq \sigma^\vee$. Thus, $\chi^{u'} \to \chi^{F^*u'}$ defines algebra homomorphisms

$$\psi_F\colon \mathbb{C}[(\sigma')^\vee \cap M] \to \mathbb{C}[\sigma^\vee \cap M], \qquad \tilde{\psi}_F\colon \mathbb{C}[M'] \to \mathbb{C}[M].$$

Passing to the spectra, we obtain a toric morphism $(\varphi_F, \tilde{\varphi}_F)$ from the toric variety $(X, \mathbb{T}, x_0)$ associated with (σ, N) to the toric variety $(X', \mathbb{T}', x_0')$ associated with (σ', N').

Theorem 3.25. *We have covariant functors being essentially inverse to each other:*

$$\{\textit{pointed lattices cones}\} \;\longleftrightarrow\; \{\textit{normal affine toric varieties}\}$$

$$\begin{array}{rcl} (\sigma, N) & \mapsto & (X_\sigma, \mathbb{T}_N, x_0) \\ F & \mapsto & (\varphi_F, \tilde{\varphi}_F) \\[1em] (\sigma(X), \Lambda(\mathbb{T})) & \leftarrow\!\shortmid & (X, \mathbb{T}, x_0) \\ \tilde{\varphi}_* & \leftarrow\!\shortmid & (\varphi, \tilde{\varphi}) \end{array}$$

Part 3-B: Exercises

Exercise 3.26. Consider a cone $\omega = \operatorname{cone}(u_1, \ldots, u_r)$ in a lattice M, where $u_i \in M$. Show that the monoid $\omega \cap M$ is generated by the finitely many lattice vectors in the parallelepiped

$$B(u_1, \ldots, u_r) \;=\; \{a_1u_1 + \ldots + a_ru_r;\; 0 \le a_1, \ldots, a_r \le 1\} \;\subseteq\; M_{\mathbb{Q}}.$$

Exercise 3.27. Show that the monoid $\omega \cap M$ of a full dimensional lattice cone (ω, M) generates M as a group.

Exercise 3.28. Show that the monoid algebra $\mathbb{C}[\omega \cap M]$ of a full dimensional lattice cone (ω, M) is integral normal. *Hints:*

(i) If $\omega = \operatorname{posloc}(v)$ with a linear form v, then $\mathbb{C}[\omega \cap M]$ is isomorphic to a localization $\mathbb{C}[T_1, \ldots, T_n]_{z_2 \cdots z_n}$.

(ii) If $\omega = \operatorname{posloc}(v_1, \ldots, v_s)$, then, in its quotient field, $\mathbb{C}[\omega \cap M]$ is the intersection of the $\mathbb{C}[\omega_j \cap M]$ with $\omega_j = \operatorname{posloc}(v_j)$.

Exercise 3.29. Verify the details of Construction 3.20.

(i) Show that the point $x_0 \in X_\sigma$ is well defined, that means that $\langle \chi^u - 1;\ u \in \sigma^\vee \cap M \rangle$ is a maximal ideal in $\mathbb{C}[\sigma^\vee \cap M]$.

(ii) Show that $\chi^u \mapsto \chi^u \otimes \chi^u$ defines an algebra homomorphism which is the comorphism of an action $\mathbb{T}_N \times X_\sigma \to X_\sigma$.

Exercise 3.30. Consider the lattice cone $(\sigma, \mathbb{Z}^3)$, where the cone σ is generated by the vectors

$$u_1 = (1,0,0,), \quad u_2 = (0,1,0,), \quad u_3 = (0,1,1,), \quad u_4 = (1,0,1,).$$

Following the lines of Example 3.21, produce an embedding $X_\sigma \subseteq \mathbb{C}^4$ and describe the $\mathbb{T}^3$-action. Give a defining equation for X_σ.

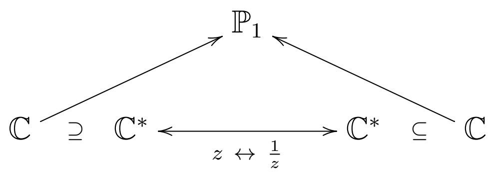
Exercise 3.31. Show that the embedding $X_\sigma \to \mathbb{C}^r$ provided in Fact 3.22 is even a toric morphism from $(X_\sigma, \mathbb{T}^n, x_0)$ to $(\mathbb{C}^r, \mathbb{T}^r, \mathbb{1}_r)$.

Part 3-C. We take look at toric localization, perform equivariant gluing of normal affine toric varieties and state in detail the covariant equivalence between lattice fans and normal toric varieties.

Clip 3-C **Notes 3-C** **Exercises 3-C**

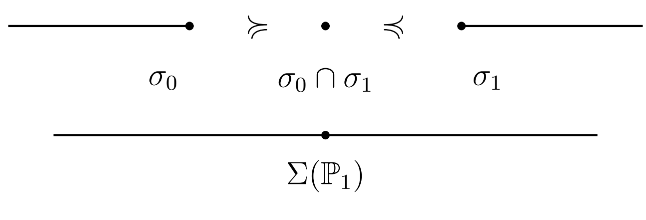
Part 3-C: Short Notes

Example 3.32. The projective line $\mathbb{P}_1$ is obtained by gluing two copies of $\mathbb{C}$ along the open subset $\mathbb{C}^* \subseteq \mathbb{C}$ in an appropriate manner:



Toric gluing hides the transition function $z \to 1/z$ by replacing one of the two copies of $\mathbb{C}$ with an isomorphic one:

$$\{\infty\} \cup \mathbb{C}^* \quad \supseteq \quad \mathbb{C}^* \quad \subseteq \quad \mathbb{C}^* \cup \{0\}$$



So, the transition functions of the gluing process are managed by the fan $\{\sigma_0,\ \sigma_0 \cap \sigma_1,\ \sigma_1\}$.

Remark 3.33. Let (σ, N) be a lattice cone, $\tau \preccurlyeq \sigma$ and fix $u \in \sigma^\vee \cap M$ with $\tau = u^\perp \cap \sigma$. Then we have

$$\tau^\vee \ = \ \mathbb{Q}u + \sigma^\vee, \qquad\qquad \tau^\vee \cap M \ = \ \mathbb{Z}u + \sigma^\vee \cap M.$$

For the associated monoid algebras and their spectra this means

$$\mathbb{C}[\tau^\vee \cap M] \ = \ \mathbb{C}[\sigma^\vee \cap M]_{\chi^u}, \qquad X_\tau \ = \ (X_\sigma)_{\chi^u}.$$

Moreover, with the inclusion map $\iota\colon X_\tau \to X_\sigma$, we have a toric open inclusion $(\iota, \mathrm{id}_{\mathbb{T}_N})$ morphism from $(X_\tau, \mathbb{T}_N, x_0)$ to $(X_\sigma, \mathbb{T}_N, x_0)$.

Construction 3.34. Let (Σ, N) be a lattice fan. Then for any two cones $\sigma, \sigma' \in \Sigma$, we have the open inclusions

$$X_\sigma \ \supseteq \ X_{\sigma\cap\sigma'} \ \subseteq \ X_{\sigma'}.$$

We define X_Σ to be the gluing of all the X_σ and $X_{\sigma'}$ along the common open subsets $X_{\sigma\cap\sigma'}$. By construction, we have

$$X_\sigma \ \subseteq \ X_\Sigma, \qquad X_{\sigma\cap\sigma'} \ = \ X_\sigma \cap X_{\sigma'} \ \subseteq \ X_\Sigma,$$

where the affine variety X_σ is open in X_Σ and the intersection $X_\sigma \cap X_{\sigma'}$ is taken in X_Σ.

Remark 3.35. Let (Σ, N) be a lattice fan. Then for any two $\sigma, \sigma' \in \Sigma$, the is the linear form $u \in M$ with

$$u|_\sigma \geq 0, \qquad u|_{\sigma'} \leq 0, \qquad u^\perp \cap \sigma = \sigma \cap \sigma' = u^\perp \cap \sigma'.$$

Thus, $X_{\sigma\cap\sigma'}$ is the localization of X_σ by χ^u and, as well, of $X_{\sigma'}$ by χ^{-u}. Due to $\chi^{-u} \in \mathcal{O}(X_{\sigma'})$, we have surjective map

$$\mathcal{O}(X_\sigma) \otimes \mathcal{O}(X'_\sigma) \ \to \ \mathcal{O}(X_\sigma \cap X_{\sigma'}), \qquad f \otimes f' \ \mapsto \ ff'.$$

In other words, the fan properties of Σ ensure that the gluing X_Σ of all the X_σ is separated.

Remark 3.36. Let (Σ, N) be a lattice fan. Then all inclusion maps $X_{\sigma\cap\sigma'} \subseteq X_\sigma$ and $X_{\sigma\cap\sigma'} \subseteq X_{\sigma'}$ in the gluing process are toric morphisms with the identity map $\mathbb{T}_N \to \mathbb{T}_N$ as accompanying homomorphism. Thus, the gluing produces a toric variety $(X_\Sigma, \mathbb{T}_N, x_0)$.

Remark 3.37. Let $F\colon N \to N'$ be a map of lattice fans (Σ, N) and (Σ', N'). Then we obtain a toric morphism

$$(X_\sigma, \mathbb{T}_N, x_0) \xrightarrow{(\varphi_F, \tilde{\varphi}_F)} (X_{\sigma'}, \mathbb{T}_{N'}, x'_0)$$

whenever $\sigma \in \Sigma$ and $\sigma' \in \Sigma'$ satisfy $F(\sigma) \subseteq \sigma'$. These toric morphisms patch together to a toric morphism

$$(X_\Sigma, \mathbb{T}_N, x_0) \xrightarrow{(\varphi_F, \tilde{\varphi}_F)} (X_{\Sigma'}, \mathbb{T}_{N'}, x'_0)$$

Theorem 3.38. *We have covariant functors being essentially inverse to each other:*

$$\{\textit{lattice fans}\} \quad \longleftrightarrow \quad \{\textit{normal toric varieties}\}$$

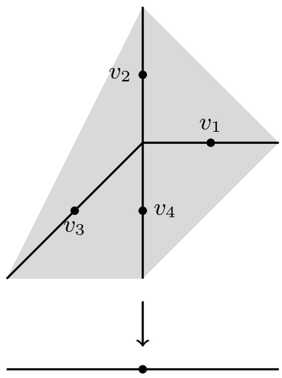
$$\begin{array}{rcl} (\Sigma, N) & \mapsto & (X_\Sigma, \mathbb{T}_N, x_0) \\ F & \mapsto & (\varphi_F, \tilde{\varphi}_F) \end{array}$$

$$\begin{array}{rcl} (\Sigma(X), \Lambda(\mathbb{T})) & \leftarrow\!\shortmid & (X, \mathbb{T}, x_0) \\ \tilde{\varphi}_* & \leftarrow\!\shortmid & (\varphi, \tilde{\varphi}) \end{array}$$

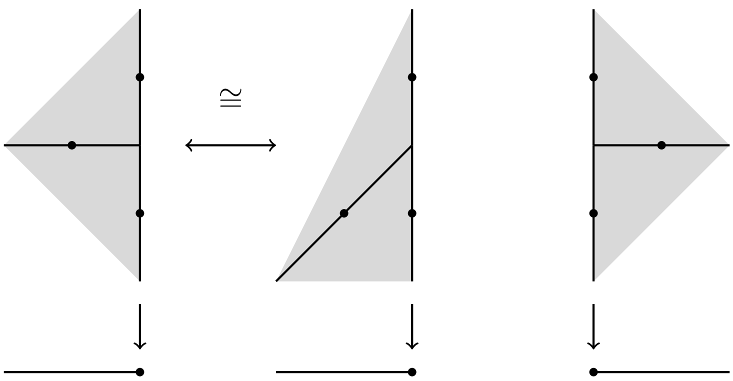
Example 3.39. Fix a positive integer $a \in \mathbb{Z}$ and consider the following four vectors

$$v_1 = (1, 0), \quad v_2 = (0, 1), \quad v_3 = (-1, -a), \quad v_4 = (0, -1).$$

Moreover, consider the lattice fan $(\Sigma_a, \mathbb{Z}^2)$ and the map of fans indicated below



The picture decomposes into two parts, shown in the middle and at the right below. Both parts define open subsets of the involved toric varieties.



At the left and at the right we find projections from product fans onto a factor. The isomorphism between the middle and the left fans arises from multiplication with

$$\begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix}$$

Geometrically this means that the morphism $X_{\Sigma_a} \to \mathbb{P}_1$ given by our map of fans is locally trivial with fiber $\mathbb{P}_1$. The surface $X_a := X_{\Sigma_a}$ is known as the *a-th Hirzebruch surface.*

Part 3-C: Exercises

Exercise 3.40. Show in terms of fans that the blowing up of the projective plane at a point is isomorphic to the first Hirzebruch surface.

Exercise 3.41. Show in terms of fans that for every $a \in \mathbb{Z}_{\geq 1}$, there is a variety Y_a which is the blowing up of X_a at a point and, as well, the blowing up of X_{a+1} at a point.

Exercise 3.42. Show in terms of fans that two Hirzebruch surfaces X_a and $X_{a'}$ are isomorphic to each other as toric varieties if and only if $a = a'$ holds. *Hint:* Look at $\det(v_i, v_j)$, where v_i and v_j are as in Example 3.39.

Exercise 3.43. Let v_1, v_2 and v_3 be as in Example 3.39, but now with $a \in \mathbb{Z}$. Consider the fan Σ'_a in $\mathbb{Z}^2$ with the maximal cones

$$\sigma_1 \;=\; \mathrm{cone}(v_1, v_2), \qquad \sigma_2 \;=\; \mathrm{cone}(v_2, v_3)$$

and the map to the fan $\Sigma(\mathbb{P}_1)$ in $\mathbb{Z}$ given by the projection $\mathrm{pr}_1 \colon \mathbb{Z}^2 \to \mathbb{Z}$ onto the first coordinate. Let

$$(L_a, \mathbb{T}^2, x_0) \xrightarrow{(\pi, \tilde{\pi})} (\mathbb{P}_1, \mathbb{T}^1, [1, 1])$$

denote the associated toric morphism. Show that $\pi \colon L_a \to \mathbb{P}_1$ is trivial with fiber $\mathbb{C}$ over $U_i = \mathbb{P}_1 \setminus V(z_i)$ and has transition map

$$(t, z) \;\mapsto\; (t, t^a z)$$

over $U_0 \cap U_1 \cong \mathbb{C}^*$. Thus, $\pi \colon L_a \to \mathbb{P}_1$ is a geometric line bundle. Show that $L_a \cong L_{a'}$ as toric varieties if and only if $a = a'$ holds.

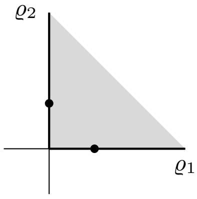
4. The orbit decomposition and around it

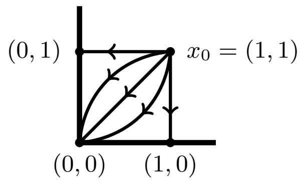
Part 4-A. For the affine toric variety defined by a lattice cone, we consider limit points, orbit cones and we present the order reversing correspondence between faces and orbits.

Clip 4-A **Notes 4-A** **Exercises 4-A**

Part 4-A: Short Notes

Example 4.1. Consider the lattice cone σ in $\mathbb{Z}^2$ generated by the vectors $(1,0)$ and $(0,1)$.





The associated affine toric variety is $(X_\sigma, \mathbb{T}^2, x_0)$ with $X_\sigma = \mathbb{C}^2$ and $x_0 = (1,1)$. We look at the possible limits

$$\lim_{t\to 0} \lambda_v(t)\cdot x_0 \;=\; \lim_{t\to 0}(t^{v_1}, t^{v_2}) \;\in\; X_\sigma \;=\; \mathbb{C}^2,$$

where $v = (v_1, v_2) \in \mathbb{Z}^2$. Two 1-psg λ_v and$\lambda_{v'}$ share the same limit if and only if v and v' lie in the relative interior of the same face of σ:

λ_v	limit
$v \in \sigma^\circ$: $v_1 > 0,\ v_2 > 0$	$x_\sigma = (0,0)$
$v \in \varrho_1^\circ$: $v_1 > 0,\ v_2 = 0$	$x_{\varrho_1} = (0,1)$
$v \in \varrho_2^\circ$: $v_1 = 0,\ v_2 > 0$	$x_{\varrho_2} = (1,0)$
$v \in \{0\}$: $v_1 = 0,\ v_2 = 0$	$x_0 = (1,1)$

Moreover, we see that the orbit decomposition of $X_\sigma = \mathbb{C}^2$ can be expressed in terms of the faces $\tau \preccurlyeq \sigma$: we have

$$X_\sigma \;=\; \bigsqcup_{\tau\preccurlyeq\sigma} \mathbb{T}^2\cdot x_\tau.$$

Remark 4.2. Let N be a lattice and $M := \mathrm{Hom}(N,\mathbb{Z})$. We regard N and M as mutually dual and, for $v \in N$ and $u \in M$, we write

$$\langle u, v\rangle \;:=\; u(v) \;=:\; v(u).$$

Reminder 4.3. For the torus $\mathbb{T}_N$ associated with a lattice N, we have the isomorphisms

$$M \to \mathbb{X}(\mathbb{T}_N),\quad u \mapsto \chi^u, \qquad\qquad N \to \Lambda(\mathbb{T}_N),\quad v \mapsto \lambda_v.$$

The bilinear pairing between the character lattice of $\mathbb{T}_N$ and the lattice of 1-psg of $\mathbb{T}_N$ is given by

$$\chi^u \circ \lambda_v(t) \;=\; t^{\langle u,v\rangle}.$$

Remark 4.4. Consider an affine toric variety $(X_\sigma, \mathbb{T}_N, x_0)$. By construction, the algebra of functions on X_σ is given by

$$\mathcal{O}(X_\sigma) \;=\; \bigoplus_{u \in \sigma^\vee \cap M} \mathbb{C}\chi^u.$$

Using the definition of the $\mathbb{T}_N$-action on X_σ, we see that the character functions $\chi^u \in \mathcal{O}(X)$ are $\mathbb{T}_N$-*homogeneous:*

$$\chi^u(t \cdot x) \;=\; \chi^u \otimes \chi^u\,(t, x) \;=\; \chi^u(t)\chi^u(x).$$

Remark 4.5. Let $\sigma \subseteq N_\mathbb{Q}$ be a cone. The *relative interior* $\sigma^\circ \subseteq \sigma$ is the interior of σ with respect to $\mathrm{lin}(\sigma) \subseteq N_\mathbb{Q}$. We have

$$\sigma \setminus \sigma^\circ \;=\; \bigcup_{\tau \prec \sigma} \tau, \qquad \sigma \;=\; \bigsqcup_{\tau \preccurlyeq \sigma} \tau^\circ.$$

Remark 4.6. Consider a cone $\sigma \subseteq N_\mathbb{Q}$ and its dual $\sigma^\vee \subseteq M_\mathbb{Q}$. Then we have mutually inverse inclusion reversing bijections

$$\begin{array}{ccc} \{\text{faces of } \sigma\} & \longleftrightarrow & \{\text{faces of } \sigma^\vee\} \\ \tau & \mapsto & \tau^\perp \cap \sigma^\vee \\ \kappa^\perp \cap \sigma & \mapsfrom & \kappa \end{array}$$

Moreover, given any vector $v \in \sigma^\circ$, then every $u \in \sigma^\vee$ evaluates at v as follows:

$$\langle u, v\rangle = 0 \text{ if } u \in \tau^\perp \cap \sigma^\vee, \qquad \langle u, v\rangle > 0 \text{ if } u \in \sigma^\vee \setminus \tau^\perp.$$

Remark 4.7. Consider an affine toric variety $(X_\sigma, \mathbb{T}_N, x_0)$. Then, for every $u \in \sigma^\vee \cap M$ and every $v \in N$, we have

$$\chi^u(\lambda_v(t) \cdot x_0) \;=\; \chi^u(\lambda_v(t))\chi^u(x_0) \;=\; t^{\langle u, v\rangle}.$$

In particular, a 1-psg λ_v converges in X_σ if and only if $\langle u, v\rangle \ge 0$ holds for all $u \in \sigma^\vee \cap M$ which in turn just means

$$v \;\in (\sigma^\vee)^\vee \;=\; \sigma.$$

Moreover, let $\tau \preccurlyeq \sigma$ and take any $v \in \tau^\circ$. We denote the *associated limit point* by $x_\tau \in X_\sigma$. It is characterized by

$$\chi^u(x_\tau) = 1 \text{ if } u \in \tau^\perp \cap \sigma^\vee, \qquad \chi^u(x_\tau) = 0 \text{ if } u \in \sigma^\vee \setminus \tau^\perp.$$

Remark 4.8. Let $(X_\sigma, \mathbb{T}_N, x_0)$ be the toric variety arising from a pointed lattice cone (σ, N). The *orbit cone* of $x \in X_\sigma$ is the face

$$\omega(x) \;:=\; \mathrm{cone}(u \in M;\; \chi^u(x) \ne 0) \;\preccurlyeq\; \sigma^\vee \;\subseteq\; M_\mathbb{Q}.$$

For every $x \in X$, the vanishing ideal of the torus orbit $\mathbb{T}_N \cdot x \subseteq X_\sigma$ in the algebra $\mathcal{O}(X_\sigma)$ is given by

$$I(\mathbb{T}_N \cdot x) \;=\; \langle \chi^u;\; u \in (\sigma^\vee \cap M) \setminus \omega(x)\rangle \;\subseteq\; \mathbb{C}[\sigma^\vee \cap M] \;=\; \mathcal{O}(X_\sigma).$$

Theorem 4.9. *Consider the toric variety* $(X_\sigma, \mathbb{T}_N, x_0)$ *arising from a pointed lattice cone* (σ, N)*. Then we have mutually inverse bijections*

$$\begin{array}{ccc} \{\textit{faces of } \sigma\} & \longleftrightarrow & \{\mathbb{T}_N\textit{-orbits of } X_\sigma\} \\ \tau & \mapsto & \mathbb{T}_N \cdot x_\tau \\ \omega(x)^\perp \cap \sigma & \leftarrow\!\shortmid & \mathbb{T}_N \cdot x \end{array}$$

Moreover, this correspondence is order reversing in the sense that for any two faces $\tau, \tau' \preccurlyeq \sigma$ *we have*

$$\tau \preccurlyeq \tau' \iff \mathbb{T}_N \cdot x_{\tau'} \subseteq \overline{\mathbb{T}_N \cdot x_\tau}.$$

Part 4-A: Exercises

Exercise 4.10. Let $\sigma \subseteq V$ be a cone in a rational vector space. Show that a subset $\tau \subseteq \sigma$ is a face of σ if and only if it has the following property: for any two $v, v' \in \sigma$ we have $v, v' \in \tau \Leftrightarrow v + v' \in \tau$.

Exercise 4.11. Consider an affine toric variety $(X_\sigma, \mathbb{T}_N, x_0)$ and let $x \in X_\sigma$. Prove the statements of Remark 4.8. *Hints:*

(i) Show that for all $u, u' \in \sigma^\vee \cap M$, we have $u, u' \in \omega(x)$ if and only if $u + u' \in \omega(x)$. Then use Exercise 4.10.

(ii) Use $\mathbb{T}_N$-homogeneity to see that $f = a_1\chi^{u_1} + \ldots + a_r\chi^{u_r}$ vanishes along $\mathbb{T}_N \cdot x$ if and only if all χ^{u_i} with $a_i \neq 0$ do so.

Exercise 4.12. Prove Theorem 4.9. *Hints:* Use Remark 4.7 as well as Remark 4.8. Observe that the vanishing ideal of $\mathbb{T}_N \cdot x$ equals the vanishing ideal of $\mathbb{T}_N \cdot x_\tau$ with the face $\tau \preccurlyeq \sigma$ corresponding to the orbit cone $\omega(x) \preccurlyeq \sigma^\vee$.

Exercise 4.13. Consider an affine toric variety $(X_\sigma, \mathbb{T}_N, x_0)$.

(i) Show that for every face $\tau \preceq \sigma$, the convergency cone of the associated limit point $x_\tau \in X_\sigma$ equals τ.

(ii) Show that $\mathbb{T}_N \cdot x_\sigma$ is closed in X_σ and contained in the closure of any other $\mathbb{T}_N$-orbit of X_σ.

Part 4-B. For toric varieties given by a lattice fan, we present the orbit decomposition, discuss isotropy groups and their dimension, orbit closures and their structure and describe the fibers of toric morphisms.

Clip 4-B **Notes 4-B** **Exercises 4-B**

Part 4-B: Short Notes

Remark 4.14. Consider a toric variety $(X_\Sigma, \mathbb{T}_N, x_0)$. Every $\sigma \in \Sigma$ gives a well-defined *limit point*, also called *distinguished point*:

$$x_\sigma \; = \; \lim_{t \to 0} \lambda_v(t) \cdot x_0 \; \in \; X_\sigma \; \subseteq \; X_\Sigma, \qquad \text{where } v \in \sigma^\circ \cap N.$$

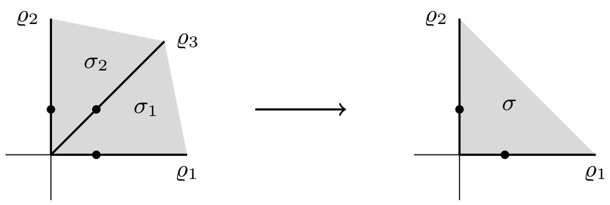
Theorem 4.15. *Consider the toric variety $(X_\Sigma, \mathbb{T}_N, x_0)$ arising from a lattice fan (Σ, N). Then we have a bijection*

$$\Sigma \;\to\; \{\mathbb{T}_N\text{-orbits of } X_\Sigma\}, \qquad \sigma \;\mapsto\; \mathbb{T}_N \cdot x_\sigma.$$

Moreover, the above map is order reversing in the sense that for any two cones $\sigma, \sigma' \in \Sigma$ we have

$$\sigma \preccurlyeq \sigma' \quad \Longleftrightarrow \quad \mathbb{T}_N \cdot x_{\sigma'} \subseteq \overline{\mathbb{T}_N \cdot x_\sigma}.$$

Remark 4.16. Consider the blow up morphism $\pi\colon \mathrm{bl}(2) \to \mathbb{C}^2$, sending $((x,y),[z,w])$ to (x,y). This is a toric morphism and the associated map of lattice fans looks as indicated below:



We know, for instance, $\pi^{-1}(x_\sigma) = \pi^{-1}(0) \cong \mathbb{P}_1$ by direct computation. The aim is to extract information like this in general from the fan picture, using the orbit decomposition.

Remark 4.17. Let N be a lattice and $N_0 \subseteq N$ a *primitive* sublattice, that means that we have

$$N_0 \;=\; \mathbb{Q}N_0 \cap N \;\subseteq\; N_\mathbb{Q}.$$

Then the corresponding homomorphism $\mathbb{T}_{N_0} \to \mathbb{T}_N$ of tori is a closed embedding with image

$$\mathbb{T}_{N_0} \;=\; \bigcap_{u \in N_0^\perp} \ker(\chi^u) \;\subseteq\; \mathbb{T}_N.$$

Remark 4.18. Consider an affine toric variety $(X_\sigma, \mathbb{T}_N, x_0)$. The character functions $\chi^u \in \mathcal{O}(X_\sigma)$ evaluate along $\mathbb{T}_N \cdot x_\sigma$ as follows:

$$\chi^u(t \cdot x_\sigma) \;=\; \chi^u(t)\chi^u(x_\sigma), \qquad \chi^u(x_\sigma) \;=\; \begin{cases} 1, & u \in \sigma^\perp \cap \sigma^\vee, \\ 0, & u \in \sigma^\vee \setminus \sigma^\perp. \end{cases}$$

Thus, $t \in \mathbb{T}_N$ satisfies $t \cdot x_\sigma = x_\sigma$ if and only if $t \in \ker(\chi^u)$ for all $u \in \sigma^\perp \cap M$. Hence the isotropy group of $\mathbb{T}_N$ at x_σ is

$$(\mathbb{T}_N)_{x_\sigma} \;=\; \bigcap_{u \in N_\sigma^\perp} \ker(\chi^u) \;=\; \mathbb{T}_{N_\sigma}, \qquad N_\sigma \;:=\; \mathrm{lin}(\sigma) \cap N.$$

Remark 4.19. Consider a toric variety $(X_\Sigma, \mathbb{T}_N, x_0)$ and let $\sigma \in \Sigma$. Then, for the isotropy group of $\mathbb{T}_N$ at x_σ, we have

$$(\mathbb{T}_N)_{x_\sigma} \;=\; \mathbb{T}_{N_\sigma}, \qquad \dim(\mathbb{T}_N)_{x_\sigma} \;=\; \dim(\sigma).$$

Moreover, $\mathbb{T}_N \cdot x_\sigma \cong \mathbb{T}_N/\mathbb{T}_{N_\sigma}$ holds. Thus, the dimensions of the orbit and its closure are given as

$$\dim\left(\overline{\mathbb{T}_N \cdot x_\sigma}\right) \;=\; \dim\left(\mathbb{T}_N \cdot x_\sigma\right) \;=\; \dim(N_\mathbb{Q}) - \dim(\sigma).$$

Remark 4.20. Consider $(X_\Sigma, \mathbb{T}_N, x_0)$ and $\sigma \in \Sigma$. The closure of the $\mathbb{T}_N$-orbit through x_σ naturally comes as a toric variety

$$\left(\overline{\mathbb{T}_N \cdot x_\sigma},\, \mathbb{T}_N/\mathbb{T}_{N_\sigma},\, x_\sigma\right),$$

where $\mathbb{T}_N/\mathbb{T}_{N_\sigma}$, the factor group of $\mathbb{T}_N$ by the isotropy group $\mathbb{T}_{N_\sigma}$ of the point x_σ is a torus and acts via $t\mathbb{T}_{N_\sigma} \cdot x := t \cdot x$.

Remark 4.21. Let Σ be a fan in a lattice N. The *star* of a cone $\tau \in \Sigma$ is the subset

$$\operatorname{star}(\tau) \;:=\; \{\sigma \in \Sigma;\; \tau \preccurlyeq \sigma\} \;\subseteq\; \Sigma.$$

The star of $\tau \in \Sigma$ gives us the closure of the orbit $\mathbb{T}_N \cdot x_\tau$ in the toric variety $(X_\Sigma, \mathbb{T}_N, x_0)$, namely

$$\overline{\mathbb{T}_N \cdot x_\tau} \;=\; \bigsqcup_{\sigma \in \operatorname{star}(\tau)} \mathbb{T}_N \cdot x_\sigma \;\subseteq\; X_\Sigma.$$

Remark 4.22. Consider $(X_\Sigma, \mathbb{T}_N, x_0)$ and $\tau \in \Sigma$. Then $N(\tau) := N/N_\tau$ is a lattice. Let $P\colon N \to N(\tau)$ be the projection and set

$$\Sigma(\tau) \;:=\; \{P(\sigma);\; \sigma \in \operatorname{star}(\tau)\}.$$

Then $(\Sigma(\tau), N(\tau))$ is a lattice fan, describing the natural toric structure on the orbit closure $\overline{\mathbb{T}_N \cdot x_\tau}$.

Remark 4.23. Consider a map $F\colon N \to N'$ of lattice fans (Σ, N) and (Σ', N') and the associated toric morphism

$$(X_\Sigma, \mathbb{T}_N, x_0) \xrightarrow{(\varphi_F, \tilde{\varphi}_F)} (X_{\Sigma'}, \mathbb{T}_{N'}, x_0').$$

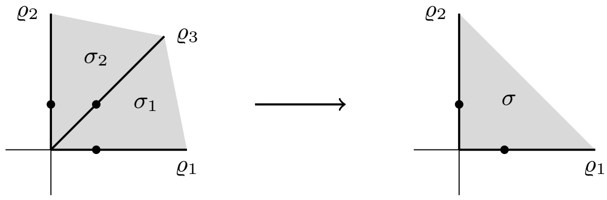
For $\sigma \in \Sigma$, let $\sigma' \in \Sigma'$ be the minimal cone containing $F(\sigma)$. Then $F(\sigma^\circ) \subseteq (\sigma')^\circ$ holds and, taking any $v \in \sigma^\circ$, we see

$$\varphi_F(x_\sigma) \;=\; \varphi_F\left(\lim_{t \to 0} \lambda_v(t) \cdot x_0\right) \;=\; \lim_{t \to 0} \lambda_{F(v)}(t) \cdot x_0' \;=\; x_{\sigma'}\,.$$

Moreover, for every cone $\sigma' \in \Sigma'$, the fiber of φ_F over the corresponding limit point $x_{\sigma'} \in X_{\Sigma'}$ is given by

$$\varphi_F^{-1}(x_{\sigma'}) \;=\; \bigcup_{F(\sigma^\circ) \subseteq (\sigma')^\circ} \tilde{\varphi}_F^{-1}(\mathbb{T}_{N_{\sigma'}}) \cdot x_\sigma.$$

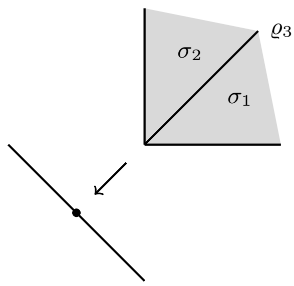
Example 4.24. Consider again the blow up morphism $\pi\colon \operatorname{bl}(2) \to \mathbb{C}^2$ and its presentation as a map of fans:



Using Remarks 4.23 and 4.21, we see that the fiber $\pi^{-1}(0)$ is the union of three torus orbits which together form an orbit closure:

$$\pi^{-1}(0) \;=\; \pi^{-1}(x_\sigma) \;=\; \mathbb{T}^2 \cdot x_{\sigma_1} \cup \mathbb{T}^2 \cdot x_{\varrho_3} \cup \mathbb{T}^2 \cdot x_{\sigma_2} \;=\; \overline{\mathbb{T}^2 \cdot x_{\varrho_3}}.$$

By Remark 4.22, the fan of $\overline{\mathbb{T}^2 \cdot x_{\varrho_3}}$ is obtained by projecting $\mathrm{star}(\varrho_3)$ along $\mathrm{lin}(\varrho_3) = \mathbb{Q}(1,1)$:



Thus we saw that, in the present example, the fiber $\pi^{-1}(0)$ is a toric variety and, as expected, we end up with $\Sigma(\pi^{-1}(0)) = \Sigma(\mathbb{P}_1)$.

Part 4-B: Exercises

Exercise 4.25. Prove Theorem 4.15. *Hint:* Reduce to the affine case and apply Theorem 4.9.

Exercise 4.26. In the setting of Remark 4.22, let Σ' denote the fan in N consisting of all faces of cones of $\mathrm{star}(\tau)$. Show that $P\colon N \to N(\tau)$ is a map of the lattice fans (Σ', N) and $(\Sigma(\tau), N(\tau))$.

Exercise 4.27. Prove Remark 4.22. *Hint:* Reduce the problem to the case of an affine toric variety $(X_\sigma, \mathbb{T}_N, x_0)$ and a face $\tau \preccurlyeq \sigma$. Then proceed as follows:

(i) Observe that the projection $N \to N(\tau)$ induces an isomorphism of tori $\mathbb{T}_N/\mathbb{T}_{N_\tau} \to \mathbb{T}_{N(\tau)}$.

(ii) Show that $M(\tau) = \tau^\perp \cap M$ is the dual lattice of $N(\tau)$.

(iii) Show that the algebra of functions $\mathcal{O}(\overline{\mathbb{T}}_N \cdot x_\tau)$ is isomorphic to $\mathbb{C}[\omega(x_\tau) \cap M(\tau)]$.

Exercise 4.28. Consider a map $F\colon N \to N'$ of lattice fans (Σ, N) and (Σ', N') and the associated toric morphism. For $\sigma \in \Sigma$ and $\sigma' \in \Sigma'$ show $\varphi_F(X_\sigma) \subseteq X_{\sigma'}$, provided $F(\sigma) \subseteq \sigma'$. Moreover, show

$$\varphi_F^{-1}(X_{\sigma'}) \;=\; \bigcup_{F(\sigma)\subseteq\sigma'} X_\sigma\,.$$

Exercise 4.29. Verify the formula for the fibers of toric morphisms given in Remark 4.23.

Exercise 4.30. Consider the affine toric variety $(X_\sigma, \mathbb{T}^2, x_0)$ arising from the lattice cone $(\sigma, \mathbb{Z}^2)$, where $\sigma \subseteq \mathbb{Q}^2$ is generated by the columns of the matrix

$$P \;=\; \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

The homomorphism $P\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ maps the positive quadrant onto σ and thus defines a toric morphism from $(\mathbb{C}^2, \mathbb{T}^2, \mathbb{1}_2)$ to $(X_\sigma, \mathbb{T}^2, x_0)$. Determine all the fibers of this toric morphism.

Exercise 4.31. Consider the affine toric variety $(X_\sigma, \mathbb{T}^3, x_0)$ arising from the lattice cone $(\sigma, \mathbb{Z}^3)$, where $\sigma \subseteq \mathbb{Q}^3$ is generated by the columns of the matrix

$$P \;=\; \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The homomorphism $P\colon \mathbb{Z}^4 \to \mathbb{Z}^3$ maps $\mathbb{Q}^4_{\geq 0}$ onto σ and thus defines a toric morphism from $(\mathbb{C}^4, \mathbb{T}^4, \mathbb{1}_4)$ to $(X_\sigma, \mathbb{T}^3, x_0)$. Determine the fibers and their dimensions for this toric morphism.

Exercise 4.32. Show that a toric morphism between toric surfaces has open image or it contracts at least a curve to a point.

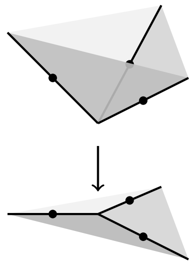
Part 4-C. We discuss the quotient presentation of a toric variety given by a non-degenerate lattice fan and we introduce homogeneous coordinates on such toric varieties.

Clip 4-C **Notes 4-C** **Exercises 4-C**

Part 4-C: Short Notes

Remark 4.33. Somewhat earlier, we have discussed the tautological map $p\colon \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}_2$, as a toric morphism and as a map of fans.

$$\begin{array}{c} \mathbb{C}^3 \setminus \{0\} \\ z \mapsto [z] \ \Big\downarrow p \\ \mathbb{P}_2 \end{array}$$



The aim is to generalize this picture to toric varieties. This allows us in particular to introduce homogeneous coordinates on toric varieties.

Definition 4.34. The *primitive generators* of a lattice fan (Σ, N) are the shortest non-zero vectors $v_1, \ldots, v_r \in N$ of the rays $\varrho_1, \ldots, \varrho_r \in \Sigma$.

Definition 4.35. A lattice fan (Σ, N) is *non-degenerate* if its primitive generators span $N_{\mathbb{Q}}$ as a vector space.

Construction 4.36. Let (Σ, N) be a non-degenerate lattice fan with primitive generators $v_1, \ldots, v_r$. Define a linear map

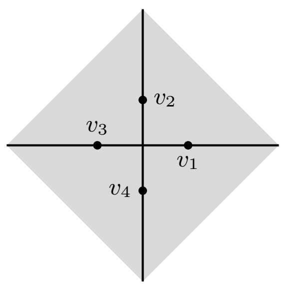
$$P \colon \mathbb{Z}^r \ \to \ N, \qquad e_i \ \mapsto \ v_i.$$

Moreover, write $\delta^r := \mathbb{Q}^r_{\geq 0}$ for the positive orthant and define a subfan $\hat{\Sigma}$ of the fan $\bar{\Sigma}$ of faces of δ^r by

$$\hat{\Sigma} \ := \ \{\delta \preccurlyeq \delta^r; \ P(\delta) \subseteq \sigma \text{ for some } \sigma \in \Sigma\}.$$

Then $\hat{\Sigma}$ and $\bar{\Sigma}$ are fans in the lattice $\mathbb{Z}^r$. Moreover, $P \colon \mathbb{Z}^r \to N$ is a map of fans from $(\hat{\Sigma}, \mathbb{Z}^r)$ to (Σ, N).

Example 4.37. Consider the fan Σ on $\mathbb{Z}^2$ having $\mathbb{P}_1 \times \mathbb{P}_1$ as associated toric variety:



The linear map $P \colon \mathbb{Z}^4 \to \mathbb{Z}^2$ sending the canonical basis vector e_i to the primitive generator v_i is given by the matrix

$$P \ = \ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

The fan $\hat{\Sigma}$ in $\mathbb{Z}^4$ consists of the faces $\delta \preccurlyeq \delta^4 = \mathbb{Q}^4_{\geq 0}$ mapping into cones of Σ; these are

$$\operatorname{cone}(e_1, e_2), \ \operatorname{cone}(e_2, e_3), \ \operatorname{cone}(e_3, e_4), \ \operatorname{cone}(e_4, e_1),$$
$$\operatorname{cone}(e_1), \ \operatorname{cone}(e_2), \ \operatorname{cone}(e_3), \ \operatorname{cone}(e_4), \ \{0\}.$$

Remark 4.38. Consider a non-degenerate fan Σ in N and the fans $\hat{\Sigma}$ and $\bar{\Sigma}$ in $\mathbb{Z}^r$. For the associated toric varieties we shortly write

$$(X, \mathbb{T}_N, x_0), \qquad (\hat{X}, \mathbb{T}^r, \mathbb{1}_r), \qquad (\bar{X}, \mathbb{T}^r, \mathbb{1}_r).$$

The toric morphism arising from the map of fans $P\colon \mathbb{Z}^r \to N$, $e_i \mapsto v_i$ is denoted by $(p, \tilde{p})$. The situation is summarized as follows:

$$\begin{array}{ccccc} \hat{X} & \subseteq & \bar{X} & = & \mathbb{C}^r \\ {\scriptstyle p}\downarrow & & & & \\ X & & & & \end{array}$$

where $\hat{X} \subseteq \bar{X}$ is an open toric embedding and $p\colon \hat{X} \to X$ is constant on the orbits of $H := \ker(\tilde{p}) \subseteq \mathbb{T}^r$ on $\hat{X}$.

Remark 4.39. For every cone $\sigma \in \Sigma$, there is a unique cone $\hat{\sigma} \in \hat{\Sigma}$ with $P(\hat{\sigma}) = \sigma$, namely

$$\hat{\sigma} \ = \ \mathrm{cone}(e_i;\ v_i \in \sigma) \ \preccurlyeq \ \delta^r.$$

Every face $\delta \preccurlyeq \delta^r$ with $P(\delta) \subseteq \sigma$ is necessarily a face of $\hat{\sigma}$. Moreover, we have an injection

$$\Sigma \ \to \ \hat{\Sigma}, \qquad \sigma \ \mapsto \ \hat{\sigma}.$$

This is a bijection if and only if each $\sigma \in \Sigma$ is *simplicial* that means that $(v_i;\ v_i \in \sigma)$ is linearly independent.

Remark 4.40. Consider a cone $\sigma \in \Sigma$ and the corresponding cone $\hat{\sigma} \in \hat{\Sigma}$. The limit point $x_{\hat{\sigma}} \in \hat{X} \subseteq \mathbb{C}^r$ is given by

$$x_{\hat{\sigma}} \ = \ (z_{\hat{\sigma},1}, \ldots, z_{\hat{\sigma},r}), \qquad\qquad z_{\hat{\sigma},i} \ = \ \begin{cases} 0, & v_i \in \sigma, \\ 1, & v_i \notin \sigma. \end{cases}$$

We have $p(x_{\hat{\sigma}}) = x_\sigma$. Moreover, as the vectors $v_i = P(e_i)$ span $N_{\mathbb{Q}}$, the homomorphism $\tilde{p}\colon \mathbb{T}^r \to \mathbb{T}_N$ is surjective and thus

$$p(\mathbb{T}^r \cdot x_{\hat{\sigma}}) \ = \ \tilde{p}(\mathbb{T}^r) \cdot p(x_{\hat{\sigma}}) \ = \ \mathbb{T}_N \cdot x_\sigma.$$

Definition 4.41. Let $(X_\Sigma, \mathbb{T}_N, x_0)$ be non-degenerate. The presentation of a point $x \in X := X_\Sigma$ in *homogeneous coordinates* is

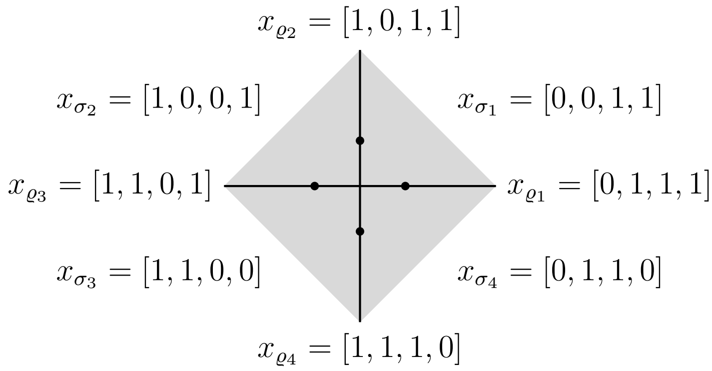
$$x \ = \ [z] \ = \ [z_1, \ldots, z_r],$$

where $x \in \mathbb{T}_N \cdot x_\sigma \subseteq X$ with a unique cone $\sigma \in \Sigma$ and $z \in \mathbb{T}^r \cdot x_{\hat{\sigma}} \subseteq \hat{X}$ is any point with $p(z) = x$.

Remark 4.42. As before, let $H = \ker(p) \subseteq \mathbb{Z}^r$. Then, for any two points $z, z' \in \mathbb{T}^r \cdot x_{\hat{\sigma}} \subseteq \hat{X}$, we have

$$[z] \ = \ [z'] \quad \Longleftrightarrow \quad z' = h \cdot z \text{ for some } h \in H.$$

Example 4.43. For $\mathbb{P}_1 \times \mathbb{P}_1$, the presentations of the limit points, apart from $x_0 = [1, 1, 1, 1]$, in homogeneous coordinates are



Remark 4.44. For the projective space $\mathbb{P}_n$, the toric homogeneous coordinates are just the usual ones.

Part 4-C: Exercises

Exercise 4.45. Show that a lattice fan is non-degenerate if and only if its associated toric variety is not a product of a torus and another toric variety.

Exercise 4.46. Consider a lattice fan (Σ, N). Show that $\hat{\Sigma} = \bar{\Sigma}$ holds if and only if X_Σ is an affine variety.

Exercise 4.47. Show that $\ker(\tilde{p}) \subseteq \mathbb{T}^r$ need not be a torus. *Hint:* Look at Exercise 4.30.

Exercise 4.48. Show that the inverse image of the $\mathbb{T}_N$-orbit through the limit point $x_\sigma \in X$ with respect to $p\colon \hat{X} \to X$ is given as

$$p^{-1}(\mathbb{T}_N \cdot x_\sigma) \ = \bigcup_{\substack{\delta \preccurlyeq \hat{\sigma} \\ P(\delta^\circ) \subseteq \sigma^\circ}} \mathbb{T}^r \cdot x_\delta.$$

Exercise 4.49. Consider $p\colon \hat{X} \to X$, a limit point $x_\sigma \in X$ and the limit point $x_{\hat{\sigma}} \in \hat{X}$. Show that the following statements are equivalent:

(i) The cone σ is simplicial.
(ii) Whenever $v_{i_1}, \ldots, v_{i_k} \in \sigma$, then $\operatorname{cone}(v_{i_1}, \ldots, v_{i_k}) \preccurlyeq \sigma$.
(iii) $\mathbb{T}^r \cdot x_{\hat{\sigma}}$ is the only $\mathbb{T}^r$-orbit in $p^{-1}(\mathbb{T}_N \cdot x_\sigma)$.

Exercise 4.50. Show that $p^{-1}(\mathbb{T}_N \cdot x_\sigma)$ may indeed contain $\mathbb{T}^r$-orbits differing from $\mathbb{T}^r \cdot x_{\hat{\sigma}}$. *Hint:* Look at Exercise 4.31.

5. Divisors and their sections

Part 5-A. We describe invariant prime divisors and divisors of character functions on a toric variety arising from a lattice fan, and we determine the divisor class group.

Clip 5-A **Notes 5-A** **Exercises 5-A**

Part 5-A: Short Notes

Reminder 5.1. Let X be any variety. A *prime divisor* on X is an irreducible closed subset $Y \subseteq X$ with

$$\dim(Y) \;=\; \dim(X) - 1.$$

The *Weil divisor group* of X is the free abelian group over all prime divisors of X:

$$\mathrm{WDiv}(X) \;=\; \bigoplus_{\substack{Y \subseteq X \text{ prime} \\ \text{divisor}}} \mathbb{Z}Y.$$

The elements are integral linear combinations of prime divisors, called *Weil divisors* and mostly denoted as

$$D \;=\; a_1 D_1 + \ldots + a_s D_s \;\in\; \mathrm{WDiv}(X).$$

Remark 5.2. Let $(X, \mathbb{T}, x_0)$ arise from a lattice fan (Σ, N) with the primitive generators $v_1, \ldots, v_r$. Then we have prime divisors

$$D_i := D_{\varrho_i} := \overline{\mathbb{T} \cdot x_{\varrho_i}} \subseteq X, \quad \varrho_i = \mathrm{cone}(v_i), \quad i = 1, \ldots, r.$$

Observe that we always have $\mathbb{T} \cdot D_i = D_i$ and the prime divisors form the boundary of $\mathbb{T}$ in X the following sense:

$$X \setminus \mathbb{T} \cdot x_0 \;=\; D_1 \cup \ldots \cup D_r.$$

The prime divisors $D_1, \ldots, D_r$ generate the subgroup of *invariant Weil divisors* of X:

$$\mathrm{WDiv}^{\mathbb{T}}(X) \;=\; \mathbb{Z}D_1 + \ldots + \mathbb{Z}D_r \;\subseteq\; \mathrm{WDiv}(X).$$

Reminder 5.3. Let X be a any normal variety. Then every prime divisor $Y \subseteq X$ admits an affine open set $U \subseteq X$ with

$$Y \cap U \;\neq\; \emptyset, \qquad I(Y \cap U) \;=\; \langle g_Y \rangle \;\subseteq\; \mathcal{O}(U).$$

This allows to define for any non-zero rational function $f = f_1/f_2$ with $f_1, f_2 \in \mathcal{O}(U)$ its *order* along Y by

$$\mathrm{ord}_Y(f) \;=\; \mathrm{ord}_Y(f_1) - \mathrm{ord}_Y(f_2), \qquad \mathrm{ord}_Y(f_i) \;=\; \max_{k \in \mathbb{Z}_{\geq 0}} g_Y^k \mid f_i.$$

Example 5.4. Consider a monomial $\chi^u = z_1^{u_1} \cdots z_r^{u_r}$. Then χ^u is a rational function on $\mathbb{C}^n$ and its order along $D_i = V(z_i)$ is given by

$$\mathrm{ord}_{D_i}(\chi^u) \; = \; u_i \; = \; \langle u, e_i \rangle.$$

Similarly, on an affine toric variety X_ϱ arising from a ray (ϱ, N) with primitive generator $v \in N$, the order of $\chi^u \in \mathbb{C}(X_\varrho)$ along D_ϱ is

$$\mathrm{ord}_{D_\varrho}(\chi^u) \; = \; \langle u, v \rangle.$$

Reminder 5.5. Let X be a normal variety. The *divisor* of a non-zero function $f \in \mathbb{C}(X)$ is the linear combination over all prime divisors $Y \subseteq X$ with the respective orders as coefficients:

$$\mathrm{div}(f) \; := \; \sum_{Y \subseteq X} \mathrm{ord}_Y(f) Y \; \in \; \mathrm{WDiv}(X).$$

Remark 5.6. Consider $(X, \mathbb{T}, x_0)$ the toric variety arising from a lattice fan (Σ, N) and a character function $\chi^u \in \mathbb{K}(X)$. Then we have

$$\mathrm{ord}_{D_i}(\chi^u) \; = \; \langle u, v_i \rangle.$$

for the order of χ^u along the toric prime divisor D_i defined by a primitive generator v_i of Σ. Thus, the divisor of χ^u is given as

$$\mathrm{div}(\chi^u) \; = \; \langle u, v_1 \rangle D_1 + \ldots + \langle u, v_r \rangle D_r \; \in \; \mathrm{WDiv}^{\mathbb{T}}(X).$$

Reminder 5.7. Let X be any normal variety. Then passing to the divisor defines a group homomorphism

$$\mathbb{C}(X)^* \; \to \; \mathrm{WDiv}(X), \qquad f \; \mapsto \; \mathrm{div}(f).$$

The image $\mathrm{PDiv}(X) \subseteq \mathrm{WDiv}(X)$ is the group of *principal divisors*. The *divisor class group* is the factor group

$$\mathrm{Cl}(X) \; = \; \mathrm{WDiv}(X) / \mathrm{PDiv}(X).$$

Remark 5.8. Consider a toric variety $(X, \mathbb{T}, x_0)$ arising from a lattice fan (Σ, M). As $\mathcal{O}(\mathbb{T})$ is a factorial ring, every Weil divisor on $\mathbb{T} \cdot x_0 \cong \mathbb{T}$ is principal. Thus

$$\mathrm{Cl}(X) \; = \; \mathrm{WDiv}^{\mathbb{T}}(X) / \mathrm{PDiv}^{\mathbb{T}}(X)$$

holds for the divisor class group, where the group $\mathrm{PDiv}^{\mathbb{T}}(X)$ of invariant principal divisors consists of precisely of the divisors $\mathrm{div}(\chi^u)$ of the character functions χ^u with $u \in M$.

Theorem 5.9. *Let $(X, \mathbb{T}, x_0)$ be the toric variety arising from a lattice fan (Σ, N). Then we have a commutative diagram with exact rows:*

$$\begin{array}{ccccccc}
0 \longleftarrow & K & \xleftarrow{\quad Q \quad} & \mathbb{Z}^r & \xleftarrow{\quad P^* \quad} & M \\
 & \cong \big\downarrow & & a \mapsto \sum a_i D_i \big\downarrow \cong & & \cong \big\downarrow u \mapsto \chi^u \\
0 \longleftarrow & \mathrm{Cl}(X) & \xleftarrow[{[D] \leftarrow\!\shortmid D}]{} & \mathrm{WDiv}^{\mathbb{T}}(X) & \xleftarrow[{\mathrm{div}(\chi^u) \leftarrow\!\shortmid \chi^u}]{} & \mathbb{X}(\mathbb{T})
\end{array}$$

where $P\colon \mathbb{Z}^r \to N$ is the linear map sending the i-th canonical basis vector e_i to the i-th primitive generator v_i and $K = \mathbb{Z}^r/P^*M$.

Part 5-A: Exercises

Exercise 5.10. Verify $\mathrm{ord}_{D_i}(\chi^u) = u_i$ from Example 5.4 using the definition from Reminder 5.3. Moreover, prove $\mathrm{ord}_{D_\varrho}(\chi^u) = \langle u, v\rangle$ as stated in Example 5.4. *Hint:* Since $v \in N$ is primitive, there is an lattice isomorphism $N \to \mathbb{Z}^n$ sending u to e_1.

Exercise 5.11. Verify all statements made in Remark 5.6 and in Remark 5.8.

Exercise 5.12. Prove Theorem 5.9. *Hint:* For the commutativity of the right rectangle use $\langle u, P(e_i)\rangle = \langle P^*(u), e_i\rangle$.

Exercise 5.13. Compute the divisor class group for the projective space $\mathbb{P}_n$, the product $\mathbb{P}_1 \times \mathbb{P}_1$ and the Hirzebruch surfaces X_a, where $a \in \mathbb{Z}_{\geq 1}$.

Exercise 5.14. Compute the divisor class group for the affine varieties $V(z_1z_2 - z_3^2) \subseteq \mathbb{C}^3$ and $V(z_1z_2 - z_3z_4) \subseteq \mathbb{C}^4$.

Exercise 5.15. Let $X = X_\sigma$ be the affine toric variety arrising from a pointed lattice cone (σ, N) with $\dim(\sigma) = \dim(N_{\mathbb{Q}})$. Show that $\mathrm{Cl}(X) = 0$ holds if and only if $X \cong \mathbb{C}^n$.

Part 5-B. We study sections of invariant divisors and the associated polyhedra for toric varieties arising from lattice fans and show how the divisorial polyhedra pop up in the homogeneous coordinate ring.

Clip 5-B **Notes 5-B** **Exercises 5-B**

Part 5-B: Short Notes

Reminder 5.16. Let X be a normal variety and D a Weil divisor on X. The space of *sections* of D is the vector subspace

$$\Gamma(X, \mathcal{O}(D)) \;=\; \{0\} \cup \{f \in \mathbb{C}(X)^*;\ \mathrm{div}(f) + D \geq 0\} \;\subseteq\; \mathbb{C}(X).$$

Example 5.17. Consider $\mathbb{P}_2$ and the prime divisor $D = V(z_0)$ on $\mathbb{P}_2$. The space of sections of D is

$$\Gamma(\mathbb{P}_2, \mathcal{O}(D)) \;=\; \mathbb{C} \oplus \mathbb{C}\frac{z_1}{z_0} \oplus \mathbb{C}\frac{z_2}{z_0}.$$

Construction 5.18. Let $(X, \mathbb{T}, x_0)$ arise from a lattice fan (Σ, N) with primitive generators $v_1, \ldots, v_r$. Given $a \in \mathbb{Z}^r$, we obtain the divisor

$$D \;=\; a_1 D_1 + \ldots + a_r D_r \;\in\; \mathrm{WDiv}^{\mathbb{T}}(X),$$

where $D_1, \ldots, D_r$ are the prime divisors associated with the primitive generators. The *divisorial polyhedron* defined by D is

$$B_D \;:=\; \{u \in M_{\mathbb{Q}};\ \langle u, v_i \rangle \geq -a_i,\ i = 1, \ldots, r\} \;\subseteq\; M_{\mathbb{Q}}.$$

Remark 5.19. Let $(X, \mathbb{T}, x_0)$ arise from a lattice fan (Σ, N). Consider $D = a_1 D_1 + \ldots + a_r D_r \in \mathrm{WDiv}^{\mathbb{T}}(X)$. For any $\chi^u \in \mathbb{C}(X)$, we have

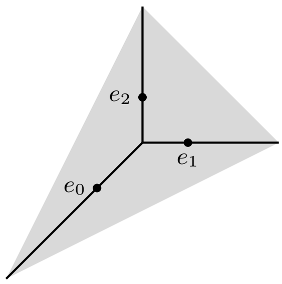
$$\mathrm{div}(\chi^u) + D \geq 0 \iff \langle u, v_i \rangle + a_i \geq 0,\ i = 1, \ldots, r \iff u \in B_D.$$

As D is invariant, $\mathbb{T}$ acts on $\Gamma(X, \mathcal{O}(D))$. As a consequence, the space of sections is generated by character functions:

$$\Gamma(X, \mathcal{O}(D)) \;=\; \bigoplus_{u \in B_D \cap M} \mathbb{C}\chi^u$$

Example 5.20. Consider again $\mathbb{P}_2$, this time as a toric variety. The defining fan Σ in $\mathbb{Z}^2$ and its primitive generators are given by

$$e_0 \;=\; (-1, -1) \qquad e_1 \;=\; (1, 0) \qquad e_2 \;=\; (0, 1)$$



Let $a = (1, 0, 0) \in \mathbb{Z}^3$. Then the corresponding invariant divisor on $\mathbb{P}_2$ is $D = V(z_0) \subseteq \mathbb{P}_2$. Moreover, we obtain

$$\langle u, e_0 \rangle \geq -1, \qquad \langle u, e_1 \rangle \geq 0, \qquad \langle u, e_2 \rangle \geq 0$$

as the defining inequalities for the divisorial polyhedron B_D. This leaves us with $u_1 + u_2 \leq 1$ and $u_1 \geq 0$ as well as $u_2 \geq 0$. Hence

$$B_D \;=\; \mathrm{conv}((0,0),\ (1,0),\ (0,1)) \;\subseteq\; \mathbb{Q}^2$$

is the 2-dimensional standard simplex. For the space of sections of the divisor D, this tells us

$$\Gamma(\mathbb{P}_2, \mathcal{O}(D)) \;=\; \mathbb{C}\chi^{(0,0)} \oplus \mathbb{C}\chi^{(1,0)} \oplus \mathbb{C}\chi^{(0,1)}.$$

Construction 5.21. Let the toric variety $(X, \mathbb{T}, x_0)$ arise from a non-degenerate lattice fan (Σ, N) with primitive generators $v_1, \ldots, v_r$. Then we have the linear map

$$P\colon \mathbb{Z}^r \;\to\; N, \qquad e_i \;\to\; v_i.$$

Let $K = \mathbb{Z}^r / P^* M$ be the factor group by the image of the dual map and $Q\colon \mathbb{Z}^r \to K$ the projection. The *homogeneous coordinate ring* of X is the K-graded polynomial ring.

$$\mathcal{R}(X) = \mathbb{C}[z_1, \ldots, z_r], \qquad \deg(z_i) := Q(e_i) \in K.$$

Remark 5.22. Let $(X, \mathbb{T}, x_0)$ arise from a non-degenerate lattice fan (Σ, N). For $w \in K$, the *class polyhedron* is

$$B_w \ = \ \operatorname{aff}(Q^{-1}(w)) \cap \mathbb{Q}^r_{\geq 0} \ \subseteq \ \mathbb{Q}^r.$$

The lattice points inside the class polyhedron correspond to the monomials of degree w in the homogeneous coordinate ring:

$$\mathcal{R}(X) \ = \ \bigoplus_{w \in K} \mathcal{R}(X)_w, \qquad \mathcal{R}(X)_w \ = \ \bigoplus_{\mu \in B_w \cap \mathbb{Z}^r} \mathbb{C} z^\mu.$$

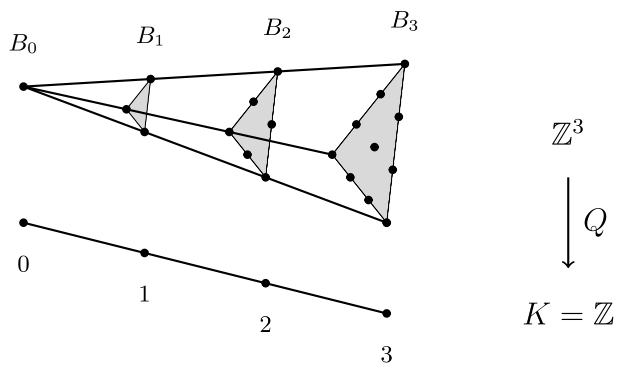
Example 5.23. Consider again the toric variety $\mathbb{P}_2$ and its defining fan Σ. Then $P\colon \mathbb{Z}^3 \to \mathbb{Z}^2$ and $Q\colon \mathbb{Z}^3 \to \mathbb{Z}$ are given by

$$P \ = \ \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \qquad Q \ = \ \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

We have $B_W = \emptyset$ and thus $\mathcal{R}(\mathbb{P}_2)_w = \{0\}$ for $w \in \mathbb{Z}_{<0}$. For $w \in \mathbb{Z}_{\geq 0}$, the class polyhedron is the simplex

$$B_w \ = \ \operatorname{conv}((w, 0, 0), \, (0, w, 0), \, (0, 0, w)) \ \subseteq \ \mathbb{Q}^3.$$

The lattice points inside the B_w represent bases of the components $\mathcal{R}(X)_w$ of $\mathcal{R}(X)$ and the whole situation can be visualized:



Remark 5.24. Let $(X, \mathbb{T}, x_0)$ be the toric variety arising from a non-degenerate lattice fan (Σ, N). Fix

$$a \ \in \ \mathbb{Z}^r, \qquad w \ := \ Q(a) \ \in \ K.$$

Then a defines an invariant Weil divisor D on X having w as its associated divisor class:

$$D = a_1 D_1 + \ldots + a_r D_r \in \operatorname{WDiv}^{\mathbb{T}}(X), \qquad \operatorname{Cl}(X) \ni [D] = w \in K.$$

Consider the injection $M \to \mathbb{Z}^r$, $u \mapsto P^* u + a$. Then, for every $u \in M_{\mathbb{Q}}$ and any primitive generator v_i of Σ, we have

$$\langle u, v_i \rangle \geq -a_i \iff \langle P^* u + a, e_i \rangle \geq 0.$$

Thus, the above injection identifies the divisorial polyhedron $B_D \subseteq M_{\mathbb{Q}}$ with the class polyhedron $B_w \subseteq \mathbb{Q}^r$:

$$P^* B_D + a \ = \ B_w \ \subseteq \ \mathbb{Q}^r.$$

Moreover, the space of sections of D is isomorphic to the component of degree w of the homogeneous coordinate ring via

$$\Gamma(X, \mathcal{O}(D)) \ \to \ \mathcal{R}(X)_w, \qquad \chi^u \ \mapsto \ z^{P^*u+a}.$$

Finally, for a section χ^u of D and the corresponding monomial z^{P^*u+a} of $\mathcal{R}(X)_w$, the divisors are related to each other via

$$\operatorname{div}(\chi^u) + D \ = \ p_* \operatorname{div}(z^{P^*u+a}),$$

where $p\colon \hat{X} \to X$ is the toric morphism associated with $P\colon \mathbb{Z}^r \to N$ and p_* is the push forward, sending $V(z_i)$ to D_i.

Part 5-B: Exercises

Exercise 5.25. Show that the homogeneous coordinate ring of the projective space $\mathbb{P}_n$ is given by

$$\mathcal{R}(\mathbb{P}_n) \ = \ \mathbb{C}[z_0, \ldots, z_n], \qquad\qquad \deg(z_i) = 1 \in K = \mathbb{Z}.$$

Exercise 5.26. Consider $\mathbb{P}_1 \times \mathbb{P}_1$ and its defining $(\Sigma, \mathbb{Z}^2)$. The homomorphisms $P\colon \mathbb{Z}^4 \to \mathbb{Z}^2$ and $Q\colon \mathbb{Z}^4 \to \mathbb{Z}^2$ are given by

$$P \ = \ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \qquad\qquad Q \ = \ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Verify $K = \mathbb{Z}^2$ and the shape of Q. Moreover, determine all class polyhedra B_w of $\mathbb{P}_1 \times \mathbb{P}_1$, where $w \in K$.

Exercise 5.27. Let $(X, \mathbb{T}, x_0)$ arise from a non-degenerate lattice fan (Σ, N). Consider the toric morphism

$$(\hat{X}, \mathbb{T}^r, \mathbb{1}_r) \xrightarrow{(p,\tilde{p})} (X, \mathbb{T}, x_0)$$

provided by Construction 4.36 and Remark 4.38, the closed subgroup $H := \ker(\tilde{p}) \subseteq \mathbb{T}^r$ and its action on $\hat{X} \subseteq \bar{X} = \mathbb{C}^r$.

(i) Show that H is a quasitorus, i.e., a direct product of a torus and a finite abelian group. *Hint:* Smith normal form of P.

(ii) Observe that $\mathcal{O}(H)$ is generated by the characters of H and conclude $H = \operatorname{Spec} \mathbb{C}[K]$ by verifying

$$\mathbb{X}(H) = \mathbb{X}(\mathbb{T}^r)/\tilde{p}^* \mathbb{X}(\mathbb{T}) \ = \ \mathbb{Z}/P^*M \ = \ K.$$

(iii) Show that, with $\chi_i := z_i|_H$, the action of H on $\bar{X} = \mathbb{C}^r$ is given by the K-grading of $\mathcal{R}(X)$ in the following sense:

$$h \cdot z \ = \ (\chi_1(h) z_1, \ldots, \chi_r(h) z_r).$$

(iv) For a cone $\sigma \in \Sigma$ and the corresponding cone $\hat{\sigma} \in \hat{\Sigma}$ observe $p^{-1}(X_\sigma) = X_{\hat{\sigma}}$ and show

$$p^*(\mathcal{O}(X_\sigma)) \;=\; \mathbb{C}[\hat{\sigma} \cap \mathbb{Z}^r]_0 \;=\; \mathcal{O}(X_{\hat{\sigma}})^H,$$

where the subscript "0" refers to degree zero with respect to the K-grading and the superscript "H" to H-invariant functions.

Property (iv) shows that $p\colon \hat{X} \to X$ is a *good quotient* for the H-action on $\hat{X}$: the morphism p is affine and $\mathcal{O}_X$ is the sheaf of invariants $p_*\mathcal{O}_{\hat{X}}^H$.

Exercise 5.28. Verify all statements made in Remark 5.24. *Hint:* For $P^*B_D + a = B_w$ use $\ker(Q) = P^*M$.

Part 5-C. We determine the Picard group, base loci, and the cones of effective, movable, semiample, as well as ample divisor classes for toric varieties in terms of their defining fan.

Clip 5-C **Notes 5-C** **Exercises 5-C**

Part 5-C: Short Notes

Reminder 5.29. A Weil divisor D on a normal variety X is *principal at* $x \in X$, if there are an open set $U \subseteq X$ and $f \in \mathbb{C}(X)$ with

$$x \;\in\; U, \qquad\qquad D|_U \;=\; \mathrm{div}(f) \;\in\; \mathrm{WDiv}(U).$$

Moreover, D is a *Cartier divisor* if it is principal near every $x \in X$. The Cartier divisors form a subgroup

$$\mathrm{CDiv}(X) \;\subseteq\; \mathrm{WDiv}(X),$$

containing the group $\mathrm{PDiv}(X)$ of principal divisors. The *Picard group* of X is the factor group

$$\mathrm{Pic}(X) \;=\; \mathrm{CDiv}(X)/\,\mathrm{PDiv}(X) \;\subseteq\; \mathrm{WDiv}(X)/\,\mathrm{PDiv}(X) \;=\; \mathrm{Cl}(X).$$

Remark 5.30. Let $(X, \mathbb{T}, x_0)$ arise from a lattice fan (Σ, N) and fix $\sigma \in \Sigma$. Then $D \in \mathrm{WDiv}^{\mathbb{T}}(X)$ is principal near x_σ if and only if

$$D|_{X_\sigma} \;=\; \mathrm{div}(\chi^u)$$

holds for some $u \in M$. In this case, look at $D' := D - \mathrm{div}(\chi^u)$. We have $D' = a_1' D_1 + \ldots + a_r' D_r$ with $a_i' = 0$ if $v_i \in \sigma$. Thus,

$$[D] \;=\; [D'] \;\in\; Q(\mathrm{lin}(\hat{\sigma}^*) \cap \mathbb{Z}^r) \;\subseteq\; K \;=\; \mathrm{Cl}(X)$$

with $\hat{\sigma}^* = \mathrm{cone}(e_j;\ v_j \notin \sigma) \preccurlyeq \mathbb{Q}^r_{\geq 0}$. Conversely, if $[D] \in Q(\mathrm{lin}(\hat{\sigma}^*) \cap \mathbb{Z}^r)$ holds, then D is principal at $x_\sigma \in X$.

Theorem 5.31. *Let $(X, \mathbb{T}, x_0)$ arise from a lattice fan (Σ, N). Then the Picard group of X is given as*

$$\mathrm{Pic}(X) \;=\; \bigcap_{\sigma \in \Sigma} Q(\mathrm{lin}(\hat{\sigma}^*) \cap \mathbb{Z}^r) \;\subseteq\; K \;=\; \mathrm{Cl}(X).$$

Remark 5.32. Let $(X, \mathbb{T}, x_0)$ arise from a lattice fan (Σ, N). The following scheme allows us to transfer information on cones $\sigma \in \Sigma$ from $N = \Lambda(\mathbb{T})$ to $K = \mathrm{Cl}(X)$:

$$\begin{array}{ccccccc}
0 & \longrightarrow & L & \longrightarrow & \mathbb{Z}^r & \xrightarrow{P\colon e_i \to v_i} & N \\
 & & & & \delta^r \succcurlyeq \hat{\sigma} & \longleftarrow & \sigma \\
 & & & & \downarrow & & \\
 & & Q(\hat{\sigma}^*) & \longleftarrow & \hat{\sigma}^* \preccurlyeq \gamma^r & & \\
0 & \longleftarrow & K & \xleftarrow[Q]{} & \mathbb{Z}^r & \xleftarrow[P^*]{} & M
\end{array}$$

where the orthants $\delta^r = \mathbb{Q}^r_{\ge 0}$ and $\gamma^r = \mathbb{Q}^r_{\ge 0}$ are regarded as dual to each other and $\hat{\sigma}^* = \mathrm{cone}(e_j;\ v_j \not\in \sigma) = \hat{\sigma}^\perp \cap \gamma^r \preccurlyeq \gamma^r$ corresponds to $\hat{\sigma} = \mathrm{cone}(e_i;\ v_j \in \sigma) \preccurlyeq \delta^r$. Moreover, $Q(\hat{\sigma}^*)$ lives in $K_\mathbb{Q} = \mathbb{Q} \otimes_\mathbb{Z} K$.

Reminder 5.33. The *base locus* and the *stable base locus* of a Weil divisor D on a normal variety X are:

$$\mathrm{Bs}(D) \;:=\; \bigcap_{f \in S_D} \mathrm{supp}(\mathrm{div}(f) + D), \qquad \mathbf{B}(D) \;:=\; \bigcap_{k \in \mathbb{Z}_{\ge 1}} \mathrm{Bs}(kD),$$

where $\mathrm{supp}(\mathrm{div}(f) + D)$ is the union of all prime divisors showing up with non-zero coefficient in $\mathrm{div}(f) + D$ and $S_D = \Gamma(X, \mathcal{O}(D)) \setminus \{0\}$.

Theorem 5.34. *Let $(X, \mathbb{T}, x_0)$ arise from a lattice fan (Σ, N). Given any $D \in \mathrm{WDiv}(X)$, set*

$$w \;:=\; [D] \;\in\; \mathrm{Cl}(X) \;=\; K.$$

Then the base locus and the stable base locus of the divisor D are given as

$$\mathrm{Bs}(D) \;:=\; \bigcup_{\substack{\sigma \in \Sigma \\ w \not\in Q(\hat{\sigma}^* \cap \mathbb{Z}^r)}} \mathbb{T} \cdot x_\sigma, \qquad \mathbf{B}(D) \;:=\; \bigcup_{\substack{\sigma \in \Sigma \\ w \not\in Q(\hat{\sigma}^*)}} \mathbb{T} \cdot x_\sigma.$$

Reminder 5.35. Let X be any normal variety. Then a Weil divisor D on X is called

(i) *effective* if D admits non-zero sections,
(ii) *movable* if $\mathrm{Bs}(D) \subseteq X$ is of codimension at least two.
(iii) *semiample* if D has empty stable base locus,
(iv) *ample* if X is covered by affine open sets of the form

$$X_f \;=\; X \setminus \mathrm{supp}(\mathrm{div}(f) + kD)$$

with $k \in \mathbb{Z}_{\ge 1}$ and a non-zero section $f \in \Gamma(X, \mathcal{O}(kD))$.

Theorem 5.36. *Let $(X, \mathbb{T}, x_0)$ arise from a lattice fan (Σ, N). Then the cones generated by the effective, movable, semiample and ample divisor classes of X in $K_{\mathbb{Q}} = \mathrm{Cl}_{\mathbb{Q}}(X)$ are given by*

$$\mathrm{Eff}(X) = Q(\gamma^r), \qquad \mathrm{Mov}(X) = \bigcap_{i=1}^{r} \mathrm{cone}(e_j;\ j \neq i),$$

$$\mathrm{SAmple}(X) = \bigcap_{\sigma \in \Sigma} Q(\hat{\sigma}^*), \qquad \mathrm{Ample}(X) = \bigcap_{\sigma \in \Sigma} Q(\hat{\sigma}^*)^\circ.$$

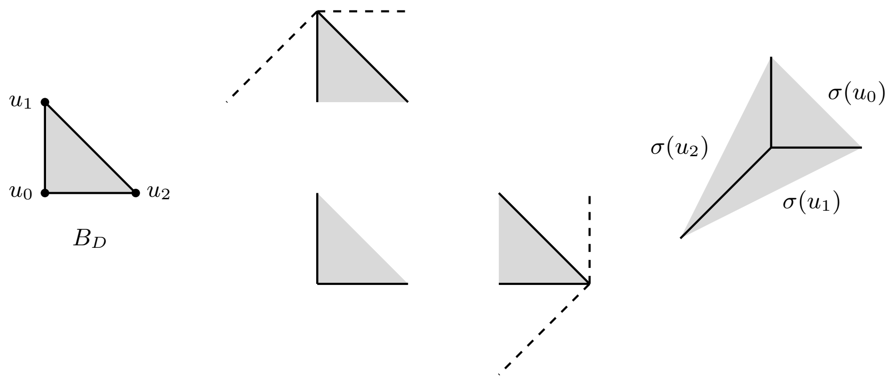
Fact 5.37. Let $(X, \mathbb{T}, x_0)$ arise from a lattice fan (Σ, N). The variety X is complete if and only if the fan Σ is complete, that means that $N_{\mathbb{Q}}$ is covered by the cones of Σ.

Remark 5.38. Let $(X, \mathbb{T}, x_0)$ arise from a lattice fan (Σ, N). The variety X is projective if and only if Σ is complete and $\mathrm{Ample}(X)$ is non-empty.

Remark 5.39. Let $(X, \mathbb{T}, x_0)$ arise from a complete lattice fan (Σ, N). A divisor $D \in \mathrm{WDiv}^{\mathbb{T}}(X)$ is ample if and only if $\Sigma = \Sigma(B_D)$ with the *normal fan* of the divisorial polyhedron $B_D \subseteq M_{\mathbb{Q}}$:

$$\begin{aligned} \Sigma(B_D) &= \{\sigma(B);\ B \preccurlyeq B_D\}, \\ \sigma(B) &= \mathrm{cone}(u - u')^\vee;\ u \in B_D,\ u' \in B). \end{aligned}$$

Example 5.40. Consider the coordinate line $D = V(z_0)$ in the projective plane $\mathbb{P}_2$. The divisorial poyhedron B_D is the standard 2-simplex.



The normal fan $\Sigma(B_D)$ equals the fan of $\mathbb{P}_2$ drawn on the right hand side. In particular, this confirms that D is an ample divisor on $\mathbb{P}_2$.

Part 5-C: Exercises

Exercise 5.41. Verify all statements made in Remark 5.30 and prove Theorem 5.31. *Hint:* On a toric variety, every Cartier divisor class is represented by an invariant Cartier divisor.

Exercise 5.42. Consider a lattice fan $(\Sigma, \mathbb{Z}^n)$ with primitive generators $v_0, \ldots, v_n$ such that $K = \mathbb{Z}$ and Q is given by

$$Q \;=\; [c_0, \ldots, c_n], \qquad c_0, \ldots, c_n \;\in\; \mathbb{Z}_{>0}.$$

Let $(X, \mathbb{T}^n, x_0)$ be the associated toric variety. Then X equals the *weighted projective space* $\mathbb{P}_{c_0,\ldots,c_n}$. Show that we have

$$\operatorname{Pic}(X) \;=\; \operatorname{lcm}(c_0, \ldots, c_n)\mathbb{Z} \;\subseteq\; \mathbb{Z} \;=\; \operatorname{Cl}(X).$$

Exercise 5.43. Show that every affine toric variety has trivial Picard group.

Exercise 5.44. Let $X(X, \mathbb{T}, x_0)$ arise from a non-degenerate lattice fan (Σ, N). Given $D \in \operatorname{WDiv}^{\mathbb{T}}(X)$, let $w := [D]$ denote the class in $K = \operatorname{Cl}(X)$. Show that the base locus of D is given in terms of the class polyhedron B_w as

$$\operatorname{Bs}(D) \;=\; \bigcap_{\mu \in B_w \cap \mathbb{Z}^r} p(V(z^\mu)) \;=\; p\left(\bigcap_{\mu \in B_w \cap \mathbb{Z}^r} V(z^\mu)\right).$$

Exercise 5.45. Show that the base locus of a Weil divisor D on a normal variety X only depends on the class $[D] \in \operatorname{Cl}(X)$. Prove Theorem 5.34.

Exercise 5.46. Show that for a Weil divisor D on a normal variety X, the properties of being effective, movable, semiample or ample only depend on the class $[D] \in \operatorname{Cl}(X)$. Prove Theorem 5.36.

Exercise 5.47. Let (Σ, N) be a complete lattice fan with $\dim(N_{\mathbb{Q}}) = 2$ and $(X, \mathbb{T}, x_0)$ the associated toric variety. Show that X is a projective surface and that $\operatorname{Mov}(X)$ equals $\operatorname{SAmple}(X)$. *Hint:* Use the fact that for a simplicial cone $\sigma \in \Sigma$, the cone $Q(\hat{\sigma}^*)$ is of full dimension in $K_{\mathbb{Q}}$.

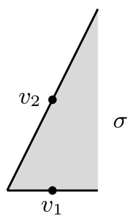
6. Singularities and resolution

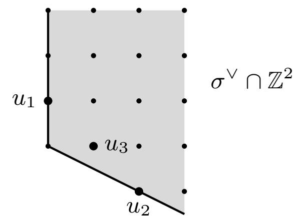
Part 6-A. We characterize of smoothness and $\mathbb{Q}$-factoriality of a toric variety in terms of its defining fan and we observe that $\mathbb{Q}$-factorial toric singularities are finite abelian quotient singularities.

Clip 6-A **Notes 6-A** **Exercises 6-A**

Part 6-A: Short Notes

Example 6.1. We look for singularities of the surface X_σ arising from the cone σ in $\mathbb{Z}^2$ generated by $v_1 = (1,0)$ and $v_2 = (1,2)$.





The cone monoid $\sigma^\vee \cap M$ has three generators: $u_1 = (0,1)$, $u_2 = (2,-1)$ and $u_3 = (1,0)$. This provides us with an epimorphism

$$\mathbb{C}[z_1, z_2, z_3] \ \to \ \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2], \qquad z_i \ \mapsto \ \chi^{u_i}.$$

The associated morphism of varieties $\varphi\colon X_\sigma \to \mathbb{C}^3$ is a closed embedding and the image is given by

$$\varphi(X_\sigma) \ = \ V(z_1z_2 - z_3^2) \ \subseteq \ \mathbb{C}^3.$$

As $g := z_1z_2 - z_3^2$ is prime and $\mathrm{grad}(g)$ vanishes only in $\varphi(x_\sigma) = 0 \in \mathbb{C}^3$, we see that $x_\sigma \in X_\sigma$ is the only singularity.

Remark 6.2. Let (σ, N) be a lattice cone. If $\dim(\sigma) < \dim(N_\mathbb{Q})$, then we find a decomposition

$$N \ = \ N' \oplus N'', \qquad N' \ := \ \mathrm{lin}(\sigma) \cap N.$$

This gives us a new lattice cone (σ', N'), where $\sigma' = \sigma$, and a product decomposition $X_\sigma = X_{\sigma'} \times \mathbb{T}_{N''}$. For the singular loci, we have

$$\mathrm{sing}(X_\sigma) \ = \ \mathrm{sing}(X_{\sigma'}) \times \mathbb{T}_{N''}.$$

Remark 6.3. Let (ω, M) be a pointed lattice cone. The cone monoid $\omega \cap M$ has a unique minimal generator system, the *Hilbert basis*:

$$\mathcal{H} \ := \ \mathcal{H}(\omega) \ := \ \{u \in \omega \cap M;\ u \text{ indecomposable}\} \ \subseteq \ \omega \cap M.$$

Here we call an element $u \in \omega \cap M$ *indecomposable* if for any two $u', u'' \in \omega \cap M$ with $u = u' + u''$, one has $u' = u$ or $u'' = u$.

Remark 6.4. Let (σ, N) be a pointed lattice cone with $\dim(\sigma)$ equal to $\dim(N_{\mathbb{Q}})$. Then $(\sigma^\vee, M)$ is pointed and of full dimension. Setting

$$F\colon \mathbb{Z}^s \rightarrow M, \qquad e_i \mapsto u_i,$$

where $u_1, \ldots, u_s$ form the Hilbert basis of $\sigma^\vee \cap M$, defines a linear map. As F sends $\mathbb{Z}^s_{\geq 0}$ onto $\sigma^\vee \cap M$ and, we obtain an epimorphism

$$\psi\colon \mathbb{C}[z_1, \ldots, z_s] \rightarrow \mathbb{C}[\sigma^\vee \cap M], \qquad z_i \mapsto \chi^{u_i}$$

The associated morphism of varieties $\varphi\colon X_\sigma \to \mathbb{C}^s$ is a closed embedding. The vanishing ideal of $\varphi(X_\sigma) \subseteq \mathbb{C}^s$ is the binomial ideal

$$I(\varphi(X_\sigma)) = \ker(\psi) = \langle z^{\mu_+} - z^{\mu_-};\ \mu \in \ker(F)\rangle,$$

where $\mu = \mu_+ - \mu_-$ with $\mu_+, \mu_- \in \mathbb{Z}^s_{\geq 0}$ is the unique decomposition such that the coordinates of μ_+ and μ_- satisfy

$$\mu_{+,i} = 0 \text{ or } \mu_{-,i} = 0 \text{ for } i = 1, \ldots, s.$$

Since $u_1, \ldots, u_s$ are indecomposable in $\sigma^\vee \cap M$, all the terms z^{μ_+} and z^{μ_-} are at least quadratic. Consequently, for all $\mu \in \ker(F)$, we have

$$\operatorname{grad}(z^{\mu_+} - z^{\mu_-})(0) = 0.$$

Thus, if X_σ is smooth, then $0 = \varphi(x_\sigma)$ is a smooth point of $\varphi(X_\sigma)$, hence $I(\varphi(X_\sigma))$ and $\ker(F)$ must be trival, showing

$$X_\sigma \cong \mathbb{C}^n, \qquad M = \mathbb{Z}u_1 \oplus \ldots \oplus \mathbb{Z}u_s.$$

Definition 6.5. We say that a pointed lattice cone (σ, N) is *regular* if its primitive generators $v_1, \ldots, v_k$ are part of $\mathbb{Z}$-basis for N.

Theorem 6.6. *Let (σ, N) be a pointed lattice cone of dimension $n = \dim(N_{\mathbb{Q}})$. Then the following statements are equivalent.*

(i) *The variety X_σ is smooth.*
(ii) *We have $X_\sigma \cong \mathbb{C}^n$.*
(iii) *The cone (σ, N) is regular.*
(iv) *The cone $(\sigma^\vee, M)$ is regular.*

Theorem 6.7. *Let $(X, \mathbb{T}, x_0)$ arise from a lattice fan (Σ, N). Then X is smooth if and only if (σ, N) is regular for every $\sigma \in \Sigma$.*

Reminder 6.8. A pointed lattice cone (σ, N) is called *simplical* if its primitive generators $v_1, \ldots, v_r$ are part of a $\mathbb{Q}$-basis of $N_{\mathbb{Q}}$.

Reminder 6.9. A normal variety X is called $\mathbb{Q}$-*factorial*, if for every Weil divisor D on X some multiple kD with $k \in \mathbb{Z}_{>0}$ is Cartier.

Theorem 6.10. *Let (σ, N) be a pointed lattice cone of dimension $n = \dim(N_{\mathbb{Q}})$. Then the following statements are equivalent.*

(i) *The variety X_σ is $\mathbb{Q}$-factorial.*
(ii) *The divisor class group $\mathrm{Cl}(X_\sigma)$ is finite.*
(iii) *The cone $\sigma \subseteq N_{\mathbb{Q}}$ is simplicial.*
(iv) *The cone $\sigma^\vee \subseteq M_{\mathbb{Q}}$ is simplicial.*

Remark 6.11. Let (σ, N) be a simplicial pointed lattice cone of dimension $n = \dim(N_{\mathbb{Q}})$. Consider the linear map

$$P\colon \mathbb{Z}^n \;\to\; N, \qquad e_i \;\mapsto\; v_i$$

and the toric morphism $(p, \tilde{p})$ from $(\mathbb{C}^n, \mathbb{T}^n, \mathbb{1}_n)$ to $(X_\sigma, \mathbb{T}_N, x_0)$ given by P. Choose $S, T \in \mathrm{Gl}(n, \mathbb{Z})$ such that

$$P \;=\; S \cdot D \cdot T, \qquad D \;=\; \mathrm{diag}(d_1, \ldots, d_n),$$

where $d_i \neq 0$ for $i = 1, \ldots, n$ and even $d_i \mid d_{i+1}$ is achievable. For the kernel of $\tilde{p}\colon \mathbb{T}^n \to \mathbb{T}^n$, this means

$$H \;:=\; \ker(\tilde{p}) \;\cong\; \mathbb{Z}/d_1\mathbb{Z} \times \ldots \times \mathbb{Z}/d_r\mathbb{Z}.$$

In particular, H is finite abelian and, as $X_\sigma = \mathbb{C}^n/H$, we see that X_σ has at most *finite abelian quotient singularities.*

Theorem 6.12. *Let $(X, \mathbb{T}, x_0)$ arise from a lattice fan (Σ, N). Then the following statements are equivalent.*

(i) *The variety X is $\mathbb{Q}$-factorial.*
(ii) *All cones of Σ are simplicial.*

Moreover, if (Σ, N) is non-degenerate, then each of (i) and (ii) is equivalent to the following statement.

(iii) *The fibers of $p\colon \hat{X} \to X$ equal the orbits of $H = \ker(\tilde{p})$ in $\hat{X}$.*

Here, $(p, \tilde{p})$ is the toric morphism from $(\hat{X}, \mathbb{T}^r, \mathbb{1}_r)$ to $X, \mathbb{T}, x_0)$ defined by $P\colon \mathbb{Z}^r \to N$, $e_i \to v_i$.

Part 6-A: Exercises

Exercise 6.13. Convince yourself about the fact that a cone is pointed if and only if its dual cone is of full dimension.

Exercise 6.14. Show that every pointed lattice cone has a finite Hilbert basis.

Exercise 6.15. Verify all details of Remark 6.4 except the description of $I(\varphi(X_\sigma))$ as a binomial ideal. Use Remark 6.4 to prove Theorem 6.6.

Exercise 6.16. Let $(X, \mathbb{T}, x_0)$ be a normal, affine toric variety. Show that X is smooth if and only if $X \cong \mathbb{C}^k \times \mathbb{T}^l$ holds for some $k, l \in \mathbb{Z}_{\geq}0$.

Exercise 6.17. Let (σ, N) be a simplicial lattice cone with primitive generators $v_1, \ldots, v_r$. Given an invariant divisor $D = a_1 D_1 + \ldots + a_r D_r$ on X_σ, find $k \in \mathbb{Z}_{\geq 0}$ and $u \in M$ with $kD = \operatorname{div}(\chi^u)$.

Exercise 6.18. Prove Theorems 6.10 and 6.12. *Hint:* Use the descrition of the divisor class group provided by Theorem 5.9.

Exercise 6.19. Show that a toric variety is $\mathbb{Q}$-factorial if and only if its Picard group is of finite index in its divisor class group.

Exercise 6.20. Prove Remark 6.11. *Hint:* Use Remark 4.23 to verify $X_\sigma = \mathbb{C}^n / H$.

Part 6-B. We resolve singularities of affine toric surfaces by subdividing the defining lattice cone through the Hilbert basis and we discuss the geometry of this resolution of singularities.

Clip 6-B **Notes 6-B** **Exercises 6-B**

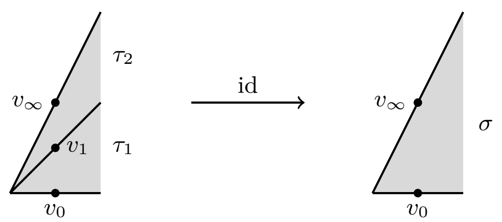
Part 6-B: Short Notes

Definition 6.21. A *resolution of singularities* of a variety X is a proper birational morphism $X' \to X$ such that X' is a smooth variety.

Fact 6.22. The proper birational toric morphisms arise from *refinements* $(\Sigma', N) \to (\Sigma, N)$ of lattice fans, that means that $\mathrm{id}\colon N \to N$ is a map of fans and we have

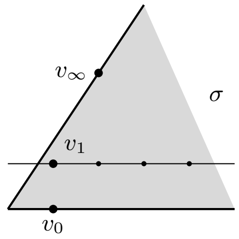
$$\operatorname{supp}(\Sigma') = \bigcup_{\sigma' \in \Sigma'} \sigma' = \bigcup_{\sigma \in \Sigma} \sigma = \operatorname{supp}(\Sigma).$$

Example 6.23. Consider the affine toric surface X_σ arising from the cone σ in $\mathbb{Z}^2$ generated by the vectors $v_0 = (1,0)$ and $v_\infty = (1,2)$. Then X_σ has a unique singularity, the limit point $x_\sigma \in X_\sigma$.



The fan Σ in $\mathbb{Z}^2$ obtained by subdividing σ at $v_1 = (1,1)$ gives us a proper birational morphism $X_\Sigma \to X_\sigma$ which moreover is a resolution of singularities.

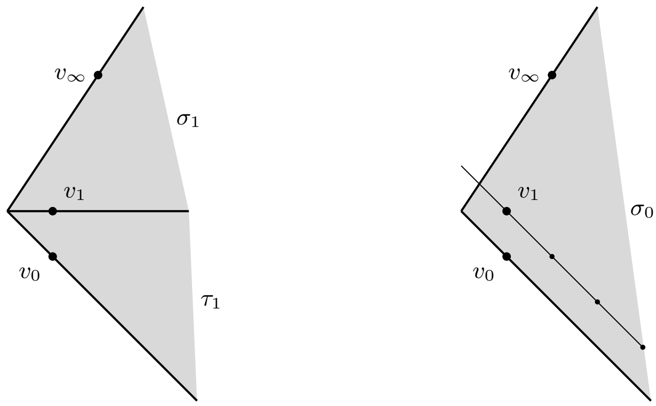
Remark 6.24. Consider $\sigma = \text{cone}(v_0, v_\infty)$ in $\mathbb{Z}^2$ with $v_0 = (1, 0)$ and $v_\infty = (a, b)$ such that $0 \leq a < b$ and $\gcd(a, b,) = 1$.



Then $v_1 = (1, 1)$ is the unique indecomposable element in $\sigma \cap \mathbb{Z}^2$ such that $\det(v_0, v_1) = 1$ holds. If $b > 0$, then we have

$$\det(v_0, v_\infty) \; = \; b \; > \; b - a \; = \; \det(v_1, v_\infty).$$

Remark 6.25. Consider a two-dimensional pointed cone σ_0 in $\mathbb{Z}^2$ and with primitive generators v_0 and v_∞, where $\det(v_0, v_\infty) > 0$.

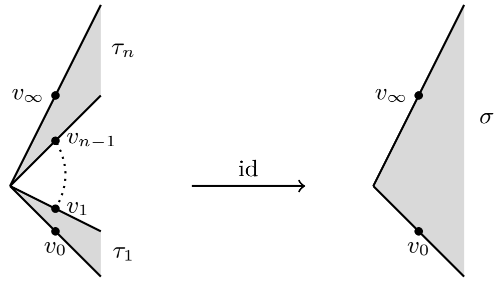


There is a unique indecomposable $v_1 \in \sigma_0 \cap \mathbb{Z}^2$ with $\det(v_0, v_1) = 1$. For $\tau_1 = \text{cone}(v_0, v_1)$ and $\sigma_1 = \text{cone}(v_1, v_\infty)$, we observe

$$\det(v_0, v_\infty) \; > \; \det(v_1, v_\infty), \qquad \mathcal{H}(\sigma_0) \; = \; \mathcal{H}(\tau_1) \cup \mathcal{H}(\sigma_1),$$

where the first statement concerns the case $v_1 \neq v_\infty$ and the second statement is about the Hilbert bases of the involved cones.

Theorem 6.26. *Let $(\sigma, \mathbb{Z}^2)$ be a two-dimensional pointed lattice cone with primitive generators v_0 and v_∞.*



Iterated application of Remark 6.25 terminates with $v_n = v_\infty$ and produces a lattice fan $(\Sigma, \mathbb{Z}^2)$ as indicated above such that

(i) *the primitive generators $v_0, v_1, \ldots, v_{n-1}, v_\infty$ of Σ are precisely the members of the Hilbert basis of $(\sigma, \mathbb{Z}^2)$,*
(ii) *the maximal cones of Σ are $\tau_i = \mathrm{cone}(v_{i-1}, v_i)$, $i = 1, \ldots, n$, and each of the lattice cones $(\tau_i, \mathbb{Z}^2)$ is regular.*

The toric morphism given by the refinement Σ of the fan of faces $\mathcal{F}(\sigma)$ of σ provides a resolution of singularities $X_\Sigma \to X_\sigma$.

Reminder 6.27. Let X be a smooth surface, $C \subseteq X$ a smooth irreducible curve and $D \in \mathrm{CDiv}(X)$. If $C \not\subseteq \mathrm{supp}(D)$, then the *restriction*

$$D|_C \;=\; c_1 Y_1 + \ldots + c_r Y_r \;\in\; \mathrm{CDiv}(C)$$

is obtained by patching together the divisors of the restrictions of locally describing rational functions of D. The *intersection number* is

$$D \cdot C \;=\; \deg\left(D|_C\right) \;=\; c_1 + \ldots + c_r \;\in\; \mathbb{Z}.$$

If the curve $C \subseteq X$ is complete and if $D' = D + \mathrm{div}(h)$ with $h \in \mathbb{K}(X)^*$ satisfies $C \not\subseteq \mathrm{supp}(D')$, then we have

$$D \cdot C \;=\; D' \cdot C.$$

This allows in particular the computation of the *self intersection number* of C: given $D = C + \mathrm{div}(h)$ with $C \not\subseteq \mathrm{supp}(D)$, we have

$$C^2 \;=\; D \cdot C.$$

Remark 6.28. Consider a fan Σ in a 2-dimensional lattice N and the invariant prime divisors $D_i \subseteq X_\Sigma$ given by the primitive generators $v_1, \ldots, v_r$. Assume that all cones of Σ are regular.

(i) For $i \neq j$, the intersection number of the curves D_i and D_j in X_Σ is given by

$$D_i \cdot D_j \;=\; \begin{cases} 1, & \mathrm{cone}(v_i, v_j) \in \Sigma, \\ 0, & \text{else.} \end{cases}$$

(ii) For pairwise distinct i, j, k with $\mathrm{cone}(v_i, v_j)$ and $\mathrm{cone}(v_j, v_k)$ belonging to Σ, the self intersection number of D_j is given by

$$D_j^2 \;=\; -m_j, \qquad v_i + v_k \;=\; m_j v_j.$$

Remark 6.29. Consider the resolution $\varphi\colon X_\Sigma \to X_\sigma$ of an affine toric surface via the Hilbert base $v_0, v_1, \ldots, v_n = v_\infty$ as in Theorem 6.26.

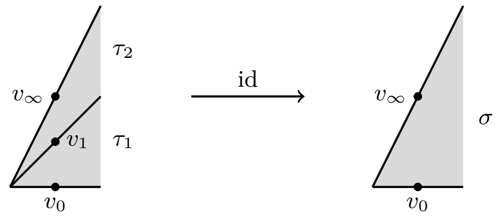
(i) The morphism $\varphi\colon X_\Sigma \to X_\sigma$ is an isomorphism over $X_\sigma \setminus x_\sigma$ and the fiber over $x_\sigma \in X_\sigma$ is given by

$$\varphi^{-1}(x_\sigma) \;=\; D_1 \cup \ldots \cup D_{n-1},$$

where each of the prime divisors $D_i \subseteq X_\Sigma$ is a smooth projective rational curve, that means $D_i \cong \mathbb{P}_1$.

(ii) For any two of the curves $D_i, D_j \subseteq X_\Sigma$ with $0 \le i < j \le n$, we have $D_i \cdot D_j = 1$ if $j = i+1$ and $D_i \cdot D_j = 0$ otherwise.
(iii) Each of the curves $D_i \subseteq X_\Sigma$ with $1 \le i \le n-1$ is of self intersection number $D_i^2 \le -2$.
(iv) The resolution $\varphi\colon X_\Sigma \to X_\sigma$ is *minimal* in the sense that any other resolution of $X' \to X$ factors through φ.

Example 6.30. Look again at the resolution of singularities $X_\Sigma \to X_\sigma$ given by the map of fans considered before:



Then the fiber over x_σ is the toric curve D_1 corresponding to the primitive generator v_1 of Σ. For the self intersection number, we have

$$D_1^2 \;=\; -m_1 \;=\; -2.$$

Theorem 6.31. *Every lattice fan (Σ, N) admits a refinement (Σ', N) such that all cones of Σ' are regular and every regular cone of Σ occurs in Σ'. The associated morphism of toric varieties is a resolution of singularities.*

Part 6-B: Exercises

Exercise 6.32. Prove Remarks 6.24 and 6.25. *Hint:* Reduce the setting of 6.25 to that of 6.24 via multiplication with a suitable matrix of $\mathrm{Gl}(2, \mathbb{Z})$.

Exercise 6.33. Show that every normal affine toric surface is a quotient of $\mathbb{C}^2$ by a diagonal action of a cyclic group. In particular, normal toric singularities are *cyclic quotient singularities.*

Exercise 6.34. Use Reminder 6.27 to prove Remark 6.28. *Hints:* For 6.28 (i) observe that if two curves don't intersect, then their intersection numer equals zero. Moreover, use that every affine smooth toric surface is isomorphic to $\mathbb{C}^2$. For 6.28 (ii) use $v_i + v_k = m_j v_j$ to find a suitable $D = D_j + \mathrm{div}(\chi^u)$ that allows to compute $D_j^2 = D \cdot D_j$.

Exercise 6.35. Consider the cone σ in $\mathbb{Z}^2$ generated by $v_0 = (1, 0)$ and $v_\infty := (1, n)$. Show that X_σ is the quotient by a cyclic group action:

$$X_\sigma = \mathbb{C}^2/G, \qquad G = \{\zeta \in \mathbb{C}^*;\ \zeta^n = 1\}, \qquad \zeta \cdot z = (\zeta z_1, \zeta^{n-1} z_2).$$

Determine the minimal resolution of singularities $\varphi\colon X_\Sigma \to X_\sigma$. Show that we have

$$\varphi^{-1}(x_\sigma) \;=\; D_1 \cup \ldots \cup D_{n-1}$$

with smooth rational curves $D_i \subseteq X_\Sigma$, all of self intersection number -2. Finally, verify

$$X_\sigma \cong V(z_1 z_2 - z_3^n) \subseteq \mathbb{C}^3.$$

Exercise 6.36. Convince yourself about the fact that every normal toric surface, affine or not, admits a resolution of singularities. How is it obtained in terms of fans?

Exercise 6.37. Compute the self intersection numbers of all the invariant curves on the a-th Hirzebruch surface X_a.

Exercise 6.38. Show that for $a \in \mathbb{Z}_{\geq 2}$, we have a resolution of singularities $X_a \to \mathbb{P}_{1,1,a}$, of the weighted projective plane $\mathbb{P}_{1,1,a}$ is $X_a \to \mathbb{P}_{1,1,a}$ by the a-th Hirzebruch surface X_a.

Exercise 6.39. We say that a morphism $\pi \colon X \to Y$ of smooth complete surfaces *smoothly contracts* an irreducible curve $C \subseteq X$ if the image $\pi(C)$ is a point $y \in Y$ and

$$\pi \colon X \setminus D \ \to \ Y \setminus \{y\}$$

is an isomorphism. Show that for a smooth complete toric surface $(X, \mathbb{T}, x_0)$ and an invariant curve $C \subseteq X$, the following statements are equivalent:

(i) There is a toric morphism contracting C smoothly.
(ii) The curve $C \subseteq X$ satisfies $C^2 = -1$.

This is the toric version of the more general *Castelnuovo criterion*, saying that a curve $C \subseteq X$ on smooth complete surface is smoothly contractible if and only if $C \cong \mathbb{P}_1$ and $C^2 = -1$.

Part 6-C. We discuss canonical divisors and Gorenstein, terminal, canonical singularities for toric varieties and we take a look at toric Fano varieties.

Clip 6-C **Notes 6-C** **Exercises 6-C**

Part 6-C: Short Notes

Reminder 6.40. Let X be a normal variety of dimension n. Over the set of smooth points $X_0 \subseteq X$, we have the cotangent bundle TX_0^* which in turn gives us a line bundle

$$L_0 \ := \ \bigwedge^n TX_0^*.$$

Any rational section s of L_0 defines a divisor $\mathrm{div}(s)$ on X_0. By normality, $X \setminus X_0$ is of codimension at least two in X. Thus $\mathrm{div}(s)$ uniquely extends to $\mathcal{K}(s) \in \mathrm{WDiv}(X)$, called a *canonical divisor*.

Fact 6.41. Let the toric variety $(X, \mathbb{T}, X_0)$ arise from a lattice fan (Σ, N). Then we have an invariant canonical divisor

$$\mathcal{K}_X \ = \ -D_1 - \ldots - D_r \ \in \ \mathrm{WDiv}^{\mathbb{T}}(X),$$

where $D_1, \ldots, D_r \subseteq X$ are the invariant prime divisors associated with the primitive generators $v_1, \ldots, v_r$ of (Σ, N).

Definition 6.42. A normal variety X is ($\mathbb{Q}$-)*Gorenstein* if (a non-zero multiple of) some canonical divisor on X is Cartier.

Remark 6.43. Every smooth variety is Gorenstein and every $\mathbb{Q}$-factorial variety is $\mathbb{Q}$-Gorenstein.

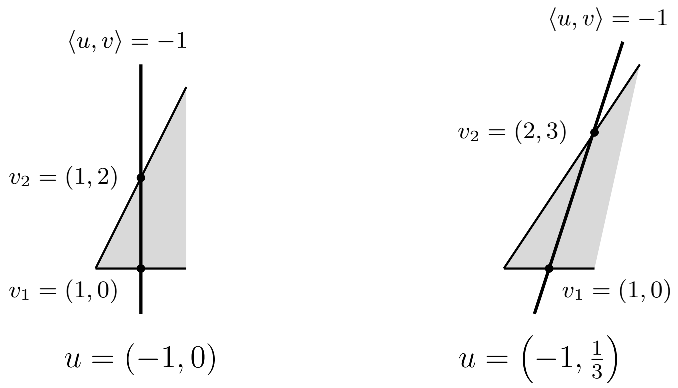
Remark 6.44. Consider a lattice cone (σ, N) of with primitive generators $v_1, \ldots, v_r$. Then the following statements are equivalent:

(i) X_σ is Gorenstein,
(ii) $\mathcal{K}_{X_\sigma} = \mathrm{div}(\chi^u)$ for some $u \in M$,
(iii) there is a $u \in M$ with $\langle u, v_i \rangle = -1$ for $i = 1, \ldots, r$.

The equivalence of (i) and (ii) relies on $\mathrm{Pic}(X_\sigma) = 0$, as it holds for any affine toric variety. Moreover, the following statements are equivalent:

(i) X_σ is $\mathbb{Q}$-Gorenstein,
(ii) $l\mathcal{K}_{X_\sigma} = \mathrm{div}(\chi^u)$ for some $u \in M$ and some $l \in \mathbb{Z}_{>0}$,
(iii) there is a $u \in M_{\mathbb{Q}}$ with $\langle u, v_i \rangle = -1$ for $i = 1, \ldots, r$.

Example 6.45. Consider the two lattice cones σ in $\mathbb{Z}^2$ and the unique linear form $u \in M_{\mathbb{Q}}$ evaluating to -1 on the primitive generators v_1 and v_2 as indicated below:



On the left hand side, u is integral, $\mathcal{K}_X = \mathrm{div}(\chi^u)$ and $X = X_\sigma$ is Gorenstein. On the right hand side, u is not integral and thus X is not Gorenstein; however, we have $3\mathcal{K}_X = \mathrm{div}(\chi^{3u})$.

Example 6.46. Consider the two affine non $\mathbb{Q}$-factorial toric threefolds $X = X_\sigma$ given by the lattice cones σ in $\mathbb{Z}^3$ indicated below:

The cone on the left hand side has primitive generators v_1, v_2, v_3, v_4 and the one on the right hand side v_1, v_2, v_3, v_4', where

$$v_1 = (0,0,1), \quad v_2 = (1,0,1), \quad v_3 = (1,1,1), \quad \begin{array}{l} v_4 = (0,1,1), \\ v_4' = (0,1,2). \end{array}$$

Now, $u = (0,0,-1)$ evaluates to -1 on v_1, v_2, v_3, v_4 but not on v_4'. Thus, the l.h.s. X is Gorenstein, and the r.h.s. X is not $\mathbb{Q}$-Gorenstein.

Definition 6.47. A *Fano variety* is a normal projective variety X such that some anticanonical divisor $-\mathcal{K}_X$ is ample.

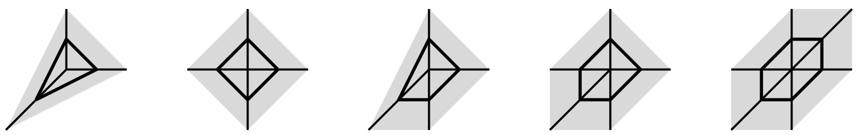
Remark 6.48. Every Fano variety is $\mathbb{Q}$-Gorenstein, because $\mathcal{K}_X$, being ample, has a Cartier multiple.

Construction 6.49. Let N be a lattice and $A \subseteq N_\mathbb{Q}$ a polytope having the origin as an interior point. The *face fan* of A in N is

$$\Sigma \ = \ \{\sigma(A');\ A' \prec A\}, \qquad\qquad \sigma(A') \ = \ \mathrm{cone}(A') \ \subseteq \ N_\mathbb{Q}.$$

Fact 6.50. Let $(X, \mathbb{T}, x_0)$ arise from a lattice fan (Σ, N) with primitive generators $v_1, \ldots, v_r$. Then X is a Fano variety if and only if Σ is the face fan of $A = \mathrm{conv}(v_1, \ldots, v_r)$.

Example 6.51. The following pictures indicate the polytopes A and their face fans in $\mathbb{Z}^2$ for the smooth toric del Pezzo surfaces:



where del *Pezzo surface* just refers to a 2-dimensional Fano variety. The vertices of the above polytopes A are taken from the list

$$(1,0), \quad (1,1), \quad (0,1), \quad (-1,1), \quad (-1,0), \quad (-1,-1), \quad (0,-1).$$

Remark 6.52. Let X be a $\mathbb{Q}$-Gorenstein variety. Given a resolution of singularities $\varphi\colon X' \to X$ and a canonical divisor $\mathcal{K}_{X'}$ on X', one finds $\mathcal{K}_X$ on X such that we have the *ramification formula*

$$\mathcal{K}_{X'} \ = \ \varphi^*\mathcal{K}_X + \sum a_i E_i,$$

where $E_i \subseteq X'$ are the *exceptional prime divisors*, that means that $\varphi(E_i) \subseteq X$ is of codimension at least two, and the rational numbers a_i are called the *discrepancies*.

Definition 6.53. A $\mathbb{Q}$-Gorenstein variety has at most *terminal (canonical)* singularities if it admits a resolution of singularities such that all discrepancies are positive (non-negative).

Remark 6.54. Let $X' \to X$ be a resolution of singularities of a $\mathbb{Q}$-Gorenstein toric variety given by a refinement of lattice fans

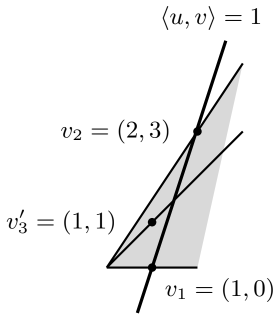
$$(\Sigma', N) \ \to \ (\Sigma, N).$$

Then any exceptional divisor $E_i \subseteq X$ is invariant and thus given by a primitive generator v_i' of Σ'. The discrepancy of E_i is

$$a_i \ = \ \langle u, v_i' \rangle - 1,$$

where we pick a cone $\sigma \in \Sigma$ satisfying $v_i' \in \sigma$ and rational linear form $u \in M_{\mathbb{Q}}$ evaluating to 1 on each primitive generator of σ.

Example 6.55. Consider the cone σ in $\mathbb{Z}^2$ shown below, the linear form $u = (1, -1/3)$ and the member $v_3' = (1, 1)$ of the Hilbert basis:



Then the discrepancy of the exceptional divisor E_3 defined by v_3' is given as $a_3 = -1/3$. In particular, X_σ is not canonical.

Theorem 6.56. *Let* $(X, \mathbb{T}, x_0)$ *be a toric Fano variety arising from a lattice fan* (Σ, N) *with primitive generators* $v_1, \ldots, v_r$ *and consider* $A = \operatorname{conv}(v_1, \ldots, v_r) \subseteq N_{\mathbb{Q}}$.

(i) X *has at most terminal singularities if and only if* 0 *and* $v_1, \ldots, v_r$ *are the only lattice points of* A.
(ii) X *has at most canonical singularities if and only if* 0 *is the only lattice point in the interior of* A.

Part 6-C: Exercises

Exercise 6.57. Show that, up to isomorphy, the Gorenstein singular toric surfaces arise from the lattice cones $\sigma = \operatorname{cone}(v_0, v_\infty)$ in $\mathbb{Z}^2$ with $v_0 = (1, 0)$ and $v_\infty = (1, n)$.

Exercise 6.58. Show that all weighted projective spaces are Fano varieties. Determine all Gorenstein weighted projective planes.

Exercise 6.59. Consider the Hirzebruch surfaces X_a, where $a \in \mathbb{Z}_{\ge 1}$ from Example 3.39. Compute their ample cones according to Theorem 5.36 and check iths way which of them are Fano varieties.

Exercise 6.60. Convince yourself about the fact that the surfaces defined by the lattice fans from Example 6.51 are $\mathbb{P}_2$, $\mathbb{P}_1 \times \mathbb{P}_1$ and the blowing up of $\mathbb{P}_2$ in one, two or three points, where in the last case, these points do not lie on a common line in $\mathbb{P}_2$.

Exercise 6.61. Show that the fans listed in Example 6.51 deliver indeed all smooth toric del Pezzo surfaces. *Hint:* One may assume that the defining fan Σ lives in $\mathbb{Z}^2$ and contains $\operatorname{cone}(-e_2, e_1)$. Use Fact 6.50 to locate the primitive generators.

Exercise 6.62. Prove Remark 6.54. *Hint:* For a suitable $l \in \mathbb{Z}_{>0}$, we have $\varphi^*(-l\mathcal{K}_X) = \operatorname{div}(\chi^{lu})$ on $\varphi^{-1}(X_\sigma)$, where $\varphi\colon X' \to X$ denotes the resolution of singularities.

Exercise 6.63. Consider a 2-dimensional lattice cone σ in $\mathbb{Z}^2$ and $X = X_\sigma$ and prove the following.

(i) The surface X is smooth if and only if it has at most terminal singularities.
(ii) The surface X is Gorenstein if and only if it has at most canonical singularities.

Hint: Use the normal form of two-dimensional lattice cones from Remark 6.24 and look at the position of the Hilbert basis member v_1.

Exercise 6.64. Show, using Fact 6.50, that for a toric Fano variety, the proper faces of the polytope A are cut out by the equations $u_\sigma = 1$, where $u_\sigma \in M_{\mathbb{Q}}$ and $\sigma \in \Sigma$. *Hint:* Look at Remark 6.44.

Exercise 6.65. Prove Theorem 6.56. *Hint:* Use Exercise 6.64 and the fact that if a resolution of singularities $X' \to X$ has only positive (non-negative) discrepancies, then this holds as well for any other resolutions of singularities $X'' \to X$.

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