Tropical Geometry

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(To be continued.)

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1 Introduction

1.1 Some generalities: algebraic geometry and a choice of numbers

The two words “Algebraic Geometry” may invoke different images for different people today. For some it might be something superficially abstract and formalistic (à la Grothendieck) adapted to work in the most general setting possible. For many others, including the authors of this book, Algebraic Geometry is just a special kind of geometry — geometry of spaces defined by polynomial equations.

Note that (unlike many algebraic properties) the resulting geometry depends not only on the type of the defining equations, but also on the choice of the numbers where we look for solutions. The two most classical choices are the field $\mathbb{R}$ of real numbers and the field $\mathbb{C}$ of complex numbers. Both these fields come naturally enhanced with the so-called “Euclidean topology” induced by the metric $|x - y|$ between two points $x, y \in \mathbb{R}$ (or in $\mathbb{C}$). Furthermore, real algebraic varieties are differentiable manifolds (perhaps with singularities) from a topological viewpoint, and complex algebraic varieties are special kind of real algebraic varieties of twice the dimension.

To illustrate the two parallel classical theories let us recall the classical example of the so-called elliptic curve. Namely, consider a cubic curve in the complex projective plane $\mathbb{CP}^2$. As long as the defining cubic polynomial is chosen generically, the resulting curve is topologically a 2-dimensional torus (see Figure 1.1). This torus is embedded in the complex projective plane $\mathbb{CP}^2$. As $\mathbb{CP}^2$ is 4-dimensional such embedding is beyond our imagination tools.

![Figure 1.1: A complex elliptic curve.](image-url)
Now consider the case of real coefficients. Even if the defining polynomial is chosen generically, the topological type of the curve is not fixed. But there are only two possible cases, see Figure 1.2.

![Figure 1.2: Two real elliptic curves.](image)

As the ambient real projective plane $\mathbb{RP}^2$ is indeed 2-dimensional, we can actually draw how the curve is embedded there. Recall that topologically $\mathbb{RP}^2$ can be obtained from a disc $D^2$ by identifying the antipodal points on its boundary circle, see Figure 1.3.

![Figure 1.3: Real projective plane.](image)

Figure 1.4 depicts embeddings of cubic curves in the real projective plane. Note that there might be different pictures inside $D^2$ before the self-identification of its boundary, but we get one of the two pictures above in $\mathbb{RP}^2 = D^2/\sim$ for any smooth real curve. An example is given in Figure 1.5.
Furthermore, the inclusion $\mathbb{R} \subset \mathbb{C}$ gives us an inclusion $\mathbb{RP}^2 \subset \mathbb{CP}^2$ as well as an inclusion of the real curve into the corresponding complex curve (see Figure 1.6).

We see that the same equation may yield quite different geometric spaces. At the same time, real and complex numbers may be the only examples of fields where algebraic geometry is that much geometric. There is also a continuation of this series with the quaternions $\mathbb{H}$ and octonions $\mathbb{O}$ which leads to very interesting geometry no longer based on fields as we lose commutativity in the case of $\mathbb{H}$ or even associativity in the case of $\mathbb{O}$.

In this book we study geometry based on a predecessor of the entire series $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, called the tropical numbers $\mathbb{T}$. 

Figure 1.4: Elliptic curves in the real projective plane.

Figure 1.5: An elliptic curve in another presentation of $\mathbb{RP}^2$ by a disk.
1.2 Tropical Numbers

We consider the set

\[ T = [-\infty, +\infty) = \mathbb{R} \cup \{-\infty\} \]

enhanced with the arithmetic operations

\[
\begin{align*}
"x + y" &= \max\{x, y\}, \\
"xy" &= x + y,
\end{align*}
\]

where we set "\(( -\infty ) + x\)" = "\(x + ( -\infty )\)" = \(x\) and "\((-\infty)x\)" = "\(x(-\infty)\)" = \(-\infty\). These operations are called tropical arithmetic operations. We use quotation marks to distinguish them from the usual operations on \(\mathbb{R}\).

**Definition 1.2.1**

The set \(T\) enhanced with these arithmetic operations is called the set of tropical numbers.

**Remark 1.2.2**

There are papers where \(\min\{x, y\}\) is taken for tropical addition. In such case one has to modify the set of tropical numbers to include \(+\infty\) and exclude \(-\infty\). It is hard to say which choice is better. The choice of \(\max\) may be more natural from the mathematical viewpoint as we are more used to taking the logarithm whose base is greater than 1, cf. equation (1.4). Also when we add two numbers in this way the sum does not get

\[
\text{Figure 1.6: Complex elliptic curve with its real locus.}
\]
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smaller. However in some consideration in Computer Science and Physics (cf. [IM]) taking the minimum is more natural. Clearly it does not really matter as the two choices are isomorphic under $x \mapsto -x$.

The term “tropical” was borrowed from Computer Science (where it was reportedly introduced to commemorate contributions of the Brazilian computer scientist Imre Simon).

The set $\mathbb{T}$ is a semigroup with respect to tropical addition. It is commutative, associative and admits the neutral element $0_{\mathbb{T}} = -\infty$. Nevertheless we do not get a group as there is no room for subtraction. Indeed, if “$x + y = 0_{\mathbb{T}}$”, then either $x = 0_{\mathbb{T}}$ or $y = 0_{\mathbb{T}}$. Thus the only element admitting an inverse with respect to tropical addition is the neutral element $0_{\mathbb{T}} = -\infty$.

We may note that tropical addition is idempotent, i.e. we have “$x + x$” = $x$ for all $x \in \mathbb{T}$. Idempotency makes tropical numbers non-Archimedean. Let us recall the Archimedes axiom (stated for the case of real numbers).

**Axiom 1.2.3 (Archimedes)**

*For any positive real numbers $a, b \in \mathbb{R}$, $a, b > 0$ there exists a natural number $n \in \mathbb{N}$ such that

\[
\underbrace{a + a + \ldots + a}_{n \text{ times}} > b.
\]

Clearly, the conventional linear order $>$ on $\mathbb{T} = \mathbb{R} \cup \{ -\infty \}$ makes perfect sense tropically. Furthermore, we can express it in terms of tropical addition: we have $a \geq b$ if and only if “$a + b$” = $b$. However, we have $\underbrace{a + a + \ldots + a}_{n \text{ times}} = a$ independently of $n$ and thus the Archimedes axiom does not hold for tropical numbers.

**Remark 1.2.4**

Note that the Euclidean topology on $\mathbb{T}$ is determined by the linear order on $\mathbb{T}$: it is generated by the sets $U_a = \{ x \in \mathbb{T} \mid x < a \}$ and $V_a = \{ x \in \mathbb{T} \mid a < x \}$ (as a subbase) for all possible $a \in \mathbb{T}$.

The tropical non-zero numbers are $\mathbb{T}^* = \mathbb{R}$. Of course, they form an honest group with respect to tropical multiplication as it coincides with the conventional addition. It is easy to check that the tropical arithmetic operations satisfy the distribution law

“$(x + y)z$” = “$xz + yz$”.

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1.3 Tropical monomials and integer affine geometry

So we see that the only defect of tropical arithmetics is the missing subtraction which makes the tropical numbers a semifield instead of a field. It does not stop us from defining polynomials. Let $A \subset \mathbb{N}^n$ be a finite set of integer vectors with non-negative entries. The function

$$a \sum_{j=(j_1,\ldots,j_n) \in A} a_j x_1^{j_1} \cdots x_n^{j_n} = \max_{j \in A} \{a_j + jx\} : \mathbb{T}^n \to \mathbb{T}$$

of $x = (x_1,\ldots,x_n) \in \mathbb{T}^n$ is a *tropical polynomial* in $n$ variables. Here, $a_j \in \mathbb{T}$ and $jx$ denotes the standard scalar product $\sum_j jx_i$. If we also allow negative exponents, i.e. $A \subset \mathbb{Z}^n$, we get a *tropical Laurent polynomial*. If we even drop the finiteness condition of $A$, we obtain a *tropical Laurent series*.

Tropical polynomials are globally well-defined continuous (with respect to the Euclidean topology) functions $F : \mathbb{T}^n \to \mathbb{T}$ with $F(\infty,\ldots,\infty) = a_0$. Laurent polynomials are always defined on $\mathbb{R}^n \subset \mathbb{T}^n$, but not necessarily on

$$\partial \mathbb{T}^n = \mathbb{T}^n \setminus \mathbb{R}^n = \{(x_1,\ldots,x_n) \in \mathbb{T}^n \mid x_j = -\infty \text{ for some } j\}.$$

Indeed, if $x \in \mathbb{T}^n$ is such that $x_1 = -\infty$, then all monomials “$a_j x^j$” in a Laurent polynomial $F$ have to satisfy $j_i \geq 0$ whenever $a_j \neq -\infty$ as otherwise the value of such monomial is $+\infty \notin \mathbb{T}$.

An infinite tropical Laurent series does not have to be well-defined even on $\mathbb{R}^n$. However, it is easy to check that the domain of a tropical Laurent series is convex (though not necessarily open or closed).

### 1.3 Tropical monomials and integer affine geometry

Each tropical monomial “$a_j x^j = a_j + jx$ with $a_j \neq -\infty$ is an affine function $\lambda : \mathbb{R}^n \to \mathbb{R}$, $x = (x_1,\ldots,x_n)$, $j = (j_1,\ldots,j_n)$. We may extend it to a part of $\partial \mathbb{T}^n$ by continuity. Namely, we define a domain $D$ with

$$\mathbb{R}^n \subseteq D \subseteq \mathbb{T}^n$$

as $\mathbb{T}^n$ minus all points $x = (x_1,\ldots,x_n)$ with $x_i = -\infty$ whenever $j_i < 0$. Clearly, $\lambda$ gets naturally extended to a continuous function $\bar{\lambda} : D \to \mathbb{T}$ and $D$ is the maximal set in $\mathbb{T}^n$ where such a continuous extension exists.

The *linear part* $jx$ of the monomial $\lambda$ is defined by an integer vector $j \in \mathbb{Z}^n$ of the dual vector space $(\mathbb{R}^n)^*$ (as we have a classical pairing
Thus we may invert our construction and define tropical monomials (and hereby also polynomials, series, etc.) starting from an arbitrary real affine space $A$ of dimension $n$ as long as we fix a discrete rank $n$ integer lattice $N \in V$, where $V = T_x A$ is the tangent vector space at $x \in A$ (which clearly does not depend on the choice of $x$).

Indeed, once we fixed $N \subseteq V$ we have a canonical dual lattice $M \subseteq V^*$ which consists of linear functionals $V \to \mathbb{R}$ assuming integer values on $N$. These functionals are tropical monomials. Thus to turn $A \approx \mathbb{R}^n$ into a space where tropical polynomials are defined all we need is to present it in the form of a real affine space whose tangent vector space is defined over the integers. We call such $A$ the basic tropical space of dimension $n$ (or just the tropical $n$-space when the context is confusion-free) and the relevant lattice $N_A$ the tropical lattice. For any two points $x, y \in A$ we have $x - y \in V_A$ and it may happen that $x - y \in N_A$ or $x - y \notin N_A$.

**Definition 1.3.1**

A map $\Phi : A \to B$ between basic tropical space $A, B$ is called integer affine if for any $x, y \in A$ with $x - y \in N_B$ we have $(\Phi(x) - \Phi(y)) \in N_B$. The map $\Phi$ is called an integer affine transformation of $A$ if $A = B$, it is invertible and the inverse function is an integer affine map, too.

We define the differential

$$d\Phi : V_A = T_x A \to V_B = T_{\Phi(x)} B$$

as the conventional differential. Note that $d\Phi$ takes $N_A$ to $N_B$. Clearly, an integer affine map $\Phi : A \to A$ is an integer affine transformation if and only if $d\Phi|_{N_A}$ is a bijection to $N_B$.

If we only want to define functions that locally coincide with tropical polynomials on $A$ we need the corresponding notion of affine structure on a more general underlying space.

**Definition 1.3.2**

Let $M$ be a smooth $n$-dimensional manifold. An integer affine structure on $M$ is given by an open cover $U_\alpha$ and charts $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ such that for each pair $\alpha, \beta$ the overlapping map $\phi_\beta \circ \phi_\alpha^{-1}$ is locally the relevant restriction of an integer affine-linear transformation $\Phi_{\beta\alpha} : \mathbb{R}^n \to \mathbb{R}^n$.

We identify two integer affine structures if the union of their covers and charts gives an integer affine structure as well.

**Example 1.3.3**

Let $A \approx \mathbb{R}^n$ be the basic tropical $n$-space and $\Lambda \subseteq V = T_x A$ be an arbi-
trary discrete rank \( n \) lattice (unrelated to \( N \)). We declare points \( x, y \in A \) equivalent if \( x - y \in \Lambda \). Let \( T \approx (S^1)^n \) be the quotient space of \( A \) by the equivalence. The quotient map \( A \to T \) can be inverted locally and we can use such local inverse maps to define an integer affine structure on \( T \).

We will see that the tropical structure on general tropical manifolds can be thought of as an extension of integer affine structures to polyhedral complexes. Let us preview this (without detailed explanation) by revisiting the example of a smooth cubic curve in the plane.

Now we do everything tropically borrowing the notions from the main part of the book. We will see that the tropical projective plane \( TP^2 \) can be viewed as a compactification of the basic tropical 2-space. Combinatorially, this compactification is a triangle whose interior is identified with the whole \( \mathbb{R}^2 \).

A curve in \( TP^2 \) might be represented as a picture in \( \mathbb{R}^2 \) which has to be compactified to get the full picture. Or we can consider already compact pictures in the triangle whose interior is equipped with an distorted integer affine structure inherited from an homeomorphism with \( \mathbb{R}^2 \).

As we shall see in this book, Figure 1.8 depicts a smooth cubic curve in \( \mathbb{R}^2 \) before the compactification. It is the hypersurfaces in \( \mathbb{R}^2 \) defined by a tropical cubic polynomial in two variables.

In the compactified view, the edges are no longer straight. Figure 1.9 provides a sketch (where the curvature of the edges is perhaps still not visible).

We may note that our cubic intersects each side of the triangle three times. Similarly to the situation over the real numbers, there is more than
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Figure 1.8: A tropical cubic curve in $\mathbb{R}^2$ (before compactification).

Figure 1.9: A tropical cubic curve embedded in $T\mathbb{P}^2$.

one “type” of a smooth tropical curve of given degree, but only finitely many.

To preview the relation between classical and tropical varieties we may look at Figure 1.10 which depicts the collapse of the complex elliptic curve from Figure 1.1 to the tropical cubic curve in $R\mathbb{P}^2$. The left-hand side of Figure 1.10 is topologically a 2-torus $S^1 \times S^1$ minus nine points. These
1.4 Forgetting the phase in a tropical number

To relate complex and tropical numbers consider a complex number

\[ z = ae^{i\alpha}, \quad a, \alpha \in \mathbb{R}, a \geq 0. \]

The exponent \( \alpha \) is called the argument, or the phase of \( z \). It gets ignored when passing to tropical numbers. It is convenient to measure the remaining parameter, the norm \( a = |z| \), on logarithmic scale. We set

\[ x = \log_t a = \log_t |z|, \]

where the base \( t \) of the logarithm is a large real number.

The map \( z \mapsto \log_t |z| \) is a surjection \( \mathbb{C} \rightarrow \mathbb{T} \) which we denote by \( \text{Log}_t \) (of course, we set \( \log_t 0 = -\infty \)). This map “forgets” the phase of \( z \). Furthermore, it rescales the norm. We may use \( \text{Log}_t \) to induce arithmetic operations on \( \mathbb{T} \) from \( \mathbb{C} \). However, since \( \text{Log}_t \) is not injective the resulting operations might be multivalued.

Indeed, addition on \( \mathbb{C} \) induces the following operation on \( \mathbb{T} \).

\[ x \vee_t y = \text{Log}_t(\text{Log}_t^{-1} x + \text{Log}_t^{-1} y) \quad (1.3) \]
We should stress here that $\Log^{-1}_t x + \Log^{-1}_t y$ denotes the set \( \{ z + w : \Log_t(z) = x, \Log_t(w) = y \} \) and we define \( x \vee_t y \) to be the image of this set under \( \Log_t \). As \( \Log^{-1}_t x \) is a circle \( t^x S^1 \), we get
\[
x \vee_t y = [\log_t |t^x - t^y|, \log_t(t^x + t^y)] \subset \mathbb{T},
\]
i.e. \( x \vee_t y \) is an interval in \( \mathbb{T} \) instead of a specific number. To get a unique number we define
\[
x \oplus_t y = \max \{ x \vee_t y \} = \log_t(t^x + t^y).
\] (1.4)
This expression has a well-defined limit when \( t \to +\infty \). We get
\[
"x + y" = \lim_{t \to +\infty} (x \oplus_t y) = \lim_{t \to +\infty} \log_t(t^x + t^y) = \max \{ x, y \},
\]
thus recovering tropical addition as a certain limit of addition of complex numbers with the help of rescaling by \( \log_t \) once we forgot the phase.

**Remark 1.4.1**
Note also that if \( x \neq y \), then we have
\[
"x + y" = \lim_{t \to +\infty} \min \{ x \vee_t y \} = \lim_{t \to +\infty} \log_t |t^x - t^y| = \max \{ x, y \}.
\]
We see that in this case \( "x + y" \) is the single limit of the multi-valued operation \( x \vee_t y \) and therefore independent of the chosen phases for the preimages of \( x \) and \( y \). If \( x = y \) then \( \min \{ x \vee_t y \} = -\infty \), so it is independent of \( t \). Hence, in the realm of multi-valued operations, the limit of \( \vee_t \) for \( t \to \infty \) is the **multi-valued tropical addition**
\[
a \vee b = \begin{cases} 
\max(a, b) & \text{if } a \neq b, \\
[-\infty, a] & \text{if } a = b.
\end{cases}
\]
The operations \( \vee_t \) and \( \vee \) are examples of hypergroup additions, see [V5] for the relevant treatment in the context of tropical calculus (more explicitly, the definitions of \( \vee_t \) and \( \vee \) are given in sections 5.4 and 5.3). Viro suggests an alternative viewpoint on tropical calculus using the multi-valued addition \( \vee \) instead of the more conventional single-valued version given by max.

The multiplication in \( \mathbb{C} \) induces a well-defined single-valued operation
\[
"xy" = \Log_t(\Log^{-1}_t x \cdot \Log^{-1}_t y) = \log_t(t^x t^y) = x + y,
\]
as the norm of the product of two complex numbers is independent of their phases.
1.5 Amoebas of affine algebraic varieties and their limits

The map $\text{Log}_t : \mathbb{C} \to \mathbb{T}$ can be applied coordinate-wise to generalize to the case of several variables.

$$\text{Log}_t : \mathbb{C}^n \to \mathbb{T}^n,$$

$$(z_1, \ldots, z_n) \mapsto (\log |z_1|, \ldots, \log |z_n|).$$

Clearly, $\text{Log}_t((\mathbb{C}^*)^n) = \mathbb{R}^n$. Images of algebraic subvarieties $V \subseteq (\mathbb{C}^*)^n$ under $\text{Log}_t$ are called amoebas. They were introduced in [GKZ, chapter 6]. The most well-known example is the amoeba

$$A_t = \text{Log}_t(\{(z, w) \in (\mathbb{C}^*)^2 \mid z + w + 1 = 0\}),$$

(1.5)

depicted in Figure 1.11.

![Figure 1.11: The amoeba $A_t$ and three rays inside.](image)

It is the closed set in $\mathbb{R}^2$ bordered by three arcs

$$tx + ty = 1,$$

$$ty + 1 = tx,$$

$$1 + tx = ty.$$

We add to the picture the three “asymptotics of the tentacles”: the negative part of the $x$-axis, the negative part of the $y$-axis and the diagonal ray $\{(x, x) \mid x \geq 0\}$. The union of these three rays, with the origin as vertex,
is denoted by \( \Gamma \). The tripod \( \Gamma \) separates the amoeba \( A_t \) into three equal parts. To see that these parts are equal we note that the whole picture — without the boundary points — is symmetric with respect to the linear action on \( \mathbb{R}^2 \) by the symmetric group \( S_3 \) generated by \( \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \) and this action interchanges the three parts of the amoeba. Also, the action is volume-preserving as our \( S_3 \) is a subgroup of \( \text{SL}_2(\mathbb{Z}) \). Thus the area \( A_t \) (which clearly depends on the parameter \( t > 1 \)) of each part is the same. One can show that

\[
A_t = \frac{\pi^2}{6(\log t)^2}
\]

(in the following, the logarithm without index always refers to the natural base logarithm \( \log := \log_e \)) as we see in the next remark.

**Remark 1.5.1**

It was observed by Passare (cf. [P1]) that two different ways of computing the area \( A_e \) produces yet another proof of Euler's formula \( \zeta(2) = \pi^2/6 \).

Indeed, as \( \log_t = \log_{\log(t)} \) coordinatewise it suffices to establish (1.6) for \( t = e \).

Let us compute the area of the part of \( A_e \) in the negative quadrant. It is given by

\[
\int_{-\infty}^{0} \int_{-\log(1-e^x)}^{0} dx dy = \int_{-\infty}^{0} -\log(1-e^x) dx \\
= \int_{-\infty}^{0} \sum_{i=1}^{\infty} \frac{e^{nx}}{n} dx = \sum_{i=1}^{\infty} \int_{-\infty}^{0} \frac{e^{nx}}{n} dx = \sum_{i=1}^{\infty} \frac{1}{n^2} = \zeta(2).
\]

On the other hand we will see that the area of \( A_e \) is half of the area of the corresponding coamoeba (to be defined) which takes up a quarter of the argument torus \( (\mathbb{R}/2\pi\mathbb{Z})^2 \), i.e. area\( (A_e) = \pi^2/2 \). But as \( \zeta(2) \) represents a third of \( A_e \), it follows \( \zeta(2) = \pi^2/6 \).

If we consider a another line \( \{ az + bw + c = 0 \} \) with \( a, b, c \) non-zero, then the amoeba \( \text{Log}_t \{ az + bw + c = 0 \} \) can be obtained from \( A_t \) by the translation

\[
x \mapsto x + \text{Log}_t c - \text{Log}_t a, \\
y \mapsto y + \text{Log}_t c - \text{Log}_t b,
\]

as \( \{ az + bw + c = 0 \} \) can be obtained from \( \{ z + w + 1 = 0 \} \) by the rescaling

\[
z \mapsto \frac{c}{a} z, \\
w \mapsto \frac{c}{b} w.
\]
Note that if \( V \subseteq C^n \) is fixed the only effect of varying \( t \) is the scaling of the target \( R^n \) with the coefficient \( \frac{1}{\log t} \), i.e.

\[
\text{Log}_t(V) = \frac{1}{\log t} \text{Log}(V).
\]

In particular, the limit of \( \text{Log}_t\{az + bw + c = 0\} \) when \( t \to +\infty \) does not depend on the coefficients (as long as they are non-zeroes) and is equal to \( \Gamma \), the union of the three rays inside \( A_t \), see Figure 1.11.

The situation changes if we vary \( V \) simultaneously with varying the base \( t \), i.e. if we consider a family of complex varieties \( V_t \subseteq (C^*)^n \) with a real parameter \( t \).

For example, let us take a family of lines \( V_t = \{a(t)z + b(t)w + c(t) = 0\} \) in \( (C^*)^2 \), where \( a(t) = \alpha t^A + o(t^A) \), \( b(t) = \beta t^B + o(t^B) \), \( c(t) = \gamma t^C + o(t^C) \), \( \alpha, \beta, \gamma \in C^* \), \( A, B, C \in R \) are functions for large positive values of \( t \) with highest order terms as described. Then the limit \( L = \text{Log}_t(V_t) \) for \( t \to +\infty \) (which we may consider in the topology induced by the Hausdorff metric on neighbourhoods of compact sets in \( R^n \)) depends only on \( A, B, C \) and is equal to the translation of \( \Gamma \) by

\[
x \mapsto x + C - A, \\
y \mapsto y + C - B.
\]

Thus the numbers \( A, B, C \) define asymptotics of the amoebas \( \text{Log}_t(V_t) \). Within the paradigm of tropical geometry we regard the limits \( L \) as geometric objects of their own, the so called \textit{tropical lines}, and \( A, B, C \in R \subset T \) as \textit{tropical coefficients} defining this line. The tropical lines we can get in this way only differ by translations in \( R^2 \).

\textbf{Remark 1.5.2}

Let us consider a special case when \( b(t) = -1, a(t) = t^A, c(t) = t^C \). Then the curve \( V_t \) is a graph of the function \( w(z) = t^A z + t^C \) while its amoeba can be written as

\[
\text{Log}_t(V_t) = \{y = "Ax" \cap tC\},
\]

i.e. it can be thought of as the graph of the multivalued addition (1.3) of "\( Ax" = A + x \in T \) and \( C \in R \).

We easily get some familiar properties of lines for these new piecewise linear objects: Two \textit{generic} lines intersect in a single point. And for two \textit{generic} points in \( R^2 \) there is a unique line connecting them. We get more tropical lines in \( R^2 \) if we allow the coefficients \( A, B, C \) to be \(-\infty = 0_T\) as
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Figure 1.12: Tropical lines intersect in one point. They can be used to connect any pair of points in $T^2$.

well: conventional horizontal lines, vertical lines and diagonal lines parallel to the vector $(1, 1)$, see Figure 2.22.

Remark 1.5.3
For an affine variety $V \subset \mathbb{C}^n$ we call $\text{Log}_t(V) \subset T^n$ its affine amoeba. Note that if $V \subset \mathbb{C}^n$ is irreducible and is not contained in a coordinate hyperplane $z_j = 0, j = 1, \ldots, n$ then $V$ is the topological closure of $V \cap (\mathbb{C}^*)^n$ in $\mathbb{C}^n$. Furthermore, the amoeba $\text{Log}_t(V)$ is the topological closure of $\text{Log}_t(V \cap (\mathbb{C}^*)^n)$ in $T^n$.

E.g. the affine amoeba $\bar{A}_t \subset T^2$ of the line $\{z + w + 1 = 0\} \subset \mathbb{C}^2$ can be obtained from $A_t$ (see (1.7)) by adding two points

$$\bar{A}_t = A_t \cup \{(-\infty, 0)\} \cup \{(0, -\infty)\}$$

at the far left and far lower apex of $A_t$ at Figure 1.11.

As another example let us consider the graph

$$G_t = \{(z, w) \in \mathbb{C}^2 \mid w = f_t(z)\}$$

of a polynomial

$$f_t(z) = \sum_{j=0}^d a_j(t)z^j$$

in $z$ whose coefficients $a_j(t)$ are $\mathbb{C}$-valued functions in the positive real variable $t$ (defined for sufficiently large values of $t$) such that there exist $\alpha_j \in \mathbb{C}^x$ and $A_j \in \mathbb{R}$

$$\lim_{t \to +\infty} \frac{a_j(t) - \alpha_j t^{A_j}}{t^{A_j}} = 0,$$

in other words, $a_j(t) = \alpha_j t^{A_j} + o(t^{A_j}), t \to +\infty$. 

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1.6 Amoebas of affine algebraic varieties and their limits

We may note that the amoeba $\text{Log}_t(G_t)$ is contained in the graph of the multivalued polynomial

$$F_t^{\text{subtrop}} = A_0 v_t^{A_1 x} v_1 \ldots v_{A_d x^d}$$

if $a_j(t) = \alpha_j t^{A_j}$. The case of $d = 2$, i.e. when $G_t$ is a parabola is depicted on Figure 1.13. Let us compare this multivalued $t$-subtropical polynomial against the corresponding tropical polynomial $F^{\text{trop}}(x) = \sum_{j=0}^d A_j x^j$.

By Remark 1.4.1 the set $F_t^{\text{subtrop}}(x) \subset T$ has a one-point limit $\{F^{\text{trop}}(x)\}$ when $t \to +\infty$ if $x$ is such that the collection $\{A_d x^d \}$ has a unique maximum. If this maximum is not unique then this limit is $[-\infty, F^{\text{trop}}(x)]$. Clearly the same holds if we slightly perturb the coefficients $a_j(t)$ from $\alpha_j t^{A_j}$ to $\alpha_j t^{A_j} + o(t^{A_j})$. The limiting graph of $\text{Log}_t(G_t)$ is shown on Figure 1.14.

Figure 1.13: Amoeba of a parabola.

Figure 1.14: A tropical parabola as the limit of amoebas when $t \to +\infty$. 
1.6 Patchworking and tropical geometry

We saw that if we change the coefficients \(a, b, c\) in \(az + bw + c = 0\), the amoeba only gets translated. If we choose \(a, b, c\) to be real, then the boundary of the amoeba \(A_t\) will be the image of the real locus of the line \(L\). In formulas,

\[
\partial A_t = \text{Log}_t(\mathbb{R}L)
\]

where \(\mathbb{R}L = \{(z, w) \in (\mathbb{R}^*)^2 | az + bw + c = 0\} = L \cap (\mathbb{R}^*)^2\).

Figure 1.15: \(\text{Log}_t\) maps the real line \(z + w + 1 = 0\) to the three boundary arcs of the amoeba.

If we assume \(a, b, c \neq 0\), then the three arcs in the boundary of \(A_t\) correspond to the three components of \(\mathbb{R}L \cap (\mathbb{R}^*)^2\) which in turn correspond to three out of four quadrants of \(\mathbb{R}^2\). Which arc corresponds to which quadrant is determined by signs of the coefficients \(a, b, c\). Furthermore, even if \(a, b, c\) are functions of a real positive parameter \(t \to +\infty\) of the form \(a(t) = \alpha t^{-A} + o(t^{-A})\) and the function is real we may still speak of the sign of \(a(t)\) as the sign of the leading coefficient \(\alpha \in \mathbb{R}\). Consider

\[
\Gamma = \lim_{t \to +\infty} \text{Log}_t(L_t)
\]

where \(L_t\) is given by \(a(t)z + b(t)w + c(t) = 0\) with the real coefficients \(a(t), b(t), c(t)\). We saw already that \(\Gamma\) is a graph in \(\mathbb{R}^2\) with one vertex. Now let us focus on \(\mathbb{R}L\) again. First, we split \((\mathbb{R}^*)^2\) into its four quadrants \(\mathbb{R}^2_{>0} \times \{+, -\}^2\) and compute the limits of \(\mathbb{R}L\) for each quadrant separately (see Figure 1.16). As described before, these components correspond to the three boundary arcs of \(A_t\), so the limit is easy to compute: For each quadrant, we just get a part of the tropical line consisting of two rays. In our example, the signs of \(a(t), b(t), c(t)\) are all positive, i.e. \(\alpha, \beta, \gamma > 0\).

We can summarize these pictures by drawing the whole limit as before, but now labeling the edges of \(\Gamma\) with the signs of the real quadrants whose part of \(\mathbb{R}L\) converge to the edge (see Figure 1.17).

Let us now construct \(\mathbb{R}P^2\) (as a topological space) by gluing together
1.6 Patchworking and tropical geometry

Figure 1.16: The logarithmic limit for each quadrant separately.

Figure 1.17: A tropical line whose edges are labelled with signs referring to the 4 quadrants.

four copies of $\text{T}^2$ along the sides at infinity as indicated in Figure 1.18. The four boundary sides of the picture are also glued together by identifying antipodal points. The inside of each triangle is homeomorphic to $\mathbb{R}^2$.

Now we can just redraw the quadrant pictures from above in this representation of $\mathbb{RP}^2$, see Figure 1.19. Equivalently, we could first draw a copy of $\Gamma$ in each of the quadrants and then throw away those edges which are not labelled with the corresponding sign in the previous picture.

As indicated, we denote the resulting set in $\mathbb{RP}^2$ by $R\Gamma$. Note that $R\Gamma$ is closed in $\mathbb{RP}^2$ and that topologically it is an embedded circle $S^1 = \mathbb{RP}^1 \subset \mathbb{RP}^2$. It can be considered as the limit of $RL_t$ under a certain reparameterization (the so-called quasitropical limit). Furthermore, $RL_t$ is isotopic to $R\Gamma$. A similar construction works not only for lines but for

---

In our case of a line, this is true for any $t$ (as long as the functions $a(t), b(t), c(t)$ are defined at $t$ and do not vanish simultaneously). In more general cases we get similar isotopies for large values of $t$ (whenever $R\Gamma$ is smooth as a tropical variety).
smooth algebraic curves in $\mathbb{RP}^2$. It was introduced by Viro in 1979 and is now known as Viro patchworking (cf. [V2]; for a list of references see [V4]). It is the most powerful construction tool currently known in Real Algebraic Geometry. One of the major breakthroughs obtained with the help of this technique was the construction by Itenberg of a counterexample to the so-called Ragsdale conjecture standing open since 1906 (see [R] for the conjecture and [IV] for the counterexample). Let us review the relevant background of this conjecture.

There are only two homology types of circles embedded in $\mathbb{RP}^2$. The line $RL$ is an example of the non-trivial class (as the complement $\mathbb{RP}^2 \setminus RL$ is still connected). A circle which bounds a disc in $\mathbb{RP}^2$ is zero-homologous and is called an oval. If we consider a smooth real algebraic curve $RC \subset \mathbb{RP}^2$, then its connected components are embedded circles. Note that non-trivially embedded circles must intersect by topological reasons and that their intersection points must be singular points of $RC$. Thus if the
degree of $RC$ is even then all its components are ovals; if it is odd, then all but one component are ovals.

Each oval separates its complement into two connected components, the \textit{interior} (homeomorphic to a disc) and the \textit{exterior} (homeomorphic to a Möbius band). An oval is called even if it sits in the interior of an even number of other ovals (and odd otherwise). The Ragsdale conjecture stated that the number of even ovals, denoted by $p$, of a smooth real curve of even degree $2k$ is bounded by

$$p \leq \frac{3k(k-1)}{2} + 1.$$  

It was noted by Viro [V3] that this inequality comes as a special case of the more general conjecture

$$b_1(RX) \leq h^{1,1}(X),$$  

(1.8)

where $X$ is a smooth complex algebraic surface defined over $\mathbb{R}$ and $RX$ is its real locus (as usual, $b_1$ stands for the first Betti number and $h^{1,1}$ stands for the $(1,1)$-Hodge number). Furthermore, the inequality (1.8) implies the bound

$$n \leq \frac{3k(k-1)}{2} + 1.$$  

for the number $n$ of odd ovals that is weaker than the historical Ragsdale conjecture [R] by 1.

In one of the first spectacular applications of the patchworking technique in [V3] Viro disproved the original Ragsdale conjecture for the number of odd ovals by that very one, leaving the more general conjecture (1.8) still plausible. Then the final counterexample was given by Itenberg [I] in yet another striking application of patchworking.

Figure 1.20 shows a very particular tropical curve of degree 10. As in the example of the line, this curve can be obtained as the limit of a family of real algebraic curves, given by a family of equations. After a choice of signs for the (leading terms of) the coefficients of these equations, we can again draw the quasitropical limit in the four quadrants (see Figure 1.21). The result corresponds to a smooth algebraic curve of degree 10 in $\mathbb{R}P^2$ with $p = 32$ even ovals (don’t forget the big one which crosses the line at infinity several times), which exceeds the Ragsdale bound $\frac{35.4}{2} + 1 = 31$.

Tropical geometry can be viewed as a further development and generalization of patchworking. The first volume of this book takes an intrinsic point of view — Tropical Geometry per se. In some applications (cf.
Gromov-Witten theory) in can completely replace Complex Geometry. But in the second volume we recover the phase by introducing phase-tropical varieties which sometimes come as quasitropical limits of real 1-parametric families of complex varieties.

Figure 1.20: A tropical curve of degree 10.
Figure 1.21: The quasitropical limit of the counterexample to the Ragsdale conjecture from [1].
2 The space $\mathbb{R}^n$ as a tropical ambient space

2.1 Geometric structures on $\mathbb{R}^n$

The “$n$-dimensional Euclidean space” $\mathbb{R}^n$ is probably the most well-known topological space. It serves as an underlying space for many geometries, including Euclidean geometry itself. Introducing different geometric structures enhances the space $\mathbb{R}^n$ with different geometric objects. Such objects may look quite different for different geometric structures.

Let us illustrate this by the example of Euclidean geometry. In the case of $\mathbb{R}^2$, Euclid postulated lines and circles as basic geometric objects which satisfy certain properties. For example, there exists a line through any two points in $\mathbb{R}^2$ and there exists a circle centered at any point with any choice of a positive number for a radius.

In modern terms we say that the Euclidean structure is a choice of a complete and flat Riemannian metric on $\mathbb{R}^n$. Once we have a metric we can define lines as geodesics for this metric or circles as collections of points equidistant from a given point in a plane. Different metrics have different types of lines and circles. Most notoriously, lines in hyperbolic space (also called Lobachevskii’s space) do not conform to the fifth axiom of Euclid as it was discovered already in the XIXth century.

There are also other types of geometric structures that allow to define lines and circles. Namely, lines and circles can also be viewed as algebro-geometric objects, i.e. objects defined by polynomial equations. A line is defined by a system of $n-1$ linear equations (in $n$ coordinates of our ambient space $\mathbb{R}^n$) and a circle as well as any other conic section is defined by a system of $n-2$ linear equations and one equation of degree 2. As usual, we require our linear equations to be linearly independent.

Note that e.g. the lines defined by those systems of linear equations coincide with the lines defined by means of the Euclidean metric. At the same time one can easily replace $\mathbb{R}^n$ with $\mathbb{C}^n$ and write polynomial equations there. In $\mathbb{C}^n$ we may consider polynomial equations with complex coefficients and such considerations again produce lines, conic sections, etc. in $\mathbb{C}^n$ which are somewhat different (in particularly, topologically) from lines, conic sections, etc. in $\mathbb{R}^n$.

However, if we compare e.g. lines in $\mathbb{C}^2$ against lines in $\mathbb{R}^2$ we see that
they share many properties. In particular, two lines (in \( \mathbb{R}^2 \) or \( \mathbb{C}^2 \)) intersect in a single point unless they are parallel. For any two distinct points (in \( \mathbb{R}^2 \) or \( \mathbb{C}^2 \)) there is a unique line passing through them.

To be able to write down polynomial equations in the spaces \( \mathbb{R}^n \) or \( \mathbb{C}^n \) we need a structure encoding polynomial functions. In the complex case, by the Chow theorem we can alternatively just consider holomorphic functions once we projectivize \( \mathbb{C}^n \). Let us consider the complex projective space \( \mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim \), where \((z_0, \ldots, z_n) \sim (w_0, \ldots, w_n)\) if there exists \( \lambda \in \mathbb{C}, \lambda \neq 0 \) such that \( z_j = \lambda w_j, j = 0, \ldots, n \). The points of \( \mathbb{CP}^n \) (i.e. the equivalence classes under \( \sim \)) are denoted by \((z_0 : \cdots : z_n)\). The embedding \( \mathbb{C}^n \subset \mathbb{CP}^n \) is given by \((z_1, \ldots, z_n) \mapsto (1 : z_1 : \cdots : z_n)\).

Indeed, by the Chow theorem for a closed set \( V \subset \mathbb{CP}^n \) the following conditions are equivalent.

1. For any \( z \in V \) there exists an open set \( U \subset \mathbb{CP}^n, U \ni z \) and a polynomial function \( f : U \to \mathbb{C} \) such that \( f^{-1}(0) = U \cap V \).
2. For any \( z \in V \) there exists an open set \( U \subset \mathbb{CP}^n, U \ni z \) and a holomorphic function \( f : U \to \mathbb{C} \) such that \( f^{-1}(0) = U \cap V \).

To define holomorphic functions on \( \mathbb{R}^{2n} \approx \mathbb{C}^n \) or, more generally, on a (real) \( 2n \)-dimensional smooth manifold \( M \) we only need the so-called almost complex structure on \( M \), that is an automorphism \( J : TM \to TM \) of the tangent bundle \( TM \) (viewed as a real vector bundle) such that \( J^2(v) = -v \) for every \( x \in M \) and \( v \in T_xM \). Note that the determinant of \( J^2 = -\text{Id} \) is necessarily positive and thus the dimension of each \( T_xM \) has to be even (so that we denote it with \( 2n \)).

Recall that if \( U \subset M \) is an open set then we say that a function \( f : U \to \mathbb{C} \) is holomorphic if it is smooth (so that the differential \( df_x \) is an \( \mathbb{R} \)-linear map \( T_xM \to \mathbb{C} \)) and for any \( x \in U \) and \( v \in T_xM \) we have

\[
    df_x(J(v)) = i \cdot df_x(v)
\]

(so that \( df_x \) is actually a \( \mathbb{C} \)-linear map once we turn \( T_xM \) into a complex vector space with the help of \( J \)).

If \( n > 1 \) it might happen that any holomorphic function on \( U \) is constant. Such an almost complex structure is called non-integrable. However, sometimes for any \( x \in M \) and \( v \in T_xM \) there exists an open set \( U \subset M, U \ni x \), and a holomorphic function \( f : U \to \mathbb{C} \) such that \( df_x(v) \neq 0 \). Such almost
complex structures are called integrable complex structures or just complex structures with the word “almost” dropped. Thus we see that in this case a sheaf of holomorphic functions on $M$ can be alternatively given by an automorphism $J$ of $TM$.

In the previous example, an automorphism $J$ of $TM$ was used to define the sheaf of holomorphic functions on $M$. The situation is similar for tropical structures on $\mathbb{R}^n$ or, in fact, any other smooth manifold $M$ (of arbitrary real dimension). Namely, to make $\mathbb{R}^n$ (or $M$) “tropical” we also start by equipping the tangent bundle of $M$ with some extra structure. In a second step, this extra structure is used to define the sheaf of tropical regular functions. The extra structure on the tangent bundle is given by specifying which of the tangent vectors are integer.

**Definition 2.1.1**

Let $M$ be a smooth $n$-dimensional manifold (from the point of view of differential topology). An almost tropical structure on $M$ is a choice of an $n$-dimensional integer lattice $T^Z_x M \subset T_x M$ that depends smoothly on $x \in M$. Formally speaking this means the following.

- $T^Z_x M$ is a discrete subgroup of $T_x M$ isomorphic to $\mathbb{Z}^n$.
- There exists an open neighborhood $U \ni x$ and $n$ smooth vector fields $u_j, j = 1, \ldots, n$, in $U$ such that for any $y \in U$ the lattice $T^Z_y M$ is generated by the vectors $u_j(y), j = 1, \ldots, n$.

A function $f : U \rightarrow \mathbb{R}$ is called monomial if it is smooth and for every $u \in T^Z_y M \subset T_y M$, $y \in U$, we have $df_y(u) \in \mathbb{Z}$. A function $f : U \rightarrow \mathbb{R}$ is called regular if it is obtained as a maximum of a finite collection of monomial functions on $U$.

**Example 2.1.2**

For now, the only example we will consider is the standard tropical structure on $\mathbb{R}^n$. It is given, after canonically identifying $T_x \mathbb{R}^n \cong \mathbb{R}^n$ for all $x \in \mathbb{R}^n$, by the lattice of (standard) integer vectors $\mathbb{Z}^n \subset \mathbb{R}^n$. In this case, a monomial is just the sum of an integer linear function and a real constant.

If $n > 1$, then it is easy to construct non-integrable almost tropical structures on $\mathbb{R}^n$, i.e. structures which are not locally isomorphic to the

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1 As we shall see in this book, in general tropical structures appear not only on smooth manifolds but also on certain polyhedral complexes that are non-manifolds topologically.
standard tropical structure of $\mathbb{R}^n$. For example, consider two vector fields on $\mathbb{R}^2$ which span the tangent space at any point, but have non-zero Lie bracket. Over $\mathbb{Z}$, they generate lattices which form a non-integrable tropical structure on $\mathbb{R}^2$.

Vice versa, if all Lie brackets of the vector fields $u_j$ on $U \subset M$ from Definition 2.1.1 are zero then the almost tropical structure admits $n$ monomial functions with linearly independent differentials. These $n$ monomial functions establish an isomorphism of $U$ with an open set in $\mathbb{R}^n$ enhanced with the standard tropical structure. Thus we get the following statement.

**Proposition 2.1.3**

*An almost tropical structure on a smooth manifold $M$ is locally isomorphic to the standard tropical structure on $\mathbb{R}^n$ if and only if all the pairwise Lie brackets of vector fields $u_j$, $j = 1, \ldots, n$, from Definition 2.1.1 vanish.*

Such almost tropical structures are called *integrable* or just *tropical structures* on smooth manifolds.

## 2.2 Polyhedral geometry

Let us fix a vector space $V$ with lattice $\Lambda$ (such that $V = \mathbb{R}\Lambda = \Lambda \otimes \mathbb{R}$). The dual vector space $V^\vee$ contains the dual lattice $\Lambda^\vee = \text{Hom}(\Lambda, \mathbb{Z})$ (where we obviously identify $\lambda \in \Lambda^\vee$ with the function $V \to \mathbb{R}$ given by linear extension). The functions $\lambda \in \Lambda^\vee$ are called *integer linear forms* or *integer linear functions* on $V$. They are distinguished from all linear forms on $V$ by the property that they send lattice vectors to integers. A function $\kappa : V \to \mathbb{R}$ is called *integer affine-linear* (or a *tropical monomial*) if it is the sum of an integer linear form and a real constant.

For each integer affine function $\kappa$ on $V$, we define the *rational halfspace* of $\kappa$ to be

$$H_\kappa := \{x \in V : \kappa(x) \geq 0\}.$$

A subset $P \subseteq V$ is called a *rational polyhedron* of $X$ if it is the intersection of finitely many rational halfspaces. Let us emphasize that the adjective ‘rational’ refers to the fact that the bounding inequalities have linear parts in $\Lambda^\vee$ (or, equivalently, $\Lambda^\vee \otimes \mathbb{Q}$). As we will not work with other halfspaces or polyhedra, and also to avoid conflicts with other usage of the attribute ‘rational’, we mostly drop it in the following. Figure 2.1 depicts an unbounded and a compact polyhedron.
Chapter 2: The space $\mathbb{R}^n$ as a tropical ambient space

Figure 2.1: An unbounded and a bounded polyhedron

The Minkowski sum of two polyhedra

$$P + Q = \{ x + y : x \in P, y \in Q \}$$

is a polyhedron again. A face $F$ of a polyhedron $P$ is given as $P \cap H_{-\kappa}$, where $\kappa$ is an integer affine-linear function such that $P \subset H_\kappa$. If $P \cap H_{-\kappa} \neq \emptyset$, $H_\kappa$ is called a supporting halfspace. The (relative) boundary $\partial P$ of $P$ is the union of all proper faces. The complement $P^0 = P \setminus \partial P$ is called the (relative) interior of $P$. We denote by $\mathbb{R} P$ the real subspace of $V$ spanned by $P$. More precisely, $\mathbb{R} P$ is spanned by all differences $x - y$, $x,y \in P$. The dimension of $P$ is the dimension of $\mathbb{R} P$.

A collection $P = \{ P_1, \ldots, P_m \}$ of polyhedra is called a polyhedral complex if for each $P_i$ all faces are also contained in $P$ and if each intersection $P_i \cap P_j$ produces a face of both $P_i$ and $P_j$ (if nonempty). The support of $P$ is $|P| := \bigcup_i P_i$. If $|P|$ is equal to a polyhedron $P$ (e.g. $\mathbb{R}^n$) we call $P$ a polyhedral subdivision of $P$ (or $\mathbb{R}^n$, respectively). We say that $P$ is of pure dimension $n$ if all maximal polyhedra in $P$ have dimension $n$. We will mostly deal with pure-dimensional complexes in the following. The $k$-skeleton of a polyhedral complex is the set of polyhedra

$$\mathcal{P}^{(k)} := \{ P \in \mathcal{P} : \dim(P) \leq k \}.$$  

It forms a polyhedral complex again. In practice, we will often speak of the codimension, e.g. the codimension 1 skeleton $\mathcal{P}^{(n-1)}$.

A polyhedron $P$ is called a cone if for each $x \in P$ the whole ray $\mathbb{R}_{\geq 0} x$ is contained in $P$. A polyhedral complex of cones is also called a polyhedral fan. A polyhedron $P$ in $V$ gives rise to a fan in $V^\vee$ which is called its normal fan and constructed as follows. For each face $F$ of $P$, let $\sigma_F$ be the cone in the dual space $V^\vee$ consisting of those linear forms which are bounded on $P$ from below and whose minimum on $P$ is attained on $F$,

$$\sigma_F := \{ \kappa \in V^\vee : \lambda(x) \geq \lambda(y) \text{ for all } x \in P, y \in F \}.$$
The collection of all such cones forms a polyhedral fan which is the normal fan of \( P \), denoted by \( \mathcal{N}(P) \).

A linear map \( \alpha : V \to V' \) between two vector spaces \( V = \Lambda \otimes \mathbb{R} \) and \( V' = \Lambda' \otimes \mathbb{R} \) is called an integer linear map if it is induced by an linear map \( \Lambda \to \Lambda' \), i.e. if it sends lattice vectors to lattice vectors. A map \( \beta : V \to V' \) is called integer affine-linear if it the sum of an integer linear map and a constant shift by an arbitrary vector. An integer affine-linear map \( \beta \) is called an integer affine-linear transformation or isomorphism if there exists an integer affine-linear inverse map. An integer affine-linear map \( \beta : \mathbb{R}^n \to \mathbb{R}^m \) can be written as \( x \mapsto Ax + b \), where \( A \in \text{Mat}(m \times n, \mathbb{Z}) \) is a matrix with integer entries and \( b \) is a vector with real entries. \( \beta \) is an isomorphism if and only if \( A \) is a square matrix with determinant \( \pm 1 \), i.e. if \( A \in \text{GL}(n, \mathbb{Z}) \).

### 2.3 Tropical Laurent polynomials and hypersurfaces

Let us now resume our treatment of tropical objects by focusing on the standard tropical structure on \( \mathbb{R}^n \). As explained before, this structure is given by calling a tangent vector \( v \in \mathbb{R}^n \) integer if and only if all its coordinates are integer numbers, i.e. \( v \in \mathbb{Z}^n \). We should think of this space \( \mathbb{R}^n \) as the tropical analogue of \((\mathbb{C}^\times)^n\), the algebraic torus over \( \mathbb{C} \). Indeed, when replacing \( \mathbb{C} \) by \( \mathbb{T} = \mathbb{R} \cup \{-\infty\} \), the neutral element \(-\infty\) is the only non-invertible element with respect to tropical multiplication. Therefore \( \mathbb{T}^n = \mathbb{R}^n \) and this is why \( \mathbb{R}^n \) is often called the tropical (algebraic) torus. When speaking about topological properties or differentiable functions, we always refer to the standard notions in \( \mathbb{R}^n \) equipped with Euclidean topology.

The algebraic functions defined on the \( n \)-torus are Laurent polynomials in \( n \) variables. Furthermore, Laurent polynomials are just sums of monomials. The algebro-geometric structure of \((\mathbb{C}^\times)^n\) is completely described by the selection of monomials as “base functions”. With regard to this, the tropical case is completely analogous. A tropical monomial \( \kappa \) on \( \mathbb{R}^n \) is a function of the form

\[
\kappa(x_1, \ldots, x_n) = a x_1^{j_1} \cdots x_n^{j_n},
\]

where \( a \in \mathbb{T}, j_1, \ldots, j_n \in \mathbb{Z} \) and \( x_1, \ldots, x_n \) are the coordinates of \( \mathbb{R}^n \). If we
plug in the tropical operations explicitly, we get the expression

\[ \kappa(x_1, \ldots, x_n) = a + j_1x_1 + \cdots + j_nx_n. \]

As usual we will oftentimes collect the exponents and coordinates in vectors

\[ j = (j_1, \ldots, j_n) \in \mathbb{Z}^n \quad \text{resp.} \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n \]

and write

\[ \kappa(x) = "ax^j" = a + jx, \]

where \( jx \) is the scalar product. Note that if \( a = -\infty \) then the whole function is constant and equal to \(-\infty\). If \( a \in \mathbb{R} \), then we get an integer affine-linear function, i.e. a linear function over \( \mathbb{Z} \) shifted by a real constant. Note that this definition of a monomial coincides with Definition 2.1.1.

**Lemma 2.3.1**

Let \( U \subset \mathbb{R}^n \) be open and connected and let \( f: U \to \mathbb{R} \) be a smooth function. Then the following conditions are equivalent.

(a) For all \( x \in U \) and \( v \in \mathbb{Z}^n \), we have \( df_x(v) \in \mathbb{Z} \).

(b) \( f \) is of the form

\[ f(x) = a + jx, \]

with \( a \in \mathbb{R} \) and \( j \in \mathbb{Z}^n \).

**Proof.** Let \( f \) be a function satisfying the first condition. This means \( df_x \in (\mathbb{Z}^n)^\vee = \mathbb{Z}^n \) for all \( x \in U \). As \( U \) is connected and \( \mathbb{Z}^n \) is discrete, \( df_x \) is constant and equal to a integer linear functional, say \( j \in \mathbb{Z}^n \). It follows that \( f \) is of the form \( a + jx \). The other implication is trivial.

Let us stress once more that the additive constants appearing in tropical monomials are allowed to be reals. This is in accordance with the fact that the attribute “integer” describing the tropical structure of \( \mathbb{R}^n \) does not refer to points of \( \mathbb{R}^n \), but only to tangent vectors.

Naturally, we define a tropical Laurent polynomial \( f \) on \( \mathbb{R}^n \) to be a finite sum of monomials. Namely, for each finite set \( A \subset \mathbb{Z}^n \) and coefficients \( a_j \in \mathbb{R}, j \in A \), we get a tropical Laurent polynomial

\[ f(x) = \sum_{j \in A} a_jx^j = \max_{j \in A} \{a_j + jx\}. \]

We get a convex function on \( \mathbb{R}^n \) which is piecewise integer affine-linear. Note that by lemma [2.3.1](#), tropical Laurent polynomials are just regular functions on \( \mathbb{R}^n \) according to definition [2.1.1](#). Let us consider some examples.
Example 2.3.2
If a tropical Laurent polynomial $f$ consists of only one term, then it is just an integer affine-linear function, as discussed before. If $f$ consists of two terms, say $f = a_i x^i + a_j x^j$, then it divides $\mathbb{R}^n$ into two halfspaces along the hyperplane $a_i + ix = a_j + jx$. In one half, $f$ equals $a_i + ix$, in the other $f$ is $a_j + jx$. In particular, $f$ is locally integer affine-linear except for points in the hyperplane, where $f$ is non-differentiable and strictly convex.

Example 2.3.3
We take three polynomials $f_1 = "0 + 1x + x^2"$, $f_2 = "0 + x + x^2"$ and $f_3 = "0 + (-1)x + x^2"$ in one variable $x$. First, let us point out some possibly confusing facts. Note, for example, that the constant term $0$ cannot be omitted here, because it is not tropically zero. In fact, $x^2 = "0x^2"$, but "$1x^2" \neq x = "0x^2"$. After this piece of warning, let us just draw the graphs of the three functions (see Figure 2.2). The important observation is that $f_2$ and $f_3$ are actually the same functions coming from different representations as a sum of monomials. For $f_3$, the linear term is never maximal. For $f_2$, the linear term is maximal at $0$, but only together with the other two terms. The reduced representation of both functions would be given by $f_2 = f_3 = "0 + x^2"$. Apart from that, note that again the functions are non-differentiable and strictly convex only at a finite number of breaking points. Away from these points, only one monomial attains the maximum and hence the function is locally affine linear.

Example 2.3.4
A more interesting example is the polynomial

$$f(x, y) = "(-1)x^2 + (-1)y^2 + 1xy + x + y + 0"$$

of degree 2 in two variables (see Figure 2.3).
Chapter 2: The space \( \mathbb{R}^n \) as a tropical ambient space

Classical algebraic geometry is the study of zero-sets of classical polynomials. Consequently, the object of study in this book should be zero-sets of tropical polynomials. But how do we define the “hypersurface of zeros” of a tropical polynomial? We give an answer now without further motivations. However, in the course of this book we will find more and more evidence that this definition is indeed the right one.

**Definition 2.3.5**

Let \( f \) be a tropical Laurent polynomial in \( n \) variables. Then we define the hypersurface \( V(f) \subseteq \mathbb{R}^n \) to be the set of points in \( \mathbb{R}^n \) where \( f \) is not differentiable.

In classical arithmetics, the zero element of a group is distinguished by the property of being idempotent, i.e. \( 0 + 0 = 0 \). In tropical arithmetics, this is true for any number as \( "x + x" = \max\{x, x\} = x \). However, we might consider this as a hint that a tropical sum should be called “zero” if the maximum is attained by at least two terms. This leads to an alternative definition of the tropical hypersurface of a tropical polynomial. The following lemma shows that both definitions coincide.

**Lemma 2.3.6**

Let \( f = \sum_{j \in A} a_j x^j \) be a tropical polynomial. Then the hypersurface \( V(f) \) is equal to the set of points \( x \in \mathbb{R}^n \) where the maximum \( f(x) \) is attained by at least two monomials, i.e.

\[
V(f) = \{ x \in \mathbb{R}^n : \exists i \neq j \in A \text{ such that } f(x) = a_i x^i = a_j x^j \}.
\]

**Proof.** If the maximum is attained by only one monomial, then \( f \) is locally affine linear and thus differentiable. If two monomials attain the
maximum, then $f$ is strictly convex at $x$ and therefore cannot be differentiable.

**Example 2.3.7**

For the three (in fact, two) polynomials from example 2.3.3, the hypersurfaces are just finite sets of points (see Figure 2.4).

![Figure 2.4: The “zeros” of tropical polynomials](image)

The polynomial $f$ from example 2.3.4 produces a connected hypersurface $V(f)$ consisting of 4 vertices, 3 bounded edges and 6 unbounded rays (see Figure 2.5).

![Figure 2.5: The hypersurface of “$(-1)x^2 + (-1)y^2 + 1xy + x + y + 0$”](image)

**Remark 2.3.8**

As mentioned in the introduction, there is an alternative definition of the tropical sum of two numbers given by *multivalued addition*. This was suggested by Viro (cf. [V5]) and is given, for each two numbers $a, b \in T = \mathbb{R} \cup \{-\infty\}$, by

$$a \uplus b = \begin{cases} \max(a, b) & \text{if } a \neq b, \\ [-\infty, a] & \text{if } a = b. \end{cases}$$
Chapter 2: The space $\mathbb{R}^n$ as a tropical ambient space

Note that when $a = b$, the sum $a \sqcup b$ is not just a single element but the set $[-\infty, a] := \{-\infty\} \cup \{x \in \mathbb{R} : x \leq a\}$. The set $T$ equipped with the multivalued addition $\sqcup$ and the uni-valued multiplication $\cdot$ = + forms a structure which is called a hyperfield (in our case, the tropical hyperfield).

One reason to consider such generalizations of ordinary algebra is that they allow to describe “zero-sets” in a natural way. Namely, when considering a multi-valued sum, it is natural to replace being zero by containing zero. Accordingly, if $f$ is a polynomial with hyperfield coefficients, its zero-set is defined by $f(x) \ni 0$ instead of $f(x) = 0$. In our case, a sum $a_1 \sqcup \cdots \sqcup a_n$ is zero, i.e. contains $-\infty$, if and only if we find $i \neq j$ such that $a_i = a_j \geq a_k, k = 1, \ldots, n$. (i.e. if the maximum of the summands occurs at least twice). If we replace a tropical polynomial

$$f(x) = \max(a_{j_1} + j_1 x, \ldots, a_{j_n} + j_n x)$$

by the hyperfield version

$$f^\gamma(x) = (a_{j_1} + j_1 x) \sqcup \cdots \sqcup (a_{j_n} + j_n x),$$

we find

$$V(f) = \{x \in \mathbb{R}^n : -\infty \in f^\gamma(x)\} =: V(f^\gamma).$$

Hence, we see that multi-valued tropical arithmetics lead naturally to our definition of zero-sets.

### 2.4 The polyhedral structure of hypersurfaces

So far, we described hypersurfaces as sets. We will now proceed by investigating their structure as polyhedral complexes.

For each monomial parameterized by $j \in A$, set

$$P_j := \{x \in \mathbb{R}^n : f(x) = "a_j x^i"\}$$

to be the locus of points where the chosen monomial is maximal. The sets $P_j$ subdivide $\mathbb{R}^n$ into the domains of linearity of $f$. If $P_j$ is a neighbourhood of a point $x$, then $f$ is obviously differentiable at $x$, with differential $df_x = j$ (regarding $j$ as a covector). Note that $P_j$ is a rational polyhedron in $\mathbb{R}^n$ as it is the intersection of the halfspaces $a_j + j x \geq a_i + ix$ for all $i \in A$. Moreover, the intersection of two polyhedra $P_i$ and $P_j$ is either empty or a common face (given by intersecting with the plane $a_i x^i = a_j x^j$). In other words, the collection of all $P_j, j \in A$ and all their faces forms a polyhedral subdivision of $\mathbb{R}^n$ denoted by $S(f)$. 

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Example 2.4.1
Our polynomials from the previous examples give subdivisions as indicated in Figure 2.6. Note that in the second and third case the subdiv-

visions $S(f_2)$ and $S(f_3)$ are identical, but the definition of $P_1$ depends on the chosen representation. For the polynomial in two variables, the subdivision $S(f)$ is shown in Figure 2.7.

![Figure 2.6: The subdivision induced by polynomials](image)

![Figure 2.7: Another subdivision](image)

Obviously, a point $x \in \mathbb{R}^n$ lies in codimension 1 strata of $S(f)$ if and only if at least two monomials attain the maximum $f(x)$ at $x$. We get the following corollary.

**Corollary 2.4.2**
*The hypersurface $V(f)$ is equal to the codimension 1 skeleton of $S_f$. In particular, $V(f)$ is canonically equipped with the structure of a rational polyhedral complex.*

Note also that the subdivision $S(f)$ does not depend on the monomial representation of $f$. In fact, we can define the reduced monomial representation by throwing away the monomial “$a_j x^j$” whenever $P_j$ is empty or not full-dimensional.
**Definition 2.4.3**

Let $f = \sum_{j \in A} a_j x^j$ be a tropical Laurent polynomial. The set

$$A^{\text{red}} := \{ j \in A : \dim(P_j) = n \} = \{ df \in A : x \in \mathbb{R}^n \setminus V(f) \}$$

is called the *reduced support* of $f$. The truncated polynomial

$$f^{\text{red}} = \sum_{j \in A^{\text{red}}} a_j x^j$$

is called the *reduced representation* of $f$.

**Lemma 2.4.4**

Let $f$ be a tropical Laurent polynomial. On the level of functions on $\mathbb{R}^n$, we have $f = f^{\text{red}}$. Moreover, two tropical Laurent polynomials describe the same function if and only if their reduced representations agree (as abstract polynomials). In particular, the subdivision $S_f$ is independent of the monomial representation of $f$.

*Proof.* As $S_f$ forms a subdivision of $\mathbb{R}^n$, every point $x \in \mathbb{R}^n$ is contained in some $P_j$ with $\dim(P_j) = n$, so

$$f(x) = a_j x^j = f^{\text{red}}(x).$$

Moreover, if two polynomials describe the same function, they also have the same reduced support (as its the set of covectors which appear as differentials of the function). The value $f(x)$ for any $x \in \mathbb{R}^n$ with $df = i$ determines the coefficient of $x^i$ uniquely. Thus the reduced representation of $f$, as well as $S_f$, is completely described by the underlying function.  

We have seen that each tropical Laurent polynomial subdivides $\mathbb{R}^n$ into its domains of linearity $S(f)$. In many situations, it is instructive to consider also the “dual subdivision”, a certain subdivision of the Newton polytope of $f$. The *Newton polytope* $NP(f)$ of $f = \sum_j a_j x^j$ is given by

$$NP(f) = \text{Conv}\{ j \in \mathbb{Z}^n : a_j \neq -\infty \},$$

i.e. the convex hull of all appearing exponents. Note that $NP(f)$ lives in the dual space of $\mathbb{R}^n$. We define the subdivision $SD(f)$ of $NP(f)$ as follows.

**Definition 2.4.5**

Let $f = \sum_{j \in A} a_j x^j$ be a tropical Laurent polynomial with $a_j \neq -\infty$ for all $j \in A$. We set

$$\tilde{A} := \{ (j, -a_j) \in \mathbb{Z}^n \times \mathbb{R} : j \in A \}$$
and $\bar{P} = \text{Conv}(\bar{A})$. The projection of the lower faces of $\bar{P}$ (i.e. those which are also faces of $\bar{P} + \rho$ with half-ray $\rho := \{0\} \times \mathbb{R}_{\geq 0}$) produces a subdivision $\text{SD}(f)$ of $\text{NP}(f)$, which we call the dual subdivision of $f$.

**Example 2.4.6**
Our polynomials in one variable from example 2.3.3 all have the same Newton polytope $\text{Conv}\{0,2\}$. Only for the first polynomial, $\text{SD}(f_1)$ is non-trivial, i.e. the segment is divided into two unit segments with vertex $\{1\}$ (see Figure 2.8).

For the polynomial $f$ of example 2.3.4, the dual subdivision is again more interesting. In Figure 2.9, only the lower faces of $\bar{P}$ are drawn solidly. We obtain a subdivision of the triangle of size 2 into 4 triangles of size 1. Moreover, the subdivision consists of 3 internal edges and 6 edges in the boundary of the big triangle.

In our examples, we can easily observe an inclusion-reversing duality between the cells of $S(f)$ and $\text{SD}(f)$. Let us formulate this systematically. For each cell $P \in S(f)$ we define

$$B(P) := \{j \in A : P \subseteq P_j\}.$$
In other words, $B(P)$ is the set of monomials which are maximal on $P$. We find

$$P = \bigcap_{j \in B(P)} P_j.$$  \hspace{1cm} (2.1)

Analogously, for a point $x \in \mathbb{R}^n$ we set $B(x) := \{ j \in A : x \in P_j \}$. $x$ is an interior point of $P$ if and only if $B(x) = B(P)$.

**Theorem 2.4.7**

The subdivisions $S(f)$ of $\mathbb{R}^n$ and $SD(f)$ of $NP(f)$ are dual in the following sense. There is an inclusion-reversing bijection of cells given by

$$S(f) \to SD(f)$$

$$P \mapsto \sigma_P := \text{Conv}(B(P))$$

such that $\dim P + \dim \sigma_P = n$ and $RP^\perp = R\sigma_P$. In particular, the set of vertices of $SD(f)$ is equal to the reduced support of $f$.

**Proof.** First let us show that $\sigma_P$ is indeed a cell of $SD(f)$. To do so, we pick a point $x$ in the relative interior of $P$. Let us consider the linear form $(x, -1)$ on $\tilde{P} \subset \mathbb{R}^{n+1}$. For each vertex $(j, -a_j)$ of $\tilde{P}$ we find $(x, -1)(j, -a_j) = a_j + jx$. Thus the face of $\tilde{P}$ on which $(x, -1)$ is maximal is exactly the convex hull of the points $\tilde{B}(x)$ with

$$\tilde{B}(x) := \{(j, -a_j) : j \in B(x)\}.$$  

The projection of this face to $\mathbb{R}^n$ is $\sigma_P$, and therefore $\sigma_P$ is indeed a face of $SD(f)$. Next we see that all lower faces of $\tilde{P}$ are obtained in this way for a suitable $(x, -1)$. It follows that equation (2.2) indeed describes a well-defined bijection whose inverse map is given by

$$\sigma \mapsto \bigcap_{j \in \sigma \cap A} P_j.$$  

The previous arguments also provide another way to construct $S(f)$. Let $N(\tilde{P})$ be the normal fan of $\tilde{P}$, then $S(f)$ is the subdivision obtained from intersecting $N(\tilde{P})$ with the plane $\mathbb{R}^n \times \{-1\} \cong \mathbb{R}^n$ (see Figure 2.10). Using this description the orthogonality and dimension statements follow directly from the corresponding statements for the dual cells of a polyhedron and its normal fan. \hfill $\square$

**Example 2.4.8**

Let $A \in \mathbb{Z}^n$ be a finite set and let $f = \sum_{j \in A} x^j$ be the Laurent polynomial
2.4 The polyhedral structure of hypersurfaces

![Diagram](image)

Figure 2.10: Dual subdivisions via normal fans

with only trivial coefficients \( a_j = 0, j \in A \). Then \( \text{SD}(f) \) is just the trivial subdivision of \( \text{NP}(f) = \text{Conv}(A) \) (i.e. \( \text{SD}(f) \) contains \( \text{NP}(f) \) and all its faces) and \( \text{S}(f) \) is just the normal fan of \( \text{NP}(f) \). Such piecewise linear functions whose domains of linearity form a fan appear for example in toric geometry (cf. [F, section 3]).

![Diagram](image)

Figure 2.11: The subdivision of a trivial coefficients polynomial

Remark 2.4.9

Let \( S \) be a subdivision of a polyhedron \( P \subseteq \mathbb{R}^n \). \( S \) is called a *regular subdivision* if it can be obtained by projecting down the lower faces of some polyhedron \( \bar{P} \subseteq \mathbb{R}^{n+1} \). Equivalently, \( S \) is regular if there exists a convex function on \( P \) which is affine-linear on each cell of \( S \) (to get the polyhedron in \( \mathbb{R}^{n+1} \), we take the convex hull of the graph of the function; the other way around, the union of lower faces of the polyhedron describes the graph of a suitable function). Note that both \( S(f) \) and \( \text{SD}(f) \) are regular subdivisions. Such subdivisions are sometimes also called *convex* or *coherent*. In real algebraic geometry, they appeared after the discovery of the patchworking technique by Viro (see [V4] for references). Note that not all subdivisions are regular. An example is given in Figure 2.12.
Assume there is convex function \( g \) inducing this subdivision. We can assume that \( g \) is constant zero on the inner square. Then if we fix one \( g(v_1) \) on an outer vertex, on the next vertex \( v_2 \) in clockwise direction, we need \( g(v_2) \geq g(v_1) \), due to the diagonal edge subdividing the corresponding trapezoid. Going around the square completely gives a contradiction.

Another way to describe the duality of \( S(f) \) and \( \text{SD}(f) \) can be formulated in terms of Legendre transforms. Let \( g : S \to \mathbb{R} \) be a function on an arbitrary set \( S \subseteq \mathbb{R}^n \). The Legendre transform \( g^* \) is a function on covectors of \( \mathbb{R}^n \) given by

\[
g^*(w) = \sup_{x \in S} \{wx - g(x)\}.
\]

It is easy to check that \( g^* \) takes finite values on a convex set of \( \mathbb{R}^n \) (possibly empty) and that \( g^* \) is convex. Moreover, if \( g \) is a convex function in the beginning, than \( (g^*)^* = g \). In our case, we see that a tropical Laurent polynomial \( f = \sum_{j \in A} a_j x^j \) is equal to the Legendre transform of its coefficient map \( A \to \mathbb{R}, j \mapsto -a_j \).

**Lemma 2.4.10**

Let \( f \) be a Laurent polynomial and let \( g \) be the convex function on \( \text{NP}(f) \) whose graph is the lower hull of

\[
\tilde{P} = \text{Conv}\{((j, -a_j) : j \in A)\}.
\]

Then \( f^* = g \) and equivalently \( f = g^* \).

**Proof.** We set \( g' : A \to \mathbb{R}, j \mapsto -a_j \). As \( f = g'^* \), it suffices to show \( g'^* = g^* \). For a fixed \( x \in \mathbb{R}^n \), we can compute \( g'^*(x) \) resp. \( g^*(x) \) as the maximum value of \((x, -1)\) on \( \{((j, -a_j) : j \in A)\} \) resp. \( \tilde{P} \). This maximum is always attained on at least one vertex of \( \tilde{P} \) and the vertices of \( \tilde{P} \) are contained in \( \{((j, -a_j) : j \in A)\} \). Hence the claim follows.

\[
\square
\]
2.5 The balancing condition

So far, we have seen that to each tropical Laurent polynomial $f$ we can associate a subdivision $S(f)$ of $\mathbb{R}^n$ and a dual subdivision $SD(f)$ of $NP(f)$ which is induced by the Legendre transform of $f$. We described the hypersurface $V(f)$ as a set — the points where $f$ is not differentiable — and as polyhedral complex of pure dimension $n-1$ — the codimension one skeleton of $S(f)$. We now add yet another layer to our description, namely multiplicities for the points. We will later say that smooth points of $V(f)$ are those with multiplicity 1. Moreover, these multiplicities are necessary to formulate the most fundamental structure property of tropical objects, the balancing condition.

For each polytope $\sigma$, we define the volume $\text{Vol}(\sigma)$ to be the volume of $\sigma$ measured in the affine space spanned by $\sigma$ and normalized such that $\text{Vol}(\sigma_n) = 1$, where $\sigma_n$ denotes the standard simplex in $\mathbb{R}^n$ given by $x_1 + \ldots + x_n \leq 1$ and $x_i \geq 0, i = 1, \ldots, n$. In other words, the volume of a full-dimensional $\sigma$ is the usual volume of $\mathbb{R}^n$ divided by $n!$. Let $\sigma$ be a lattice polytope, i.e. all vertices are integer points. We call $\sigma$ a minimal simplex if $\text{Vol}(\sigma) = 1$. This is the case if and only if $\sigma$ can be mapped to the standard simplex $\sigma_n$ by an integer affine-linear transformation. For any lattice polytope $\sigma$ we have $\text{Vol}(\sigma) \in \mathbb{N}$. This follows from the fact that we can always triangulate $\sigma$ into simplices (not necessarily minimal ones). The volume of a simplex with vertices $v_0, \ldots, v_n$ is given by the determinant of the vectors $v_1 - v_0, \ldots, v_n - v_0$, which is obviously integer for lattice simplices. For one-dimensional lattice polytopes (i.e. edges) we find $\text{Vol}(\sigma) = \#\{(\sigma \cap \mathbb{Z}^n)\} - 1$. For two-dimensional lattice polytopes, Pick’s theorem states $\text{Vol}(\sigma) = 2i + b - 1$, where $i := \#\{(\sigma \cap \mathbb{Z}^n)\}$ is the number of interior lattice points and $b := \#\{(\partial \sigma \cap \mathbb{Z}^n)\}$ is the number of lattice points in the boundary.

**Definition 2.5.1**

Let $f$ be a tropical Laurent polynomial. We turn its hypersurface $V(f)$ into a weighted polyhedral complex as follows. The multiplicity of each cell $P$ of $V(f)$ is the volume of the corresponding cell $\sigma_P$ in the dual subdivision $SD(f)$,

$$\text{mult}(P) := \text{Vol}(\sigma_P) \in \mathbb{N}.$$ 

For each point $x \in V(f)$ in the relative interior of $P$ we define the multiplicity $\text{mult}(x) \in \mathbb{N}$ to be $\text{mult}(P)$.

Each point resp. cell of $V(f)$ with multiplicity greater than 1 is called singular. If $V(f)$ does not have singular points resp. faces, we call it a
smooth hypersurface. This is equivalent to the condition that all cells of SD(f) are minimal simplices, in which case we call SD(f) a unimodular subdivision. Of course, it suffices to check this condition for the maximal cells of SD(f), as faces of minimal simplices are minimal simplices again.

**Remark 2.5.2**

Let F be a facet of V(f). The dual cell σ_F is an edge whose endpoints, say i and j, are the exponents of the two monomials in the reduced representation of f which attain the maximum at F. Thus the multiplicity \( \text{mult}(F) \) can be considered as a measure of the change of slope when crossing from the linearity domain \( P_i \) to \( P_j \) through F. This is precisely what happens for polynomials in one variable. In general, one should think of the change of slope relative to F.

**Example 2.5.3**

Let us revisit our running examples. The polynomials from example 2.3.3 describe hypersurfaces of either two points with multiplicity 1 or one single point with multiplicity 2. Correspondingly, the dual subdivision divides the interval \([0,2]\) in either two segments of volume 1 or one segment of volume 2 (see Figure 2.13). In the case of the planar conic from example

\[
\begin{align*}
\text{SD}(f_1) & \quad \text{V}(f_1) \\
0 & 1 & 2 & \text{mult} 1 & 1 \\
\text{SD}(f_2) = \text{SD}(f_3) & \quad \text{V}(f_2) = \text{V}(f_3) \\
0 & 2 & \text{mult} 2
\end{align*}
\]

Figure 2.13: Multiplicities of tropical zeros

the dual subdivision only consists of minimal triangles of volume 1 and therefore the conic is smooth (see Figure 2.14).

**Remark 2.5.4**

For a smooth hypersurface V(f), the reduced support of f is NP(f) \( \cap \mathbb{Z}^n \). This follows from the fact that a minimal simplex does not contain interior integer points and thus each point in NP(f) \( \cap \mathbb{Z}^n \) must be a vertex of SD(f).
2.5 The balancing condition

Figure 2.14: $V \left((-1)x^2 + (-1)y^2 + 1xy + x + y + 0\right)$ is smooth.

Example 2.5.5
The tropical hyperplane in $\mathbb{R}^n$ is the hypersurface $V(f)$ with $f = x_1 + \cdots + x_n + 0$. It is obviously smooth, as $\text{NP}(f)$ is the standard simplex itself. Figure 2.15 depicts the 2-dimensional hyperplane in $\mathbb{R}^3$.

Figure 2.15: The tropical 2-dimensional hyperplane

A hypersurface $V(f)$, equipped with multiplicities as above, satisfies the so-called balancing condition — a fundamental property of tropical varieties.

Definition 2.5.6
Let $\mathcal{X}$ be a pure-dimensional polyhedral complex in $\mathbb{R}^n$ with multiplicities $\text{mult}(\sigma)$ for each facet $\sigma$ of $\mathcal{X}$. We say $\mathcal{X}$ is balanced or satisfies the balancing condition if for every cell $\tau \in \mathcal{X}$ of codimension one the following
equation holds.
\[ \sum_{\sigma \text{ facet } \tau \subset \sigma} \text{mult}(\sigma) v_{\sigma/\tau} \equiv 0 \pmod{\mathbb{R}\tau} \]

Here \( v_{\sigma/\tau} \in \mathbb{Z}^n \) denotes a primitive generator of \( \sigma \) modulo \( \tau \), i.e. an integer vector that points from \( \tau \) to the direction of \( \sigma \) and satisfies
\[ \mathbb{Z}\sigma = \mathbb{Z}\tau + \mathbb{Z}v_{\sigma/\tau}. \]

In other words, \( v_{\sigma/\tau} \) is an arbitrary representative of the unique primitive generator \( u_{\sigma/\tau} \) of the ray \( (\sigma - \tau)/\mathbb{R}\tau \). The balancing condition requires that the sum of these primitive generators, weighted by the multiplicities of the facets, vanishes modulo \( \mathbb{R}\tau \). We could also write it as
\[ \sum_{\sigma \text{ facet } \tau \subset \sigma} \text{mult}(\sigma) u_{\sigma/\tau} = 0 \in \mathbb{R}^n/\mathbb{R}\tau. \]

Figure 2.16: The balancing condition

**Theorem 2.5.7**

Any hypersurface \( V(f) \) of a Laurent polynomial \( f = \sum_{j \in A} x^j \) forms a balanced polyhedral complex.

**Proof.** Let us first consider the two-dimensional case, i.e. assume \( f \) is a polynomial in two variables and hence \( V(f) \) is a piecewise linear curve in the plane. We have to check the balancing condition around each vertex \( V \) of \( V(f) \). Let \( \sigma_V \) be the 2-cell in the dual subdivision. By duality we see that locally around \( V \) the subdivision \( S(f) \) looks like the normal fan of \( \sigma_V \). In particular, for each edge \( E \) containing \( V \) the primitive generator \( v_{E/V} \) is orthogonal to the dual edge \( \sigma_E \) in \( \sigma_V \), and \( \text{mult}(E) \) is by definition
2.5 The balancing condition

just the integer length of $\sigma_E$. Thus, when we concatenate all the vectors $\text{mult}(E)v_{E\setminus V}$ in say clockwise order, the prescribed chain of edges is just a rotation of the boundary of $\sigma_V$. The fact that this boundary closes up is thus equivalent to the fact

$$\sum_{E \text{ edge} \ V \subset E} \text{mult}(E)v_{E\setminus V} = 0.$$ 

This finishes the proof for $\mathbb{R}^2$, and essentially the same argument can be applied in the general case. Let $P$ be a codimension one face of $V(f)$ and let $\sigma_P$ be the dual 2-cell of $\text{SD}(f)$. Then $\mathbb{R}\sigma_P$ is canonically the dual space of $\mathbb{R}^n/RP$ and for each facet $F$ containing $P$, the primitive generator $u_{F/P} \in \mathbb{R}^n/RP$ is orthogonal to the corresponding edge $\sigma_F \in \text{SD}(f)$. Thus, once again the “closedness” of $\sigma_P$ guarantees the balancing condition. 

**Remark 2.5.8**

Note that the balancing condition only involves the multiplicity of generic points, i.e. of the facets of the complex $V(f)$. As we defined multiplicities for all cells of $S(f)$, we might as well ask if the $k$-skeleton

$$S(f)^{(k)} := \{ P \in S(f) : \dim(P) \leq k \}$$

also satisfies the balancing condition. Indeed, a fundamental result from polyhedral algebra shows that $S(f)^{(k)}$ is balanced for all $k$ (cf. [M1]).

**Remark 2.5.9**

Let us revisit our question of defining the “zero set” of a tropical polynomial. Now that we know that the balancing condition is a fundamental property of tropical hypersurfaces, we might be bothered by the fact that the graph of a tropical Laurent polynomial

$$\Gamma = \{ (x, y) \in \mathbb{R}^n \times \mathbb{R} : y = f(x) \}$$

Figure 2.17: The balancing condition in two dimensions
does not satisfy the balancing condition (of course, we equip each facet with multiplicity 1 here). Note that in classical algebraic geometry, the graph of a polynomial is a hypersurface again given by an algebraic equation, namely $y = f(x)$. Tropically, however, the graph of a polynomial is obviously a polyhedral complex, but the balancing condition fails exactly at the points over $V(f)$ where the function is strictly convex. We can repair this by using the multivalued tropical addition $\gamma$ from remark 2.3.8 again. Let us consider the multivalued function $f^\gamma$ and its “graph”

$$\Gamma^\gamma = \{(x,y) \in \mathbb{R}^n \times T : y \in f^\gamma(x)\}.$$

We find that $\Gamma^\gamma$ contains the ordinary graph $\Gamma$, but additionally, over each point $x \in V(f)$, contains the half-infinite interval $\{(x,y) : y \leq f(x)\}$. Moreover, $\Gamma^\gamma$ is canonically balanced. To that end, we equip each facet of $\Gamma^\gamma$ which is not already contained in $\Gamma$ with the multiplicity of its projection to $\mathbb{R}^n$ which is a facet of $V(f)$. It is easy to see that $\Gamma^\gamma$ is the only balanced completion of $\Gamma$ if we only allow to add facets in direction $(0,-1) \in \mathbb{R}^n \times \mathbb{R}$. This is why $\Gamma^\gamma$ is sometimes called the completed graph of $f$. Note that, in contrast to $\Gamma$, the completed version $\Gamma^\gamma$ intersects the “zero-section” $\mathbb{R}^n \times \{-\infty\}$ and this intersection gives back $V(f)$ — yet another reason for our definition of the zeros of a tropical polynomial.

Let us now show the inverse of theorem 2.5.7, which states that balanced polyhedral complexes of codimension one are always tropical hypersurfaces.

**Theorem 2.5.10**

*Let $V \subset \mathbb{R}^n$ be a balanced polyhedral complex of dimension $n-1$. Then*
there exist tropical polynomials \( f, g \) such
\[
V = V(\frac{f}{g}).
\]
Moreover, if all weights of \( V \) are positive, we can assume that \( g \equiv 0 \), i.e.
\[
V = V(f)
\]
for a suitable tropical polynomial \( f \).

Proof. First note that we can easily write \( V \) as
\[
V = V_1 - V_2,
\]
where both \( V_1 \) and \( V_2 \) are positive (for example, we can take \( V_2 \) to be the sum of all affine spaces spanned by facets with negative weights). So it suffices to show the second part of the theorem: We assume that \( V \) is positive and want to show that \( V = V(f) \) for a tropical polynomial \( f \).

Let \( C_1, \ldots, C_l \) be the connected components of \( \mathbb{R}^n \setminus V \). Our plan is to find a monomial \( f_i \) for each component \( C_i \) such that \( f = \sum f_i \) and \( f_i|_{C_i} = f_i \). We start by picking one component, say \( C_1 \), and setting \( f_1 \equiv 0 \). Assume that \( C_2 \) is seperated from \( C_1 \) by a facet \( F \) of \( V \). Then we continue by defining
\[
f_2 = f_1 + \text{mult}(F)h,
\]
where \( h \) is the unique affine-linear function such that \( h|_F = 0 \) and \( h(v_{C_2/F} + x) = 1 \) (where \( v_{C_2/F} \) is a primitive generator of \( C_2 \) modulo \( F \) and \( x \in F \)).

As \( \mathbb{R}^n \setminus V^{(n-2)} \) is connected, we can continue this method to find \( f_i \) for all components \( C_i \). However, we have to check that this is well-defined, i.e. \( f_i \) does not depend on the chosen path from \( C_1 \) to \( C_i \) (in \( \mathbb{R}^n \setminus V^{(n-2)} \)).
To see this, let $s$ be a loop in $\mathbb{R}^n \setminus V^{(n-2)}$. For simplicity, in the following discussion we will assume that all loops are oriented, piecewise linear, and intersect $V$ in only finitely many points. We have to add up all the terms $\text{mult}(F)h$ each time $s$ crosses a facet $F$ and to show that the sum is zero. Of course, we can contract $s$ to a point in $\mathbb{R}^n \setminus V^{(n-3)}$ (using only the nice loops introduced above). During this deformation, the way how the loop passes the facets of $V$ can only change in two special situations. We finish by showing that in both situations the sum of functions $\text{mult}(F)h$ does not change and hence is zero, as for the trivial constant loop.

The first special situation is given by a loop in the deformation which runs into a facet $F$ of $V$, but enters and leaves it by the same connected component, say $C_i$. Let $v$ be this particular point where $s$ intersects $F$ and let $C_j$ be the second component adjacent to $F$. Then we can deform the loop by moving $v$ into $C_i$ or $C_j$, but both possibilities result in the same sum of functions, as in the second case we add $\text{mult}(F)h$ as well as $-\text{mult}(F)h$.

The second special situation is when a loop in the deformation passes through a cell $P$ of $V$ of dimension $n-2$. The difference of the two possible deformations here is just given by the sum of functions given by a small loop that turns around $P$ once. This is where we finally use that $V$ satisfies the balancing condition. Namely, let $F_1, \ldots, F_h$ be the facets adjacent to $P$, let $s$ be a small loop around $R$ passing each facet once, and let $h_j$ be linear function corresponding to the traverse of $F_j$. Then the condition

$$\sum \text{mult}(F_j)h_j = 0$$

is just the dual version (and thus equivalent to) the balancing condition of $V$ at $P$. We have seen this in more details in the proof of theorem \ref{balancing-condition}.

So far we proved that the above method produces a well-defined monomial $f_i$ for each connected component $C_i$. Thus we can define the tropical polynomial

$$f = \sum_{i=1}^l f_i.$$ 

It remains to show $f|_{C_i} = f_i$ for all $i$. Then by construction of the $f_i$, we have $V = V(f)$. Let us consider the function $f'$ defined by $f'|_{C_i} := f_i$. By construction, this function is convex on all line segments in $\mathbb{R}^n \setminus V^{(n-2)}$. By continuity, it is therefore convex on $\mathbb{R}^n$. Hence, $f'$ is equal to twice its Legendre dual, which is exactly $f$. Therefore $f|_{C_i} = f_i$ for all $i$. \hfill $\square$
This section is devoted to the study of some examples of tropical hypersurfaces in \( \mathbb{R}^2 \), i.e. planar curves. While these are easily visualizable, they still exhibit interesting combinatorial, but also geometric properties.

A tropical planar curve is the hypersurface \( C = V(f) \) of a tropical Laurent polynomial

\[
f = \text{"} \sum_{j \in \mathbb{Z}^2} a_j x^{j_1} y^{j_2} \text{"}
\]

in two variables \( x, y \). Let \( \Delta_d, d \in \mathbb{N} \) denote the convex hull of the points \((0,0), (d,0), (0,d)\). We call \( \Delta_d \) the \textit{d-fold standard simplex}. Mostly, we will consider polynomials whose Newton polytope is \( \text{NP}(f) = \Delta_d \) for a suitable \( d \). In this case we call \( V(f) \) a curve of degree \( d \). We may extend this definition to arbitrary polynomials by saying \( V(f) \) is of degree \( d \) if \( d \) is the smallest natural number such that (a shift of) \( \Delta_d \) contains \( \text{NP}(f) \). If \( \text{NP}(f) \) is not equal to this minimal (shift of) \( \Delta_d \), we call \( V(f) \) degenerated.

Of course, we start with \textit{planar lines} \( L = V(f) \), where \( f \) is a polynomial of degree 1. The general form of such a polynomial is

\[
f(x,y) = \text{"} ax + by + c \text{"}.
\]

Let us start with \( a = b = c = 0 \) (see Figure 2.20). Then \( L = V(f) \) consists of three rays \( \mathbb{R}_+ (1,1), \mathbb{R}_+(1,-1) \) and \( \mathbb{R}_+(0,-1) \). The point \((0,0)\) is the single vertex of \( L \). Each ray corresponds to two of the three terms of \( f \) being maximal, while \((0,0)\) is the single point where all three terms attain the maximum. What happens if we change \( a, b, c \)? There is still a unique point where all three monomials are maximal, \((c - a, c - b)\). Therefore
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$L_{a,b,c} = V(ax + by + c)$ is just the translation of $L$ to this point. If we decrease one coefficient at a time, we move $L$ in one of the directions $(-1,-1)$, $(1,0)$ or $(0,1)$. In the limit, i.e. when one coefficient becomes $-\infty$, we end up with an ordinary classical line whose Newton polytope is just a segment (see Figure 2.21). In total, there are a single non-degenerated and 3 degenerated types of tropical lines, and two lines of the same type are translations of each other. Let us stress that the only classical lines which show up as “tropical lines” are the lines of slope $(1,1)$, $(-1,0)$ or $(0,-1)$. Let us now decrease a second coefficient of our linear polynomial, such that in the limit two coefficients are $-\infty$. Geometrically, the line vanishes at infinity and $V(f)$ is empty. This reminds us of the fact that we work with the tropical algebraic torus $\mathbb{R}^n = (\mathbb{T}^\times)^n$ here, a non-compact space on which single monomials are “non-vanishing” functions. Later on, we will consider various compactifications of $\mathbb{R}^n$, in particular tropical projective space $\text{TP}^2$. In this space, when decreasing two of the coefficients, the corresponding moving line will attain one of the coordinate
2.6 Examples: Planar Curves

lines $x = -\infty$, $y = -\infty$ or $z = -\infty$ as limit.  

But for now, let us stick to the non-compact picture. There are two elementary properties of classical planar lines which we want to study tropically now: Two generic lines intersect in a single point, and, dually, through two different points in the plane there passes a unique line.

First, let us consider the intersection of two tropical lines. Indeed, for most pairs of lines, we get exactly one point of intersection, but there are two notable exceptions, as illustrated in the Figure 2.23. While in the first two pictures we get a perfectly nice single point of intersection, the third picture shows two parallel degenerated lines with no intersection. As in classical geometry, we will get rid of this special case by compactifying $\mathbb{R}^2$. However, a classical geometer would very likely compactify to the real projective plane $\mathbb{R}P^2$ to make the lines intersect at infinity. This is different from the tropical compactifications we will study later, e.g. $\mathbb{TP}^2$. The fourth picture shows a more interesting way of failure. Here, the two tropical lines have, instead of a single intersection point, a whole ray in common. This type of abnormality does not have a classical counterpart (except for taking the same line twice) and will later encourage us to introduce the concept of stable intersection. The stable intersection of two curves consists only of those points in the set-theoretic intersection which are stable under small deformations — small translations, in our case. E.g. for lines, a small translation of one of them will yield a unique intersection point close to the apex of the common ray. When moving the translation back to the original line, the limit of intersection points is just this single point, which we therefore call the stable intersection of the two lines. The dual problem of finding a line through two points shows a completely similar behaviour.

Figure 2.23: Intersections of tropical lines

---

2In particular, we see that the dual space of lines in $\mathbb{TP}^2$ is itself a copy of $\mathbb{TP}^2$ and inherits a stratification into affine spaces corresponding to the seven types of lines.
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Proposition 2.6.1

Any pair of points $p_1, p_2 \in \mathbb{R}^2$ can be joined by a tropical line. Furthermore, this line is unique if and only if the points do not lie on an ordinary classical line of slope $(1, 1), (-1, 0)$ or $(0, -1)$.

Proof. Even though the statement is completely elementary, let us give a short proof here. First, we set $L$ to be the tropical line whose single vertex is $p_1$. If $p_2$ lies on one of the rays of $L$, we are already done (with existence). If not, $p_2$ is contained in one of the sectors of $\mathbb{R}^2 \setminus L$ corresponding to a certain monomial being maximal. We move $L$ into this sector by decreasing the coefficient of the corresponding monomial. While doing so, the two rays of $L$ that bound the sector will sweep it out completely. Hence we just stop when the moving line meets $p_2$ (see Figure 2.25). The uniqueness statement follows from our previous discussion of the intersection of two lines. Namely, if we can find two different lines containing $p_1$ and $p_2$, then the two lines have a ray of slope $(1, 1), (-1, 0)$ or $(0, -1)$ in common, and the statement follows. $\square$
We now consider curves of degree 2 given by polynomials of the form

\[ f(x, y) = ax^2 + bxy + cy^2 + dx + ey + f \].

Contrary to the case of lines, where we found only one non-degenerated example up to translations, the variety of conics is already much richer. Not only do conics have different combinatorial types, but also do conics of the same type differ by more than just translations. Figure 2.26 gives a list of the four combinatorial types of smooth (non-degenerated) conics, given by the four unimodular subdivisions of \( \Delta_2 \). In all four cases the complement \( R^2 \setminus V(f) \) consists of six connected components — the linearity domains of the six monomials in \( f \). Let us again deform a given conic by changing the coefficient of just one monomial. This will enlarge or shrink the corresponding connected component, depending on whether we increase or decrease the coefficient. More precisely, the deformation will move all edges adjacent to this component while all other edges rest (i.e. they might at most change length). This follows from the fact that the position of each of these edges is given by an equation of the form

\[ a_i + i_1 x + i_2 y = a_j + j_1 x + j_2 y \]

with \( i, j \in \Delta_2 \cap \mathbb{Z}^2 \), and hence an edge will change its position if and only if \( a_i \) or \( a_j \) is the coefficient being changed (see Figure 2.27). This describes small changes of a single coefficient. However, if we continue the deformation, a couple of things can happen. Of particular interest is the case when eventually the combinatorial type of the conic changes. This happens if one of the edges shrinks to length zero and vanishes. In this case also the corresponding edge in the dual subdivision vanishes which merges together two of the triangles to a 2-cell of volume 2. Such a 2-cell can either be a parallelogram or a triangle with one side of length 2. Both situations can be illustrated by deforming the

![Figure 2.26: Smooth (non-degenerated) conics](image)

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conic from the previous picture. Increasing, as before, the coefficient of the monomial "$x$", we run into a conic with a singular point whose dual cell is a square. Increasing the coefficient further, we get a smooth conic again, but the combinatorial type has changed (see Figure 2.28). If we instead increase the monomial "$x^2$", we run into a singular conic with a ray of multiplicity 2 (see Figure 2.29). Note that in this case, increasing the coefficient further does not remove the multiplicity 2 edge. The parallelogram degeneration corresponds to a reducible conic which decomposes into two lines. The second degeneration will later be interpreted as a smooth conic that is tangent to the coordinate axis $y = -\infty$ at infinity.

For higher degree curves, the number of combinatorial types becomes large very quickly. Already, for curves of degree 3, there are 79 smooth non-degenerate combinatorial types. Figure 2.30 depicts particular ex-
amples of a smooth and a singular cubic. In general, a planar curve is smooth if and only if each of its vertices has exactly 3 adjacent edges (we say all vertices are 3-valent) and each pair of the primitive integer vectors $v_1, v_2, v_3$ spanning the 3 edges of a vertex form a $\mathbb{Z}$-basis of $\mathbb{Z}^2$. We now present a few special cases of combinatorial types for arbitrary degrees.

The honeycomb triangulation of $\Delta_d$ is defined by the property that all its edges are parallel to one of the three boundary edges of $\Delta_d$. In other words, it is obtained from the collection of lines $x = i, y = j, x + y = k$ with
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$i, j, k \in \{0, \ldots, d\}$. Our figure depicts the case $d = 5$. The curves of this combinatorial type are called *honeycombs*. They proved to be useful in the context of the Horn problem (e.g. [KT]).

![Figure 2.31: A honeycomb curve of degree 5](image)

The bathroom tiling curves form a similar type of curves. Their dual triangulation of $\Delta_d$ is given by $x = i, y = j$ with $i, j \in \{0, \ldots, d\}$, $x + y = k, x - y = l$ with $k, l \equiv d \mod 2$. Again, our picture shows an example of degree 5. These curves are of interest for example when constructing real algebraic curves.

![Figure 2.32: A bathroom tiling curve of degree 5](image)

Let us also recall an example of a special tropical curve which we mentioned in the introduction. Figure 2.33 shows the curve of degree 10 together with its subdivision. It is this curve (together with the extra data of suitable signs) that was used by Itenberg (cf. [IV]) to disprove the famous Ragsdale conjecture (cf. [R]; the conjecture was an inequality.)
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involving the numbers of “odd” and “even” ovals of a real planar curve of
given degree).

Figure 2.33: The Itenberg-Ragsdale curve of degree 10

A fundamental theorem in the study of classical planar curves is Bé-
zout’s theorem which states that two projective curves of degree \(d\) and \(e\)
have \(de\) points of intersections (in various meanings and under various as-
sumptions). Previously we discussed the case of lines which are supposed
to intersect in a single point. It is easy to convince oneself that a more
general statement should be true also tropically. For example, a smooth
non-degenerate curve of degree \(d\) has exactly \(d\) unbounded rays in each
of the directions \((1,1)\), \((-1,0)\) and \((0,-1)\) corresponding two the \(d\) dual
segments in the boundary of \(\Delta_d\). Thus two such tropical curves of degree
\(d\) and \(e\) can be translated in such a way that they only intersect in such
rays of fixed direction (for each curve) — in this case there are exactly
de\(d\) intersection points (see Figure 2.34). It is quite surprising that we can
actually state a quite strong version of Bézout’s theorem using only our
knowledge of dual subdivisions. For this purpose, let us assume that \(C\)
and \(D\) are non-degenerate tropical planar curves of degree \(d\) resp. \(e\) and
given by the polynomials \(f = \sum_{j \in \Delta_d} a_j x^{j_1} y^{j_2}\) resp. \(g = \sum_{j \in \Delta_e} b_j x^{j_1} y^{j_2}\). Furthermore, let us assume that \(C\) and \(D\) intersect ”transversally. This
means that that \(C \cap D\) is finite and each intersection point is the intersec-
tion of the relative interiors of an edge from \(C\) and an edge from \(D\). We
count each such point with an intersection multiplicity defined as follows.

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Figure 2.34: The 6 intersection points of a conic and a cubic

Assume the intersecting edges are $\sigma$ and $\tau$, then we define

$$\text{mult}(p) := \text{mult}(\sigma) \cdot \text{mult}(\tau) [\mathbb{Z}^2 : \mathbb{Z}\sigma + \mathbb{Z}\tau]$$

$$= \text{mult}(\sigma) \cdot \text{mult}(\tau) | \det(v, w)|,$$

where $v$ and $w$ are primitive integer vectors describing the slope of $\sigma$ resp. $\tau$. This multiplicity is one if and only if $v, w$ form a lattice basis of $\mathbb{Z}^2$.

Figure 2.35: Intersection points of multiplicity 1 and 2

Theorem 2.6.2

Let $C = V(f)$ and $D = V(g)$ be two non-degenerate tropical curves of degree $d$ resp. $e$ which intersect transversally. The number of intersection points, counted with multiplicities, is equal to $de$.

$$de = \sum_{p \in C \cap D} \text{mult}(p)$$

Proof. The main trick is to consider the union of the two curves $B := C \cup D = V(“fg”).$ Note that each intersection point $p \in C \cap D$ is a vertex of $C \cup D$ and thus has a corresponding dual cell $\sigma$ in $\text{SD}(“fg”).$ This dual cell must be a parallelogram whose pairs of parallel edges correspond to
the edges of $C$ resp. $D$ that intersect. Moreover, we see immediately that the definition of the multiplicity of $p$ is directly related to the volume of $\sigma$ by

$$\text{mult}(p) = \frac{\text{Vol}(\sigma)}{2}.$$ 

If a vertex of $B$ is not an intersection point, it is just a vertex of $C$ or $D$. Note that the corresponding dual triangle in $\text{SD}(“fg”)$ is just a shift of the corresponding triangle in $\text{SD}(f)$ resp. $\text{SD}(g)$. Indeed, assume that $v$ is a vertex of $C$ and therefore by assumption $v \notin D$. Then locally around $v$ the function $g$ is affine-linear, say $b_j + jx$, and therefore “$fg$” is locally equal to $f + b_j + jx = “fb_j x^j”, which corresponds to a shift of the dual picture by $j$. Hence the maximal cells of $\text{SD}(“fg”)$ are either triangles, which are in (volume-preserving) bijection to the triangles of $\text{SD}(f)$ and $\text{SD}(g)$, or parallelograms, which are in bijection to $C \cap D$. Moreover, we have

$$\text{Vol(parallelograms of SD(“fg”))} = \text{Vol}(\Delta_{d+e}) - \text{Vol}(\Delta_d) - \text{Vol}(\Delta_e) = (d + e)^2 - d^2 - e^2 = 2de.$$ 

This, together with the above formula, proves the result. \qed

![Figure 2.36: Bézout’s theorem in the case of two tropical conics](image)

Our proof is adapted from a paper of Vigeland (cf. [V1]). An alternative approach is used in [RGST]. The authors show that the number of intersection points (counted with multiplicities) is invariant under translations of the curves. Thus we can arrange the curves such that only the infinite rays of the curves intersect, as in Figure 2.34 In this case it is obvious that the number of intersection points is $de$. 

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3 Compactifying $\mathbb{R}^n$ — Projective space and other toric varieties

In the previous chapter we studied the tropical space $\mathbb{R}^n$ and its hypersurfaces. Let us recall that the base set of tropical arithmetics — the tropical numbers — is $\mathbb{T} = \mathbb{R} \cup \{\infty\}$. The set of units (with respect to "\cdot" = +) is given by the real numbers $\mathbb{T}^\times = \mathbb{R}$. Therefore, the space $\mathbb{R}^n = (\mathbb{T}^\times)^n$ plays the role of the tropical algebraic torus with classical analogue $(\mathbb{C}^\times)^n$. In particular, $\mathbb{R}^n$ is non-compact and it is often useful (for the same reasons as classically) to consider compactifications of $\mathbb{R}^n$ such as projective space. The most straightforward way to compactify an algebraic torus is given by toric varieties. Hence, in this chapter we will introduce tropical toric varieties. In fact, the concept of toric varieties can be easily translated to the tropical world. This is due to the fact that toric varieties are glued from affine pieces by using only monomial transformations. Thus no summing is involved an the strange idempotent nature of $(\mathbb{T}, \cdot)$ does not bother us. Instead, all we have to do when carrying out the standard constructions from toric geometry is to replace $(\mathbb{C}, \cdot)$ by $(\mathbb{T}, \cdot)$.

3.1 Affine space $\mathbb{T}^n$

The first step when compactifying $\mathbb{R}^n$ is obvious. We just take tropical affine space $\mathbb{T}^n \supset \mathbb{R}^n$ — with classical analogue $\mathbb{C}^n$. An element of $\mathbb{T}^n$ is an $n$-tuple of numbers $x = (x_1, \ldots, x_n)$, but now the value $x_i = -\infty$ is also valid. If all coordinate entries $x_i$ are finite, i.e. $x \in \mathbb{R}^n$, we call $x$ a finite point. The additional points $x \in \mathbb{T}^n \setminus \mathbb{R}^n$, with one or more coordinate entries equal to $-\infty$, are called infinite points. Refining this distinction, we may define for each subset $I \subseteq [n]$

$$R_I := \{x \in \mathbb{T}^n : x_i = -\infty \forall i \in I, x_i \neq -\infty \forall i \not\in I\}.$$ 

This gives us a natural stratification

$$\mathbb{T}^n = \bigsqcup_{I \subseteq [n]} R_I.$$
Note that for all $I$ we have an identification $R_I = R^{[n]\setminus I} \cong R^{n-|I|}$, which means that all the strata in $T^n \setminus R^n$ are tropical algebraic tori of smaller dimension (see Figure 3.1).

![Diagram of $T^2$ and its stratification](image)

Figure 3.1: $T^2$ and its stratification

How does $T^n$ look like geometrically? As the geometry of a single stratum $R_I \cong R^{n-|I|}$ is well-known, we are only interested in the case when different strata interact. Let, for example, $x \in R^n$ be a finite point, let $v \in \mathbb{Z}^n$ be a primitive direction vector and consider the ray $R = x + R_{\geq 0}v \subseteq R^n$. What happens when we take the closure $R \in T^n$? A few examples of rays in $T^2$ are depicted in Figure 3.2 (the vectors labeling each ray describe the particular choice of $v$).

![Diagram of rays in $T^2$ labelled by their direction vectors](image)

Figure 3.2: Rays in $T^2$ labelled by their direction vectors

We have to distinguish two cases. Whenever $v$ contains a strictly positive entry, no limit point exists, i.e. $\overline{R} = R$. Conversely, if all entries of $v$ are non-positive, a limit point $x = \overline{x}(x,v)$ is added to $\overline{R} = R \cup \{x\}$. However, figure 3.2 might give us a wrong impression of where this point $x$ is.
located. Its coordinates are

\[ x_i = \begin{cases} x_i & \text{if } v_i = 0, \\ -\infty & \text{if } v_i < 0. \end{cases} \]

In particular, whenever \( v_i < 0 \), the coordinate \( x_i \) does not depend on \( x \) at all. In our examples, it follows that two of the rays, though having different directions, run into the same limit point \(( -\infty, -\infty )\), independent of their exact starting point (see Figure 3.3). Only the two rays with direction \((-1,0)\) resp. \((0,-1)\) have limit points different from \(( -\infty, -\infty )\).

![Figure 3.3: The closures of the rays in \( T^2 \)](image)

More formally, we can describe this behaviour as follows. Let \( \sigma \subseteq R^n \) be the cone spanned by the negative standard basis vectors \(-e_1, \ldots, -e_n\). Its faces are of the form \( \sigma_I \), where \( I \subseteq [n] \) and \( \sigma_I \) is the cone spanned by \(-e_i, i \in I\). Then the above discussion can be summarized as follows. If \( v \not\in \sigma \), then no limit point is added and \( \overline{R} = R \). Conversely, if \( v \in \sigma \), let \( \sigma_f \) be the minimal face of \( \sigma \) containing \( v \). Then \( \overline{x} \in R_f \) and its finite coordinates are just given by the projection

\[ R^n \to R^{[n]}_I. \]

We see that the stratum containing \( \overline{x} \) is determined by the minimal face containing \( v \). In particular, only for the special directions \(-e_1, \ldots, -e_n\) our given ray ends up in strata of codimension 1.

### 3.2 Glueing via monomial maps

Classical smooth toric varieties are obtained from glueing various affine patches via monomial transformations. To tropicalize this construction,
3.2 Glueing via monomial maps

Figure 3.4: Tropical affine space $\mathbb{T}^3$

Let us recall that monomial transformations of $(\mathbb{C}^\times)^n$ are in one-to-one correspondence with linear automorphisms of $\mathbb{Z}^n$. Namely, let $A = (a_{ij}) \in \text{Mat}(n \times n, \mathbb{Z})$ be a square matrix with integer entries. The associated monomial transformation is

$$
\Phi_A : (\mathbb{C}^\times)^n \to (\mathbb{C}^\times)^n,
(z_1, \ldots, z_n) \mapsto \left( \prod_i z_i^{a_{1i}}, \ldots, \prod_i z_i^{a_{ni}} \right),
$$

but of course $A$ also defines the $\mathbb{Z}$-linear map

$$
\phi_A : \mathbb{R}^n \to \mathbb{R}^n,
x = (x_1, \ldots, x_n) \mapsto Ax = \left( \sum_i a_{1i}x_i, \ldots, \sum_i a_{ni}x_i \right).
$$

For invertibility (over $\mathbb{Z}$) we require $\det(A) = \pm 1$. From the point of view of tropical geometry, the linear map $\phi_A : \mathbb{R}^n \to \mathbb{R}^n$ is just the “tropicalization” of $\Phi_A$, keeping in mind that \( \prod x_i^{a_{ki}} = \sum a_{ki}x_i \). In other words: Tropical monomial transformations of $\mathbb{R}^n$ are $\mathbb{Z}$-invertible $\mathbb{Z}$-linear maps of $\mathbb{R}^n$. Moreover, $\phi_A$ can be extended to parts of $\mathbb{T}^n$ — the only thing we have to keep in mind is that $a_{ki} \cdot (-\infty)$ is not defined when $a_{ki} < 0$. Let $\mathbb{T}^n = \bigsqcup R_I$ be the stratification of $\mathbb{T}^n$ as before. Then $\phi_A$ can be extended to the stratum $R_I$ if and only if $a_{jki} \geq 0$ for all $i \in I$. Again, this is completely analogous to the classical situation, and therefore the construction of toric varieties from affine patches can be translated to the tropical world.
Definition 3.2.1
Let $F = \{\sigma\}$ be a unimodular rational fan in $\mathbb{R}^n$ (where unimodular means that each cone is generated by parts of a $\mathbb{Z}^n$-basis). Then there exists a corresponding tropical smooth toric variety $TX(F)$ or just $X(F)$ with the following properties.

(a) $X(F)$ contains $\mathbb{R}^n$ as an open dense subset and addition on $\mathbb{R}^n$ extends to an action of $\mathbb{R}^n$ on $X(F)$,

(b) The orbits of this action can be labelled one-to-one by the cones of $F$, and for each orbit $\mathbb{R}_{\sigma}$ we have a canonical identification $\mathbb{R}_{\sigma} \cong \mathbb{R}^n/\mathbb{Z}\sigma$.

(c) Let $x \in \mathbb{R}^n$ be a finite point, and let $v \in \mathbb{R}^n$ be a direction vector. Then $x + \lambda v$ converges for $\lambda \in \mathbb{R}$, $\lambda \to \infty$ to a point $\pi$ if and only if $v$ is contained in the support of $F$. Moreover, if $v$ lies in the relative interior of $\sigma$, then $\pi \in \mathbb{R}_{\sigma}$ and, via the above identification, $\pi$ is given by the image of $x$ under the projection $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}\sigma$.

From the previous discussion it should be clear how to obtain $TX(F)$ with these properties. In fact, let $CX(F)$ be the corresponding complex smooth toric variety. Then $CX(F)$ is obtained from glueing together affine patches $U_i$ (with $(\mathbb{C}^\times)^n \subseteq U_i \subseteq \mathbb{C}^n$) via monomial transformations $\Phi_{A^{ij}} : U_i \to U_j$ given by the $\mathbb{Z}$-invertible matrices $A^{ij}$, i.e.

$$CX(F) = \bigsqcup_i U_i / z \sim \Phi_{A^{ij}}(z).$$

Then $TX(F)$ can be constructed as

$$TX(F) = \bigsqcup_i V_i / x \sim A^{ij}x,$$

with $\mathbb{R}^n \subseteq V_i \subseteq \mathbb{T}^n$ the corresponding tropical patches and glueing along the $\mathbb{Z}$-linear transformations $A^{ij} : V_i \to V_j$. Note that the compatibility of the glueing maps is inherited because $\Phi_{A^{ik}} \circ \Phi_{A^{ij}} = \Phi_{A^{ij}}$ implies $A^{jk}A^{ij} = A^{ik}$. The properties (a), (b), (c) can be shown in exactly the same fashion as in the classical case. In particular, property (c) reflects the corresponding statement about limits of (translations of) one-parameter subgroups in $(\mathbb{C}^\times)^n$ of the form $(z_1t^{v_1}, \ldots, z_n t^{v_n})$ with $z \in (\mathbb{C}^\times)^n$, $t \in \mathbb{C}^\times$, $v \in \mathbb{Z}^n$. 

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Remark 3.2.2
We restrict here to smooth toric varieties only for simplicity. In fact, any toric variety has a tropical counterpart — all we have to do is to allow more general affine models to be glued together. For a general (not necessarily unimodular) cone $\sigma$, the affine patch can be described as $\text{Hom}(S_\sigma, T)$, i.e. the set of monoid homomorphisms from $S_\sigma = \sigma^\vee \cap (\mathbb{Z}^n)^*$ to $(T, \cdot)$ (as usual, $\sigma^\vee$ denotes the dual cone). More details can be found in [P2, section 3].

3.3 Tropical projective space

To illustrate the previous discussion, let us study the most important example — projective space. We start with the symmetric description of $\mathbb{P}^n$ as a quotient and then recover the fan by using property [c] above.

Definition 3.3.1
Tropical projective space is

$$\mathbb{P}^n : = \mathbb{T}^{n+1} / \sim,$$

where the equivalence relation $\sim$ is given by

$$x \sim \lambda \cdot x \text{ for all } \lambda \in \mathbb{T}^*$$

or in other words

$$(x_0, \ldots, x_n) \sim (x_0 + \lambda, \ldots, x_n + \lambda) \text{ for all } \lambda \in \mathbb{R}.$$

As usual, projective coordinates are denoted by $(x_0 : \ldots : x_n)$. $\mathbb{P}^n$ can be covered by affine patches $U_i = \{x \in \mathbb{P}^n : x_i \neq -\infty\}$ which can be identified with $\mathbb{T}^n$ via

$$(x_0 : \ldots : x_n) \mapsto (x_0 - x_i, \ldots, x_i - x_i, x_{i+1} - x_i, \ldots, x_n - x_i).$$

On the overlaps $U_i \cap U_j$ we get induced maps glueing two copies of $\mathbb{T}^n$ via

$$(y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) \mapsto (y_0 - y_j, \ldots, y_{i-1} - y_j, -y_j, y_{i+1} - y_j, \ldots, y_n - y_j).$$

Thus $\mathbb{P}^n$ is obtained by glueing $n+1$ copies of $\mathbb{T}^n$ along these $\mathbb{Z}$-linear transformations (or tropical monomial transformations). Furthermore, $\mathbb{P}^n$ contains $\mathbb{R}^{n+1}/\mathbb{R}(1, \ldots, 1) \cong \mathbb{R}^n$ as an open dense subset and this
torus acts on $\mathbb{T}P^n$ by coordinatewise addition. The orbits of this action are of the form

$$R_I = \{(x_0 : \ldots : x_n) \in \mathbb{T}P^n : x_i = -\infty \forall i \in I, x_i \neq -\infty \forall i \notin I\}$$

with $I \subset \{0, \ldots, n\}$.

Let us consider $\mathbb{T}P^2$ in more detail. By breaking symmetry, we may specify $\mathbb{R}^2 \subset \mathbb{T}^2 \cong U_0$ as our preferred space for pictures. Then the glueing may be visualized as follows.

![Figure 3.5: Glueing $\mathbb{T}P^2$ from its three affine patches](image)

To obtain the fan of $\mathbb{T}P^2$, we may use property (c) backwards and ask, for any $v \in \mathbb{R}^2$, in which orbit the limit point of $x + \lambda v$ lies (where $x$ is an arbitrary finite point and $\lambda \to \infty$). Using the affine patches, this question reduces to the $\mathbb{T}^n$-case considered before. We find that $\mathbb{R}^2$ is subdivided into a complete fan consisting of 3 rays spanned by $(1,1), (-1,0)$ and $(0,-1)$ — the distinguished directions of $\mathbb{T}P^2$ (note that in projective coordinates we may write these vectors more symmetric as $(-1,0,0), (0,-1,0)$ and $(0,0,-1)$). If $v$ is not one of these distinguished directions, then the ray $x + \lambda v$ necessarily converges to one of the fix points of the torus action. Conversely, if $v$ for example $(1,1)$, the limit point is contained in the one-dimensional orbit $R\{0\}$ (see Figure 3.6).

**Remark 3.3.2**

Note that the fans we associate to tropical toric varieties are the reflection at the origin of fans commonly used in classical toric geometry. This because it seems more natural tropically to follow $v$ in positive direction,
3.3 Tropical projective space

![Figure 3.6: Some rays and their limit points in TP²](image)

i.e. $\lambda \to \infty$, whereas in the classical case one usually performs $t \to 0$ (for $t \in \mathbb{C}^*$).

![Figure 3.7: The stratification and the fan of TP²](image)

In our pictures we often depict TP² as a triangle, to which it is indeed homeomorphic. Even better, the stratification of the triangle given by its faces matches with stratification of TP² by orbits. However, we should keep in mind that not all (for example, metric) properties are reflected properly by this picture. The interior of the triangle represents the set of finite points $\mathbb{R}^2 \subset TP²$. The infinite points in TP² \ $\mathbb{R}^2$ (i.e. the boundary of the triangle) are infinitely far away from all finite points. Moreover, in the finite part there are 3 distinguished directions for which rays converge to a codimension one orbit (i.e. the relative interior of an edge of the triangle). All other rays converge to the corner points of the triangle.
All what has been said can be generalized to the case of higher dimensions. Indeed, $TP^n$ is topologically an $n$-simplex (with matching stratifications). The interior of the simplex corresponds to the big torus $R^n$. There are $n + 1$ distinguished directions

$$(1,\ldots,1), (-1,0,\ldots,0), \ldots, (0,\ldots,0,-1)$$

in correspondence with the $n + 1$ maximal faces of the simplex.

Figure 3.8: $TP^3$ and its distinguished directions
4 Projective tropical geometry

4.1 Compactified cycles

In chapter 2 we made the observation that tropical hypersurfaces in $\mathbb{R}^n$ are polyhedral complexes that satisfy the balancing condition (cf. 2.5.6)

$$\sum_{\sigma \text{ facet}} \sum_{\tau \subset \sigma} \text{mult}(\tau) v_{\tau \cap \sigma} = 0 \mod \mathbb{R}$$

for all cells of codimension one. Note that the balancing condition makes sense for polyhedral complexes of any dimension. Indeed, such balanced complexes are the basic objects in tropical geometry. Compared to the case of hypersurfaces, we will slightly shift our point of view by focusing on the underlying polyhedral sets and not insisting on a fixed subdivision into cells. Instead, when needed (for example to formulate the balancing condition) we choose a subdivision which fits our demands and show that a certain construction/property does not depend on the choice.

**Definition 4.1.1**

A tropical $k$-cycle $X$ in $\mathbb{R}^n$ is a weighted polyhedral complex in $\mathbb{R}^n$ of pure dimension $k$ that satisfies the balancing condition from definition [2.5.6]. Two $k$-cycles $X,X'$ are identified if $|X| = |X'|$ and if $\text{mult}(\sigma) = \text{mult}(\sigma')$ whenever $\sigma$ and $\sigma'$ are facets of $X$ resp. $X'$ such that $\dim(\sigma \cap \sigma') = k$.

Hence, a tropical cycle is an equivalence class of weighted polyhedral complexes. Note that if a weighted complex is balanced, then any equivalent weighted complex is balanced, too.

We might reformulate the definition of a $k$-cycle as follows. A polyhedral set $X$ (of pure dimension $k$) in $\mathbb{R}^n$ is just a union of polyhedra $X = \bigcup_i \sigma_i$ (of the same dimension $k$). A point $x \in X$ is called generic if, for a suitable representation of $X$ as union of polyhedra, $x$ is not contained in any of the proper faces of the $\sigma_i$. In other words, a small neighbourhood of $x$ in $X$ looks like an affine space in $\mathbb{R}^n$. To make a polyhedral set weighted, we equip each connected component of the set of generic points with a non-zero integer weight. This means we consider a locally constant function $\text{mult}(x)$ on the set of generic points. Any polyhedral set $X$ can be written as the support set of a polyhedral complex, and such a choice we call a polyhedral structure on $X$. If $X$ is weighted, each polyhedral structure
inherits weights by \( \text{mult}(\sigma) = \text{mult}(x) \), where \( x \) is a point in the relative interior of a facet \( \sigma \) (in particular, \( x \) is generic). Obviously, two such polyhedral structures are equivalent in the sense of definition 4.1.1. Of course, a balanced polyhedral complex also defines a weighted polyhedral set. Thus we can reformulate the definition in the following way.

**Definition 4.1.2**
A tropical \( k \)-cycle \( X \) in \( \mathbb{R}^n \) is a weighted polyhedral set in \( \mathbb{R}^n \) of pure dimension \( k \) such that, for any polyhedral structure of \( X \), the balancing condition is satisfied.

Now let \( Y \) be any tropical toric variety (e.g. \( \mathbb{T}^n \) or \( \mathbb{TP}^n \)). Then we can easily extend “both” definitions by taking closures. Recall that \( Y \) is a disjoint union of torus orbits \( O_i = \mathbb{R}^{n_i}, i = 1, \ldots, l \).

**Definition 4.1.3**
A tropical \( k \)-cycle \( X \) in \( Y \) of pure sedentarity \( s \) is the closure in \( Y \) of a tropical \( k \)-cycle in a torus orbit \( O_i = \mathbb{R}^{n_i} \), where \( s = \dim(Y) - n_i \). A general tropical \( k \)-cycle \( X \) in \( Y \) is a union \( X = \bigcup_{i=1}^l X_i \) of \( k \)-cycles \( X_i \) of pure sedentarity for each orbit \( O_i \) (of course, some \( X_i \) might be empty).

The set of all \( k \)-cycles of \( Y \) is denoted by \( Z_k(Y) \) and forms a group under taking unions and adding weights. That is to say, the sum \( X + X' \) is supported on \( X \cup X' \) and the weight of a generic point \( x \) is given by \( \text{mult}_X(x) + \text{mult}_{X'}(x) \) (if \( x \in X \cap X' \)). By definition, we have

\[
Z_k(Y) \cong \bigoplus_{i=1}^l Z_k(O_i),
\]

where the sum runs through all torus orbits of \( Y \).

The sedentarity of a cycle \( X \) of pure sedentarity is denoted by \( \text{sed}(X) \). It describes the number of coordinates equal to \(-\infty\) for a generic point of \( X \) in any toric chart.

Another description of cycles in \( Y \) can be obtained as follows. We define a polyhedron \( \sigma \) in \( Y \) to be the closure in \( Y \) of a (usual) polyhedron \( \sigma' \) in one of the orbits \( O_i = \mathbb{R}^{n_i} \). It is easy to see that the intersection \( \sigma \cap O_j \) of such a generalized polyhedron with any (other) torus orbit is a polyhedron again. So we can define the faces of \( \sigma \) to be the closures of all faces of all intersections \( \sigma \cap O_j, j = 1, \ldots, l \) (in particular, for \( j = i \) we get the closures of the “finite” faces of \( \sigma \)). Again we call \( \dim(Y) - n_i \) the sedentarity of \( \sigma \) and denote it by \( \text{sed}(\sigma) \).
Chapter 4: Projective tropical geometry

\[ \sigma' \]

\[ \mathbb{R}^2 \]

\[ \mathbb{T}^2 \]

With this notion of polyhedra in \( Y \), we can repeat the definitions of polyhedral complexes, polyhedral sets, etc. word by word and can define a tropical cycle in \( Y \), as before in the \( \mathbb{R}^n \) case, as an equivalence class of polyhedral complexes resp. a polyhedral set. The only further adaption concerns the balancing condition.

A (generalized) weighted polyhedral complex in \( Y \) is called balanced if at any codimension one cell \( \tau \) the balancing condition

\[ \sum_{\tau \subset \sigma, \text{facet}, \text{sed}(\sigma) = \text{sed}(\tau)} \text{mult}(\sigma) v_{\sigma/\tau} = 0 \mod R\tau \]  

(4.1)

holds. Here, we sum up over all facets \( \sigma \) containing \( \tau \) and of the same sedentarity \( \text{sed}(\sigma) = \text{sed}(\tau) \). This ensures that all the vectors \( v_{\sigma/\tau} \) make sense and lie in the same vectorspace.

A tropical \( k \)-cycle \( X \) in \( Y \) is thus the same as a weighted polyhedral set in \( Y \) of pure dimension \( k \) such that the polyhedral structures of \( X \) satisfies the generalized balancing condition above.

Remark 4.1.4

We allow our multiplicities to be integer, in particular, zero. However, in practice, facets (resp. generic points) of multiplicity \( \text{mult}(\sigma) = 0 \) are of no interest and are therefore discarded from \( X \). So, basically we assume that multiplicities are always non-zero and whenever a certain construction (e.g. summing two cycles) produces facets of weight zero, we just remove them.

Example 4.1.5

For each \( k \) with \( 0 \leq k \leq n \) there exists a \( k \)-cycle in \( \mathbb{T}P^n \) called the standard \( k \)-plane \( H_k \) of \( \mathbb{T}P^n \). It is of sedentarity 0, as it comes as the closure of
the following fan in $\mathbb{R}^n$. Let $e_0, \ldots, e_n$ be the $\text{TP}^n$-standard directions in $\mathbb{R}^n$. For any subset $I$ of $\{0, \ldots, n\}$ of $k$ or less elements, we denote by $\sigma_I$ the cone generated by the vectors $e_i, i \in I$. The collection of all these cones, with weights 1 on the facets, forms a balanced polyhedral fan $F_k$. The extremal cases are $F_0 = \{0\}$ and $F_n = \mathbb{R}^n$. For $n - 1$, we get the standard hyperplane $H_{n-1} = V(x_1 + \cdots + x_n + 0)$. The closure of $F_k$ in $\text{TP}^n$ is the standard $k$-plane $H_k$ of $\text{TP}^n$. For any torus-invariant subspace $\text{TP}^I \subseteq \text{TP}^n$, the intersection $H_k \cap \text{TP}^I$ is just the standard plane in $\text{TP}^I$ of codimension $n - k$.

![Figure 4.2: A tropical plane in $\text{TP}^3$](image)

### 4.2 Stable intersection

When we consider cycles in $\text{TP}^n$, we would like to speak about their degree. A way of defining the degree of a projective cycle is by intersecting it with a “plane” of complementary dimension. The intersection construction needed here is called stable intersection and is the topic of this section. Given two cycles $X$ and $Y$ in $\mathbb{R}^n$ of pure dimension $k$ resp. $l$, our goal is to construct a cycle $X \cdot Y$ of pure dimension $k + l - n$ which is supported on $X \cap Y$. Note that $X \cap Y$ is obviously a polyhedral set again, but might have parts of dimension bigger than $k + l - n$. So our approach is as follows: First we define $X \cdot Y$ in nice cases, namely when the intersection is transversal. Then, in the general case, we translate one of the cycles.
slightly. For generic translations, the resulting cycles are transversal and we can define the stable intersection as the limit of the transversal intersections when moving the translated cycles back to the original one. Let us start with the definition of the transversal case. In the following, whenever \(X\) and \(Y\) are cycles, we may choose some polyhedral structure on them to induce a polyhedral structure on \(X \cap Y\) whose cells are given by \(\sigma \cap \sigma', \sigma \in X, \sigma' \in Y\).

**Definition 4.2.1** (Transversal intersection)
Let \(X\) and \(Y\) be two cycles in \(\mathbb{R}^n\) of pure dimension \(k\) resp. \(l\). We say \(X\) and \(Y\) intersect transversally if \(X \cap Y\) is of dimension \(k + l - n\) and if every facet \(\tau\) of \(X \cap Y\) can be written uniquely as \(\tau = \sigma \cap \sigma'\) with facets \(\sigma, \sigma'\) of \(X\) resp. \(Y\) (for suitable polyhedral structures).

In this case, we define the transversal intersection \(X \cdot Y\) to be the polyhedral complex \(X \cap Y\) with multiplicities

\[
\omega(\tau) = \omega(\sigma) \cdot \omega(\sigma') \cdot [\mathbb{Z}^n : \mathbb{Z}\sigma + \mathbb{Z}\sigma'],
\]

(where \(\tau, \sigma, \sigma'\) are as before).

**Proposition 4.2.2**
In the transversal case, \(X \cdot Y\) is balanced, i.e. defines a cycle in \(\mathbb{R}^n\) of pure dimension \(k + l - n\).

**Proof.** As \(X\) and \(Y\) intersect transversally, any codimension one cell of \(X \cap Y\) lies in the codimension one skeleton of either \(X\) or \(Y\). We may assume the former, i.e. the codimension one cell can be written uniquely as \(\tau \cap \sigma'\), where \(\tau\) is a codimension one cell of \(X\) and \(\sigma'\) is a facet of \(Y\). Let \(\sigma \supset \tau\) be a facet of \(X\). We may compare primitive generators of \(\sigma \cap \sigma'\) (modulo \(\tau \cap \sigma'\)) and \(\sigma\) (modulo \(\tau\)). By definition we have

\[
\mathbb{Z}\nu_{\sigma \cap \sigma' / \tau \cap \sigma'} + \mathbb{Z}\tau = (\mathbb{Z}\sigma + \mathbb{Z}\sigma') + \mathbb{Z}\tau,
\]

\[
\mathbb{Z}\nu_{\sigma / \tau} + \mathbb{Z}\tau = \mathbb{Z}\sigma,
\]
4.2 Stable intersection

and therefore

\[ v_{\sigma \cap \sigma'} = [Z\sigma : (Z\sigma \cap Z\sigma') + Z\tau] \cdot v_{\sigma \cap \tau} \mod Z\tau. \]

Plugging in the identity

\[ [Z\sigma : (Z\sigma \cap Z\sigma') + Z\tau] = [Z\sigma + Z\sigma' : Z\tau + Z\sigma']. \]

and multiplying by \([Z^n : Z\sigma + Z\sigma']\) we get

\[ [Z^n : Z\sigma + Z\sigma'] \cdot v_{\sigma \cap \sigma'} = [Z^n : Z\tau + Z\sigma'] \cdot v_{\sigma \cap \tau} \mod Z\tau. \]

Now let \(\sigma_1, \ldots, \sigma_m\) be the collection of facets containing \(\tau\) (with primitive vectors \(v_i\)). Then the facets of \(X \cap Y\) containing \(\tau \cap \sigma'\) are \(\sigma_i \cap \sigma'\) (with primitive vectors \(w_i\)) and the above relation gives

\[
\sum_{i=1}^{m} \omega(\sigma_i \cap \sigma') w_i = \omega(\sigma') \sum_{i=1}^{m} \omega(\sigma_i) [Z^n : Z\tau + Z\sigma'] w_i = \omega(\sigma') \cdot [Z^n : Z\tau + Z\sigma'] \cdot \sum_{i=1}^{m} \omega(\sigma_i) v_i = 0,
\]

i.e. the balancing condition around \(\tau\) implies the balancing condition around \(\tau \cap \sigma'\).

To extend the intersection of two cycles to the non-transversal case, we need the following statements. As usual, the degree of a zero-dimensional cycle \(X\), which is just a weighted formal sum of points \(X = \sum \omega_p \{p\}\), is the sum of all multiplicities \(\deg(X) = \sum \omega_p\).

**Proposition 4.2.3**

Let \(X\) and \(Y\) be two cycles in \(\mathbb{R}^n\) of pure dimensions \(k\) resp. \(l\). Then the following holds.

(a) For a generic vector \(v \in \mathbb{R}^n\), the intersection of \(X\) and the translation \(Y + v\) is transversal.

(b) Assume \(l = n - k\). Then \(\deg(X \cdot (Y + v))\) is the same for all such generic \(v \in \mathbb{R}^n\).

**Proof.** For the first statement, we consider all pairs of cells \(\sigma \in X, \sigma' \in Y\). If \(R\sigma + R\sigma' \neq \mathbb{R}^n\), then for any \(v \in \mathbb{R}^n \setminus (R\sigma + R\sigma')\), we have \(R\sigma \cap (R\sigma' + v) = \emptyset\) and thus \(\sigma \cap (\sigma' + v) = \emptyset\). It follows that for generic \(v\), all intersections
For the second statement, let us first refine the previous consideration. For any vector $v \in \mathbb{R}^n$, the data of pairs of cells $\sigma \in X, \sigma' \in Y$ such that $\sigma \cap (\sigma' + v) \neq \emptyset$ is called the intersection type of $v$. It is easy to check that the set of vectors with given intersection type form (the interior of) a polyhedron, and this subdivides $\mathbb{R}^n$ into a complete polyhedral complex. $X$ and $Y + v$ intersect transversally if $v$ is contained in the interior of a maximal cell of this subdivision (in fact, the condition of transversal intersection is equivalent to the fact that a small perturbation of $v$ does not change the intersection type). Now assume we are given two such generic vectors. If they are contained in the same cell, the degree is obviously constant (as it can be computed in terms of $X$, $Y$ and the intersection type). If not, we can connect the two vectors by passing through at most codimension one cells of the subdivision of $\mathbb{R}^n$. In other words, it suffices to study intersection types where $\sigma \cap (\sigma' + v) = \emptyset$ whenever the sum of the codimensions of $\sigma$ and $\sigma'$ is greater than one (otherwise, the intersection type is of higher codimension). Therefore let $\tau$ be a codimension one cell, say of $X$, such that there is a (unique) facet $\sigma'$ of $Y$ with $\tau \cap (\sigma' + v) \neq \emptyset$. It is enough to show the invariance of the degree locally at $\tau$. That is to say, we may assume that $Y = \{\sigma'\}$ is a linear space and $X = \{\tau, \sigma_1, \ldots, \sigma_m\}$ is a fan with exactly one codimension one cell $\tau$ and facets $\sigma_i$. If $\dim(R\tau + R\sigma') < n - 1$, for generic $v$ we have $F \cap (G + v) = \emptyset$ and $\deg(X \cdot (Y + v)) = 0$ is invariant. So we may assume that $H = R\tau + R\sigma'$ is a hyperplane. Let $v_i$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.4}
\caption{A non-transversal intersection and two small deformations}
\end{figure}
be primitive generators for $\sigma_i \supset \tau$. The balancing condition states
\[
\sum_{i=1}^{m} \omega(\sigma_i)v_i = 0 \mod R\tau.
\]

Choose a primitive generator $w$ of $\mathbb{Z}^n/\mathbb{Z}H$ and write $v_i = \lambda_i w \mod H$

![Diagram of facets and translation vectors]

Figure 4.5: Passing a codimension one case

for (unique) $\lambda_i \in \mathbb{Z}$. It follows
\[
\sum_{i=1}^{m} \omega(\sigma_i)\lambda_i = 0. \tag{4.3}
\]

On the other hand, we may write
\[
[Z^n : Z\sigma_i + Z\sigma'] = |\lambda_i| \cdot [ZH : Z\tau + Z\sigma'].
\]

Now, if we move $Y$ towards $w$, (i.e. $v$ is a translation vector which points in the same direction as $w$ modulo $H$), then $Y + v$ intersects the facets $\sigma_i$ with $\lambda_i > 0$ and
\[
\deg(X \cdot (Y + v)) = \omega(\sigma') [ZH : Z\tau + Z\sigma'] \sum_{\lambda_i > 0}^{i} \omega(\sigma_i)\lambda_i.
\]

Contrary, when moving $Y$ towards $-w$, we must sum over $i$ with $\lambda_i < 0$ instead. By equation (4.3), both sums are equal.

We are now ready to define the stable intersection of two cycles in the general case.

**Definition 4.2.4**
Let $X$ and $Y$ be two cycles in $\mathbb{R}^n$ of pure dimension $k$ resp. $l$. We define the *stable intersection* $X \cdot Y$ to be
\[
X \cdot Y := \lim_{\epsilon \to 0} X \cdot (Y + \epsilon v),
\]
where \( v \in \mathbb{R}^n \) is a vector such that \( X \) and \( (Y + \epsilon v) \) intersect transversally for small \( \epsilon > 0 \). To be more precise, \( X \cdot Y \) is supported on the \( k + l - n \)-skeleton \( X \cap Y \) and each \( k + l - n \)-cell \( \tau \) of \( X \cap Y \) gets the multiplicity
\[
\omega(\tau) = \sum_{\sigma, \sigma'} \omega(\sigma) \omega(\sigma') [Z^n : Z\sigma + Z\sigma'],
\]
where the sum runs through the pairs of facets \( \sigma \in X, \sigma' \in Y \) with \( \tau = \sigma \cap \sigma' \) and \( \sigma \cap (\sigma + \epsilon v) \neq \emptyset \) for small \( \epsilon > 0 \).

We still have to show that \( X \cdot Y \) does not depend on \( v \) and is balanced. First, let us introduce some terminology. Let \( X \) be a polyhedral complex in \( \mathbb{R}^n \) and fix a cell \( \tau \in X \). For any cell \( \sigma \in X \) containing \( \tau \), we define \( \sigma / \tau \) to be the polyhedral cone in \( \mathbb{R}^n / \mathbb{R}\tau \) generated by all differences \( x - y, x \in \sigma, y \in \tau \). The collection of \( \sigma / \tau \) for all \( \sigma \supseteq \tau \) forms a fan in \( \mathbb{R}^n / \mathbb{R}\tau \) denoted by \( \text{Star}_X(\tau) \) and called the star of \( X \) at \( \tau \). If \( X \) is weighted, \( \text{Star}_X(\tau) \) carries induced weights. \( X \) is balanced if and only if \( \text{Star}_X(\tau) \) is balanced for all \( \tau \in X \) (in fact, \( X \) is balanced by definition if \( \text{Star}_X(\tau) \) is balanced whenever \( \tau \) is of codimension one).

**Example 4.2.5**
Let \( H_k \) be the standard \( k \)-plane in \( \mathbb{TP}^n \) and consider \( F_k = H_k \cap \mathbb{R}^n \). Let \( \sigma_I \) be a cell of \( F_k \). Then \( \text{Star}_{F_k}(\sigma_I) \) is again (the finite part of) a standard plane of codimension \( n - k \), where the standard directions in \( \mathbb{R}^n / \mathbb{R}\sigma_I \) are considered to be the images of \( e_j, j \notin I \). This should be compared to our previous observation that \( H_k \cap \mathbb{TP}^I \), where \( \mathbb{TP}^I \) is the corresponding toric coordinate plane, is a standard plane, too. In fact, \( H_k \cap \mathbb{TP}^I \) can naturally be considered as the closure of \( \text{Star}_{F_k}(\sigma_I) \).

**Proposition 4.2.6**
Let \( X \) and \( Y \) be two cycles in \( \mathbb{R}^n \) of pure dimension \( k \) resp. \( l \). Then the following holds.

(a) **The stable intersection** \( X \cdot Y \) **is well-defined**, i.e. does not depend on
the choice of \( v \in \mathbb{R}^n \).

(b) **The stable intersection can be computed locally**. In formulas, we have
\[
\omega(\tau) = \deg(\text{Star}_X(\tau) \cdot \text{Star}_Y(\tau)),
\]
and, more general, for any cell \( \rho \)
\[
\text{Star}_{X,Y}(\rho) = \text{Star}_X(\tau) \cdot \text{Star}_Y(\tau).
\]
4.2 Stable intersection

(c) $X \cdot Y$ is balanced and therefore defines a cycle in $\mathbb{R}^n$ of dimension $k + l - n$.

(d) Assume $l = n - k$. Then
\[ \deg(X \cdot Y) = \deg(X \cdot (Y + v)) \]
for all $v \in \mathbb{R}^n$.

Proof. We first prove (b) in the following sense. For a fixed generic $v$, we claim that the multiplicity of $\tau$, with respect to $v$, is
\[ \omega(\tau) = \deg(\text{Star}_X(\tau) \cdot \text{Star}_Y(\tau + [v])). \]
This follows directly from the description of $\omega(\tau)$ we gave in the definition of stable intersection (and the observation that dividing by $R\tau$ does not affect any of the lattice indices). With the help of our earlier invariance result 4.2.3, we can therefore conclude that $X \cdot Y$ is independent of $v$ and that (b) holds.

Assertion (c) is already proven for the transversal case in Proposition 4.2.2. It remains to show that the balancing condition is preserved when taking the limit. By (b) we may assume that $X$ and $Y$ are fans. Let $\rho$ be a cell in $X \cap Y$ of dimension $k + l - n - 1$. For a fixed generic $v$, let $\tau_1, \ldots, \tau_m$ be all the codimension one cells in $X \cdot (Y + v)$ which converge to $\rho$ when $v \to 0$. From the transversal case, we know that
\[ \sum_{i=1}^m \text{Star}_{X \cdot (Y + v)}(\tau_i) \] is balanced, and this sum differs from $\text{Star}_{X \cdot Y}(\rho)$ only in the following way. We might have a facet $\sigma \cap (\sigma' + v)$ of $X \cdot (Y + v)$ which degenerates in the limit, i.e. $\sigma \cap \sigma' = \rho$. However, it is easy to check that in this case $\sigma \cap (\sigma' + v)$ contains exactly two of the $\tau_i$, say $\tau_1$ and $\tau_2$, and the primitive generators of $\sigma \cap (\sigma' + v)$ with respect to $\tau_1$ and $\tau_2$ are exactly opposite. Thus we obtain $\text{Star}_{X \cdot Y}(\rho)$ from the sum in equation (4.4) by substracting (multiples) of linear subspaces (e.g. $R(\sigma \cap (\sigma' + v))$). In particular, $\text{Star}_{X \cdot Y}(\rho)$ is still balanced, which proves (c).

Finally, assertion (d) follows from the definitions and Proposition 4.2.3.

Stable intersection gives us a map
\[ \cdot : Z_k(\mathbb{R}^n) \times Z_l(\mathbb{R}^n) \to Z_{k+l-n}(\mathbb{R}^n). \]
It is worth to emphasize that this intersection product is constructed on the non-compact space $\mathbb{R}^n$ and without passing to equivalence classes of any kind — even in the case of non-transversal (or self-)intersections. The product satisfies the following properties.

**Proposition 4.2.7**

*Stable intersection is associative, commutative, bilinear and its neutral element is given by the “fundamental” cycle $\mathbb{R}^n$. We have $X \cdot Y \subseteq X \cap Y$.*

**Proof.** Commutativity, Bilinearity and the neutral element assertion follow directly from the definitions. Also by definition, $X \cdot Y$ is contained in the $k + l - n$-skeleton of $X \cap Y$ (note that this inclusion might still be proper as some of the multiplicities $\omega(\tau)$ might be zero).

It remains to show associativity. We first study the situation where all of the involved intersections are transversal. In this case, each facet $\tau$ of $X \cap Y \cap Z$ can be written uniquely as $\tau = \sigma_1 \cap \sigma_2 \cap \sigma_3$ for facets of $X$, $Y$ and $Z$ respectively. The multiplicity of $\tau$ in $(X \cdot Y) \cdot Z$ is given by

$$\omega(\tau) = \omega(\sigma_1)\omega(\sigma_2)\omega(\sigma_3) \left[ Z^n : Z\sigma_1 + Z\sigma_2 \right] \left[ Z^n : (Z\sigma_1 \cap Z\sigma_2) + Z\sigma_3 \right].$$

After plugging in the identity

$$[Z^n : (Z\sigma_1 \cap Z\sigma_2) + Z\sigma_3] = [Z^n : Z\sigma_2 + Z\sigma_3] \left[ Z\sigma_2 : (Z\sigma_1 \cap Z\sigma_2) + (Z\sigma_2 \cap Z\sigma_3) \right],$$

we obtain an expression which is symmetric in $\sigma_1$ and $\sigma_3$ and thus is equal to the multiplicity in $X \cdot (Y \cdot Z)$. For the general case, we just have to unwind our definition and observe that a triple stable intersection can be computed by moving two of the cycles simultaneously, i.e.

$$(X \cdot Y) \cdot Z = \lim_{v \to 0} \lim_{u \to 0} (X \cdot (Y + v)) \cdot (Z + u).$$

Therefore the previous argument is sufficient.

**Example 4.2.8**

Let us consider the stable intersection of (the finite parts of) standard planes $H_k$ in $\text{TP}^n$. Let us denote the finite parts by $F_k = H_k \cap \mathbb{R}^n$. We claim

$$F_k \cdot F_l = F_m,$$

with $m = k + l - n$. By definitions, the $k + l - n$-skeleton of $F_k \cap F_l$ is exactly the support of $F_m$, so we just have to show that all facets appear with multiplicity 1 in $F_k \cdot F_l$. To do this, we use locality and compute the
multiplicity of a facet by intersecting the corresponding stars instead. We mentioned before that these stars are standard planes again (cf. example [4.2.5]), so the claim can be reduced to check that

\[ F_k \cdot F_{n-k} = \{0\} \]

with multiplicity 1. This may be verified explicitly by choosing the translation vector

\[ v := (-1, \ldots, -1, 1, \ldots, 1), \]

for example. Indeed, in this case, the only facets of \( F_k \) and \( F_{n-k} + v \) that intersect are \( \sigma_{\{1,\ldots,k\}} \) and \( \sigma_{\{k+1,\ldots,n\}} + v \). The intersection point

\[ \sum_{i=1}^{k} e_i = \sum_{i=k+1}^{n} e_i + v, \]

has multiplicity 1, as \( \mathbb{Z}\sigma_{\{1,\ldots,k\}} + \mathbb{Z}\sigma_{\{k+1,\ldots,n\}} = \mathbb{Z} e_1 + \cdots + \mathbb{Z} e_n = \mathbb{Z}^n \). So the claim follows. If we denote the standard hyperplane in \( \mathbb{R}^n \) by \( F := F_{n-1} \), we get \( F_k = F \cdots F = F^{n-k} \). Hence it makes sense to use the notation \( H^k := H_{n-k} \) for the compactified standard planes as well.

Figure 4.6: Two planes in \( TP^3 \) which intersect in a line with a bounded edge
4.3 Projective degree

Using stable intersection numbers, we are now ready to define the degree of a cycle in $\mathbb{TP}^n$. Let us recall that $\mathbb{TP}^n$ is a disjoint union of torus orbits $O_I \cong \mathbb{R}^I$ labelled by proper subsets of $\{0, \ldots, n\}$. Moreover, a cycle $X$ in $\mathbb{TP}^n$ is of pure sedentarity if $X = X^o$ for some cycle $X^o \subset O_I$. We call $I$ the sedentarity type of $X$. Note that $X^o = X \cap O_I$.

**Definition 4.3.1**

Let $X$ be a $k$-cycle in $\mathbb{TP}^n$ of sedentarity zero. We define the (projective) degree of $X$ to be

$$\deg(X) := \deg(X^o \cdot (H^o)^k),$$

where $H$ denotes the standard hyperplane in $\mathbb{TP}^n$. For a $k$-cycle $X$ of pure sedentarity, we define its degree $\deg(X)$ by considering it as a sedentarity zero cycle in the respective coordinate plane $\mathbb{TP}^I \subseteq \mathbb{TP}^n$. If $X$ is of mixed sedentarity, its degree is the sum of the degrees of its components of pure sedentarity.

**Example 4.3.2**

From our computations in example 4.2.8 it follows that the standard planes $H_k$ have degree 1.

4.4 Projective hypersurfaces

In Chapter 2, we used tropical Laurent polynomials to describe hypersurfaces in $\mathbb{R}^n$. We will now see how a homogeneous tropical polynomial describes a hypersurface in $\mathbb{TP}^n$. Let us start with $\mathbb{T}^n$ first. Let

$$f(x) = \sum_{j \in \mathbb{N}^n} a_j x^j$$

be a tropical polynomial in $n$ variables (with multi-index notation). The coefficients $a_j \in \mathbb{T}$ are tropical numbers and the set $A = \{ j : a_j \neq -\infty \}$ is finite (and non-empty). As all exponents are positive now, this describes a function

$$f : \mathbb{T}^n \to \mathbb{T},$$

and moreover, the set

$$V(f) := \{ x \in \mathbb{T}^n : \exists i \neq j \in \mathbb{N}^n \text{ such that } f(x) = a_i x^i = a_j x^j \}$$

(4.5)
4.4 Projective hypersurfaces

is well-defined. Note that as long as \( f \) consists of at least two monomials, the “honest” zero-set of \( f \) satisfies

\[
\{ x : f(x) = -\infty \} \subseteq V(f).
\]

For monomials, we use the definition \( V(ax^i) := \{ x : f(x) = -\infty \} \).

Let us check that \( V(f) \) an \( n - 1 \)-cycle in a natural way. To see this, we divide by monomials if possible and rewrite \( f = x^k f' \), where \( k \in \mathbb{N} \) and \( f' \) is a polynomial which is not further divisible by monomials. Obviously, we have \( V(f) = V(x^k) \cup V(f') \) on the level of sets, and the cycle structure should feature \( V(f) = V(x^k) + V(f') \). For the sedentary zero part, we have \( V(f') \supseteq V(f'/\text{divides.alt0} \mathbb{R}^n) \), so this part already carries canonical multiplicities by taking the closure of a hypersurface cycle in \( \mathbb{R}^n \). For the part of higher sedentary, let us denote the coordinate hyperplanes of \( T^n \) by

\[
H_i := V(x_i) = \{ x \in T^n : x_i = -\infty \}.
\]

**Definition 4.4.1**

The **hypersurface** \( V(f) \subset T^n \) given by the tropical polynomial \( f \) is defined to be the cycle

\[
V(f) := \overline{V(f'|\mathbb{R}^n)} + \sum_{i=1}^{n} k_i H_i,
\]

where \( k_i \in \mathbb{N} \) is the maximal number such that \( x^k_i \) divides \( f \). As a set, this agrees with equation (4.5).

For \( TP^n \) we can proceed analogously. Let

\[
f(x) = \sum_{j \in \mathbb{N}^{n+1}} a_j x^j
\]

be a tropical homogeneous polynomial of degree \( d \in \mathbb{Z} \) in \( n + 1 \) variables (with multiindex notation). In other words, the coefficients \( a_j \in T \) are tropical numbers, the set \( A = \{ j : a_j \neq -\infty \} \) is finite and for all \( j \in A \) we have \( j_0 + \ldots + j_n = d \). In projective coordinates, this does not quite define a function to \( T \) as we have \( f(\lambda \cdot x) = f(x) + \lambda d \). However, the set

\[
V(f) = \{ x \in TP^n : \exists i \neq j \in \mathbb{N}^n \text{ such that } f(x) = a_i x^i = a_j x^j \} \quad (4.6)
\]

is still well-defined. According to our choice of embedding \( R^n \cong \{ 0 \} \times R^n \subset TP^n \), we may consider the dehomogenized polynomial \( f(x_0 = 0)|_{\mathbb{R}^n} \) with associated hypersurface in \( \mathbb{R}^n \). The following definition is nearly identical to the \( T^n \)-case.
Definition 4.4.2
The hypersurface \( V(f) \subset TP^n \) given by the tropical homogeneous polynomial \( f \) is defined to be the cycle
\[
V(f) := V(f(x_0 = 0)|_{\mathbb{R}^n}) + \sum_{i=1}^{n} k_i H_i,
\]
where \( k_i \in \mathbb{N} \) is the maximal number such that \( x_i^{k_i} \) divides \( f \). As a set, this agrees with equation (4.6).

Remark 4.4.3
Note that for any affine chart \( T^n \cong TP^n \setminus H_i \) our definitions are compatible, i.e. \( V(f) \setminus H_i = V(f(x_i = 0)) \). In fact, one may check (analogously to the classical case) that on the overlaps of two affine charts, the two dehomogenized polynomials only differ by an invertible monomial, i.e. a linear function that does not attain \(-\infty\) on the boundary.

The same statement holds for all tropical toric varieties, if we consider homogeneous polynomials \( f \) in the associated Cox ring. Therefore, for general toric varieties we may define hypersurfaces by the same method. That is to say, the hypersurface of \( f \) is the closure of the hypersurface in \( \mathbb{R}^n \) described by any dehomogenization of \( f \) plus each torus-invariant divisor \( D \) with multiplicity equal to the number of times that \( x_D \) divides \( f \).

Proposition 4.4.4
Let \( f \) be a homogeneous polynomial of degree \( d \) and \( V(f) \subset TP^n \) its associated hypersurface. Then \( \deg(V(f)) = d \) holds.

Proof. Obviously, the statement holds when \( f \) is a monomial, as \( \deg(H_i) = 1 \). We can therefore restrict to the case when \( f \) is not divisible by monomials and thus \( V(f) \) has sedentarity zero. In this case, by abuse of notation we replace \( f \) by its dehomogenization which is a degree \( d \) polynomial in \( n \) variables, and \( V(f) \) is a hypersurface in \( \mathbb{R}^n \). By definition, the degree of \( V(f) \) can be computed by intersecting the hypersurface with the tropical standard line \( L = H^{n-1} \) and counting intersection points. By Proposition 4.2.6 the degree of this intersection is translation-invariant, i.e. we can replace \( L \) by any translation \( L + v, v \in \mathbb{R}^n \). In particular, we can choose \( v \in \mathbb{R}^n \) such that the intersection is transversal. For \( p \in V(f) \cap (L + v) \), let us compute its intersection multiplicity. Recall that the facet \( \sigma \) of \( V(f) \) containing \( p \) is related by duality to two monomials of \( f \), namely \( f = "a_{j_1} x^{j_1} + a_{j_2} x^{j_2}" \) locally near \( p \). Therefore we have \( \mathbb{R}\sigma = \ker(j_1 - j_2) \)
(where we consider \( j_1 - j_2 \) as a linear form on \( \mathbb{R}^n \)) and let \( j_1 - j_2 = \omega(\sigma)l \), where \( l \) is the (consistently oriented) primitive generator of \( \mathbb{R}\sigma^\perp \). Now, let \( w \) be a primitive generator of the ray \( \rho \) of \( L + v \) at \( p \). We have \([Z^n : Z\sigma + Z\rho] = |l(w)|\) and thus

\[
\omega(p) = \omega(\sigma)[Z^n : Z\sigma + Z\rho] = |(j_1 - j_2)(w)|.
\]

In other words: \( f|_{L+v} \) is a piecewise affine-linear function which has a (convex) breaking point at \( p \), and \( \omega(p) \) is exactly the change of slope at this point. It follows that \( \deg(X) \) is equal to the sum of these slope changes on the whole line \( L + v \). This can be easily computed. Note that at the vertex \( v \) of \( L + v \), \( f \) is affine-linear in a neighbourhood and therefore the sum of all outgoing slopes of \( f|_{L+v} \) at \( v \) is zero. For each ray of \( L + v \) with direction \(-e_i, i \neq 0\), \( f \) will be constant once we are far enough at infinity. This follows from the assumption that \( f \) is not divisible by \( x_i \) and thus contains a monomial without \( x_i \)-factor. For the ray with direction \( e_0 \), the final slope will be \( d \), as \( f \) is of degree \( d \) and the slope of any monomial on this ray is exactly its degree. Therefore the sum of all changes of slope of \( f|_{L+v} \) is equal to \( d \), and the assertion follows.

Let us digress a little bit in order to analyze the homotopy type of tropical hypersurfaces.

**Proposition 4.4.5**

Let \( V(f) \) be a hypersurface in \( \mathbb{R}^n \) whose Newton polytope \( \text{NP}(f) \) is full-dimensional. Then \( V(f) \) is homotopy-equivalent to a bouquet of \( k(n-1) \)-spheres. Here \( k \) is the number of vertices of the dual subdivision \( \text{SD}(f) \) which are interior points of \( \text{NP}(f) \).

**Proof.** Recall that each vertex of \( \text{SD}(f) \) corresponds to a connected component of \( \mathbb{R}^n \setminus V(f) \). For vertices in the boundary of \( \text{NP}(f) \), the boundary of this component is contractible and the homotopy type after removing the component is unchanged. For vertices in the interior of \( \text{NP}(f) \), the boundary of the corresponding component is homeomorphic to an \( (n-1) \)-sphere. Up to homotopy equivalence, removing such a component is equivalent to removing a point of \( \mathbb{R}^n \). But \( \mathbb{R}^n \) minus \( k \) points is a bouquet of \( k(n-1) \)-spheres, as required.

**Remark 4.4.6**

Assume \( \text{NP}(f) \) is not full-dimensional, but generates the affine subspace \( A \subset \mathbb{R}^n \) in dual space. Then \( V(f) \) is translation-invariant along \( A^\perp \) and
the projection to $\mathbb{R}^n/A^4$ provides a contraction of $V(f)$ to a hypersurface in a smaller-dimensional space whose Newton polytope is full-dimensional (namely $NP(f) \subset A$). Therefore the proposition is basically valid in this case, too. The only difference is that now we count vertices in the relative interior of $NP(f)$ and $V(f)$ is a bouquet of $(m-1)$-spheres, where $m$ is the dimension of $NP(f)$.

When considering hypersurfaces in compact toric varieties, the statement remains nearly unchanged. All we have to do is to use the right notion of being an interior point now. We state the result for projective hypersurfaces here, although the generalization to other (compact) toric varieties is straightforward.

**Proposition 4.4.7**

*Let $V(f)$ be a hypersurface in $TP^n$ of sedentarity zero. We fix an affine chart and assume that the Newton polytope $NP(f)$ is full-dimensional. Let $\Delta_d$ be the smallest simplex containing $NP(f)$. Then $V(f)$ is homotopy-equivalent to a bouquet of $k$ $(n-1)$-spheres. Here $k$ is the number of vertices of the dual subdivision $SD(f)$ which are interior points of $\Delta_d$.***

**Proof.** As in the proof for $\mathbb{R}^n$, each vertex of $SD(f)$ corresponds to a connected component of $TP^n \setminus V(f)$. Again, these components differ, depending on whether the vertex is in the boundary of $\Delta_d$ or not. Accordingly, the boundary of the connected component is contractible or homeomorphic to an $(n-1)$-sphere. Thus again, $V(f)$ is homotopy equivalent to $TP^n$ minus $k$ points, which is a bouquet of $k$ $(n-1)$-spheres. \[\square\]
A tropical homogeneous polynomial in 3 variables describes a tropical curve in $\mathbb{T}P^2$. Curves of higher sedentarity are just unions of coordinate lines, so we focus our attention to sedentarity zero curves here. Such curves may equivalently be described by a not necessarily homogeneous polynomial $f$ in 2 variables which is not divisible by monomials. Here, as before, the embedding $\mathbb{R}^2 \cong \{0\} \times \mathbb{R}^2 \subset \mathbb{T}P^2$ is fixed and thus $f$ defines a projective curve $V(f) \subset \mathbb{T}P^2$. In consistency with section 2.6, the degree of $V(f)$ is the smallest number such that $\text{NP}(f) \subseteq \Delta_d$. Here $\Delta_d$ is the $d$-fold standard simplex, i.e. the convex hull of the points $(0,0), (d,0), (0,d)$. In section 2.6 we mostly considered non-degenerated curves with $\text{NP}(f) = \Delta_d$. Contrary, the special feature of degenerated curves $V(f)$ with $\text{NP}(f) \neq \Delta_d$ is that they contain at least one of the torus fixed points $(0,-\infty,-\infty), (-\infty,0,-\infty), (-\infty,-\infty,0) \in \mathbb{T}P^2$.

In order to obtain a general Bézout theorem for planar projective curves, we have to define intersection multiplicities for curves intersecting at torus fixed points.

A way to do this is as follows. Let $X_1, X_2$ be two curves in $\mathbb{T}^2$ and let $\rho_1, \rho_2$ be unbounded rays of $X_1$ resp. $X_2$ which both contain $(-\infty, -\infty)$. This means that these rays have primitive generators $v_1, v_2$ contained in the interior of $\sigma_{\{1,2\}} = \mathbb{R}_{\geq 0}(-e_1) + \mathbb{R}_{\geq 0}(-e_2)$.

We may write $v_i = (x_i, y_i), x_i, y_i \in \mathbb{Z}_{\leq 0}$. Then the contribution of this

---

Figure 4.8: A generic and a non-generic projective conic in $\mathbb{T}P^2$
pair of rays to the intersection number at \((\infty, \infty)\) is
\[
\omega(\rho_1, \rho_2) = \omega(\rho_1)\omega(\rho_2) \min\{x_1y_2, x_2y_1\}.
\]
We then define the multiplicity of \((\infty, \infty)\) to be
\[
\omega((\infty, \infty)) = \sum_{\rho_1, \rho_2} \omega(\rho_1, \rho_2),
\]
where sum is taken over all pairs of rays of \(X\) resp. \(Y\) containing \((\infty, \infty)\).

The intersection product \(X \cdot Y\) of two projective curves \(X, Y \subset TP^2\) is now defined to be stable intersection in \(R^2\) plus the three torus fixed points with multiplicities computed as above (in an affine chart).

Strange as this definition may look, it is the right choice to make Bézout’s theorem work in the case of intersection at the torus fixed points. The proof of this will shed some light on the origin of the above multiplicity formula. Indeed, after perturbing one of the curves to a non-degenerated one, the additional intersection points in \(R^2\) carry exactly the intersection multiplicities which we assigned to \((\infty, \infty)\) above.

**Theorem 4.5.1** (Tropical Bézout’s theorem)
Let \(X\) and \(Y\) be two 1-cycles in \(TP^2\). Then
\[
\deg(X \cdot Y) = \deg(X) \cdot \deg(Y)
\]
holds.

**Proof.** Any 1-cycle in \(TP^2\) may be written as \(X = X' - X''\), where both \(X'\) and \(X''\) are cycles with only positive weights. Using this decomposition and the linearity of the degree, we can assume that \(X\) and \(Y\) have only positive weights.
4.5 Projective curves

We first consider the non-degenerated case, where both $X$ and $Y$ do not contain torus fixed points. This case was already proven in Theorem 2.6.2. We can provide another proof easily by using the fact that $\deg(X \cdot Y)$ is translation-invariant. For example, we may move $X$ in direction $(1,0)$ and $Y$ in direction $(0,1)$ such that only the $(-1,0)$-directed rays of $X$ and the $(0,-1)$-directed rays of $Y$ intersect. Let us denote these rays by $\sigma_1, \ldots, \sigma_k$ resp. $\tau_1, \ldots, \tau_l$. As $X$ and $Y$ are non-degenerated, we have $\deg(X) = \sum \omega(\sigma_i)$ and $\deg(Y) = \sum \omega(\tau_j)$. At each intersection point $\sigma_i \cap \tau_j$, the intersection multiplicity is $\omega(\sigma_i) \omega(\tau_j)$. Thus the sum of all these intersection multiplicities is $\deg(X) \cdot \deg(Y)$.

In the degenerated case, let $\sigma$ be a ray of $X$ containing a torus fixed point. In the corresponding affine chart $\mathbb{T}^2$, this can be reformulated by the fact that both coordinates of the direction vector $v = (v_1, v_2)$ are negative. Now we use the following trick. We pick a finite point $p \in \sigma \cap \mathbb{R}^2$ such that the remaining unbounded part of $\sigma$ does not intersect $Y$. We replace this part by the two rays $\mathbb{R}_{\geq 0}(-1,0)$ and $\mathbb{R}_{\geq 0}(0,-1)$ with multiplicities $\omega(\sigma)v_1$ resp. $\omega(\sigma)v_2$. By construction, this deformed cycle $X'$ is still balanced and $\deg(X') = \deg(X)$ (choose a line which does not intersect $\rho$). Moreover, if we choose $p$ close enough to $(-\infty,-\infty)$, the intersection of $X'$ and $Y$ does not change, except for the following: For each ray $\tau$ of $Y$ containing $(-\infty,-\infty)$, we get exactly one new finite intersection point of $\tau$ with one of the added rays, and the intersection multiplicity at this point is

$$\omega(\sigma)\omega(\tau) \min\{v_1u_2, u_1v_2\},$$

where $u = (u_1, u_2)$ is the primitive generator of $\tau$ (with $u_1, u_2 < 0$). Here, which of the two terms is minimal corresponds to which of the two new rays intersects $\tau$. In other words, the new finite intersection points of $X' \cdot Y$ exactly compensate the loss of intersection multiplicities at $(-\infty,-\infty)$, compared to $X \cdot Y$. In particular, we see that our definition of intersection multiplicities at $(-\infty,-\infty)$ is constructed such that $\deg(X \cdot Y) = \deg(X' \cdot Y)$. Repeating this procedure for each “degenerated” ray of $X$ and $Y$, we eventually end up with two non-degenerated cycles without changing the degrees or the intersection number, and the assertion follows. \hfill $\square$

A positive tropical cycle in $\mathbb{T}P^n$ of dimension 1 and degree 1 is called a \textit{tropical line}. Let us recall a trivial fact from classical algebraic geometry: Any line in $\mathbb{C}P^n$ is isomorphic to $\mathbb{C}P^1$. In tropical geometry, however, there exist (topologically) different projective lines. Of course, the archetype is still $\mathbb{T}P^1$. But already the standard line $L$ in $\mathbb{T}P^2$ is different from $\mathbb{T}P^1$. 

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Figure 4.10: Replacing a degenerated ray by two standard ends

Figure 4.11: A tropical line in $\mathbb{TP}^3$

$L$ contains 3 infinite points and one vertex, whereas $\mathbb{TP}^1$ has 2 infinite points and no vertex at all. The following picture shows a tropical line in $\mathbb{TP}^3$.

In general, tropical lines have the following properties.

**Lemma 4.5.2**

Let $L$ be a tropical line in $\mathbb{TP}^n$ of sedentariness zero. Then $L$ has the following properties:

(a) All weights of $L$ are 1.

(b) $L$ is contractible. In other words, it is a rational graph.

(c) Let $v_0, \ldots, v_m$ be the primitive generators of the unbounded ends of $L$ (pointing towards the point at infinity). Then $m \leq n$ and

$$v_j = - \sum_{i \in I_j} e_i$$
for a suitable subset $I_j \subset \{0, \ldots, n\}$. Moreover, $\{0, \ldots, n\}$ is the disjoint union of $I_0, \ldots, I_m$.

(d) $L$ has at most $m - 3$ bounded edges (equality holds when all finite vertices are 3-valent) and at most $m - 2$ finite vertices.

Proof. By definition all weights are positive, therefore all points in the transversal intersection with a hyperplane $H$ are counted positively. In particular, an edge with weight greater than one would immediately give a contradiction to $\deg(L) = 1$, so the first statement holds.

For the second statement, assume that $L$ contains a cycle $C$. For sure we can find a hyperplane $H$ which intersects $C$ and is transversal to $L$. As $C$ is a cycle, there must be a second intersection point of $C$ and $H$, which contradicts $\deg(L) = 1$.

Let us now prove the third assertion. We start be showing that each $v_j$ is a simple sum of the standard primitive vectors $e_0, \ldots, e_n$. By relabelling the coordinates we may assume $v_j \in (\mathbb{Z}_{\geq 0})^n$. We have to show that all coordinate entries of $v_j$ are either 0 or $-1$. Assume contrarily that the first coordinate entry is lower than $-1$. Then we take a hyperplane $H$ transversal to $L$ and such that the facet of $H$ spanned by $-e_2, \ldots, -e_n$ and the ray of $L$ with direction $v_j$ intersect. As the lattice index contributing to the local intersection multiplicity is equal to the first coordinate entry of $v_j$, we get a contradiction again. Thus we showed

$$v_j = -\sum_{i \in I_j} e_i$$

for a suitable subset $I_j \subset \{0, \ldots, n\}$. Next, we show that these sets are pairwise disjoint. Assume $i \in I_j \cap I_{j'}$. Then any hyperplane $H$ which we move far towards $-e_i$ will intersect both rays generated by $v_j$ and $v_{j'}$, which is impossible. It remains to show $\{0, \ldots, n\} = \bigcup_{j=0}^m$. Equivalently, we may verify $\sum_{k=0}^n v_k = 0$. This is true (for any tropical curve) because of the balancing condition at each vertex of $L$. Indeed, if we add up all the balancing equations for all finite vertices, primitive vectors of bounded edges cancel because they appear twice with opposite signs. Only the $v_k$ remain and therefore add up to zero.

The fourth statement follows from the fact that $L$ is rational, has $m$ 1-valent ends and all other (finite) vertices have valence at least 3 (by the balancing condition).

We call $L$ in $\mathbb{TP}^n$ a generic line if it has $n+1$ infinite points (i.e. $n+1$ rays with directions $-e_0, \ldots, -e_n$). Non-generic lines are those which intersect higher codimension coordinate planes.
4.6 Floor decompositions

In this section, we describe floor decomposition with respect to a fixed projection \( \pi : \mathbb{R}^n \to \mathbb{R}^{n-1} \). For simplicity, in this section we fix the projection \( \pi \) to be given by forgetting the last coordinate. We will also frequently use the dual projection \( q : \mathbb{R}^n \to \mathbb{R} \) which forgets all except the \( n \)-th coordinate.

**Definition 4.6.1**

Let \( V(f) \subset \mathbb{R}^n \) be a hypersurface with dual subdivision \( \text{SD}(f) \). The \( V(f) \) is called *floor-decomposed* (with respect to \( \pi \)) if for each cell \( \sigma \in \text{SD}(f) \) the image \( q(\sigma) \), which is an integer interval in \( \mathbb{R} \), is of length at most 1 (i.e. of length 0 or 1).

A floor-decomposed hypersurface splits naturally into floors and elevators as follows. We assume \( q(\text{NP}(f)) = [0, m] \). For a cell \( P \) of \( V(f) \), we denote the dual cell in \( \text{SD}(f) \) by \( \sigma_P \). For \( i = 1, \ldots, m \), we define

\[
F_i := \bigcup_{P \text{ cell of } V(f)} P, \quad q(\sigma_P) = [i-1, i]
\]

and call this closed set a *floor* of \( V(f) \). For \( i = 0, \ldots, m \), we define

\[
E_i := \bigcup_{P \text{ cell of } V(f)} \text{RelInt}(P), \quad q(\sigma_P) = (i)
\]
and call this set an *elevators* of $V(f)$. (In a more refined setting, one might call the connected components of the $E_i$ elevators.) Accordingly, we can split the polynomial $f$, i.e. we can write $f$ as a polynomial of the last coordinate $x_n$

$$f = \sum_{i=0}^m f_i x_n^i,$$

with $f_i \in \mathbb{T}[x_1, \ldots, x_{n-1}]$. We have the following relation.

**Proposition 4.6.2**

The projection $\pi(E_i)$ of an each elevator is a tropical hypersurface in $\mathbb{R}^{n-1}$ given by $V(f_i)$. 

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Each floor $F_i$ projects one-to-one to $\mathbb{R}^{n-1}$ and can be described as the graph of the function $f_{i-1}/f_i = f_{i-1} - f_i$ on $\mathbb{R}^{n-1}$.

Proof: For the first part, let us fix an elevator $E_i$. We want to show that $\pi(E_i) = V(f_i)$ as sets. First, take $x \in E_i$. By definitions, the dual cell $\sigma_x$ of $x$ in $\text{SD}(f)$ satisfies $q(\sigma_x) = \{i\}$. It follows that locally at $x$ we have $f \equiv f_{\sigma_x} \equiv f_m x_1^n$. In particular, $f_i(\pi(x))$ attains its maximum at least twice and thus $\pi(x) \in V(f_i)$. For the other direction, pick $y \in \mathbb{R}^{n-1}$. We may consider the polynomial $g = \sum f_i(y) x_n^i$ in one variable $x_n$ obtained by restricting $f$ to $\pi^{-1}(x)$. $g$ is of degree $m$ and its subdivision $\text{SD}(g)$ can be computed as

$$\text{SD}(g) = \{q(\sigma_P) : \sigma_P \in \text{SD}(f) \text{ such that } P \cap \pi^{-1}(y) \neq \emptyset\}.$$ 

As $f$ is floor-decomposed, all cells of $\text{SD}(g)$ have length 1, i.e. $[0, m]$ is maximally subdivided. It follows that for any $i$, there exists $x_n$ such that the only maximal term in $g(x_n)$ is $f_i(y) x_n^i$. If we further assume $y \in V(f_i)$, the point $x = (y, x_n) \in \mathbb{R}^n$ is in $V(f)$ and the maximum of $f(x)$ is only attained on the $i$-th level. It follows $x \in E_i$ and therefore $y \in \pi(E_i)$. This proves $\pi(E_i) = V(f_i)$. Note that also the multiplicities of the two sets are compatible in the sense that $\text{mult}_{V(f)}(x) = \text{mult}_{V(f_i)}(\pi(x))$. This is true because we defined the multiplicity to be the volume of the dual cell in $\text{SD}(f)$ resp. $\text{SD}(f_i)$, which is the same (up to being embedded in different ambient spaces).

Now let us prove the second part of the assertion. First note that $x \in F_i$ is equivalent to

$$f(x) = f_i(\pi(x)) x_n^i = f_{i-1}(\pi(x)) x_n^{i-1}.$$ 

Transforming the second equality according to the rules of tropical arithmetic, we get $x_n = f_{i-1}(\pi(x)) - f_i(\pi(x))$. So $F_i$ is contained in the graph of the function $f_{i-1} - f_i$ and therefore $\pi|_{F_i}$ is injective. For surjectivity, we use again the fact that, for any $y \in \mathbb{R}^{n-1}$ and corresponding one variable polynomial $g = \sum f_i(y) x_n^i$, $\text{SD}(g)$ maximally subdivides $[0, m]$. Hence, there exists a (unique) “zero” $z$ of $g$ with $g(z) = f_i(y) z^i = f_{i-1}(y) z^{i-1}$ and therefore $(y, z) \in F_i$.

The nice feature of floor-decomposed hypersurfaces is that they are completely described by (the projection of) their elevators and the “height” of their floors. This is the statement of the following proposition. For simplicity, we restrict ourselves to the case of generic hypersurfaces of $\mathbb{T}P^n$. 


By generic we mean that the associated Newton polytope is a (dilated) simplex $\Delta_d$. Of course, for other toric varieties and different shapes of Newton polytopes, one can proceed similarly.

We make the following convention. For a generic hypersurface $V$ in $\text{TP}^n$, we denote by $f_V$ the unique polynomial in $n$ variables such that $V \cap \mathbb{R}^n = V(f_V)$, the constant term of $f_V$ is 0 (not $-\infty$) and all exponents are positive. In other words, $f_V$ is the dehomogenization (with respect to the fixed coordinates $\mathbb{R}^n \subset \text{TP}^n$) of the unique homogeneous polynomial (up to adding a constant) describing $V$.

**Proposition 4.6.3**

Let $V_0, \ldots, V_{m-1}$ be generic tropical hypersurfaces in $\text{TP}^{n-1}$ of sedentarity zero and of degree $\text{deg}(V_i) = m - i$. Moreover, let $h_1 < h_2 < \cdots < h_m$ be real numbers and set $f_i := f_{V_i} - \sum_{0 \leq j < i} h_j$ for all $i = 0, \ldots, m$ (with $h_0 = 0$ and $f_m \equiv 0$). Under the assumption $(2f_i - f_{i-1} - f_{i+1})(x) > 0$ for all $x \in \mathbb{R}^{n-1}$ and $i = 1, \ldots, m-1$, there exists a unique generic floor-decomposed hypersurface $V \subset \text{TP}^n$ such that $\pi(E_i) = V_i$ for all elevators and each floor $F_i$ intersects the $x_n$-coordinate axis at height $h_i$. This hypersurface is given by $V = V(f)$, where $f = \sum_{i=0}^{m} f_i x_n^i$. In particular, $V$ has sedentarity zero and degree $m$.

**Proof.** Let us study $V := V(f)$ with $f := \sum_{i=0}^{m} f_i x_n^i$. The assumption $(2f_i - f_{i-1} - f_{i+1})(x) > 0$ makes sure that the various floors (given by the graphs of $f_{i-1} - f_i$) do not intersect and therefore $V$ is indeed floor-decomposed. By proposition 4.6.2, the elevators of $V$ satisfy $\pi(E_i) = V(f_i) = V_i$. Moreover, again by proposition 4.6.2, the height of the intersection of $F_i$ with the $x_n$-coordinate axis is given by the the difference of the constant terms of $f_{i-1}$ and $f_i$, which is $-h_1 - \cdots - h_{i-1} + h_i + \cdots + h_{i-1} + h_m = h_m$. For the uniqueness of $V$, we note that the condition $\pi(E_i) = V_i$ already implies $f_i - f_{V_i} \equiv \text{const}$. These constants are uniquely fixed (up to adding a global constant) by the “heights” of the floors $F_i$. 

**References**

Weighted fans satisfying the balancing condition already appears in [FS], where they are called Minkowski weights. They provide a combinatorial
tool to describe the cohomology classes of toric varieties (in fact, the weight of a cell just describes the degree of the cap-product of the cohomology class with the corresponding boundary cycle). As the name Minkowski weight suggests, the emphasis is put on the weights rather than the geometry of the underlying sets. However, a precursor of stable intersection shows up, which is called the fan displacement rule. It is used to compute the product of two cohomology classes.

The first detailed proof of Bézout’s theorem for plane curves appears \cite{RGST} in the case of non-degenerated curves. A stronger version (when at least one of the curves is non-degenerated) and a proof based on the dual subdivisions can be found in \cite{V1}.

Stable intersection in the more general setting of tropical cycles was introduced in \cite{M2}. In \cite{AR}, stable intersection is defined in a more algebraic-geometric manner (by taking cartesian products and intersecting with the diagonal) and some properties of stable intersections are established.

The concept of floor decompositions first appears in \cite{BM1,BM2} in the context of providing nice combinatorial formulas for counting plane curves. See also \cite{FM,BGM} for further developments.
5 Tropical modifications and equivalence

5.1 Tropical modifications in $\mathbb{R}^n$

Let $X$ be a $k$-cycle in $\mathbb{R}^n$ and let $f : X \to \mathbb{R}$ be a piecewise linear function on $X$. This means that there exists a polyhedral structure on $X$ such that $f$ restricted to a cell is affine linear. From now on, we will only consider such polyhedral structures on $X$. For each cell $\tau$, the affine linear function $f|_\tau$ induces a linear map $\mathbb{R} \tau \to \mathbb{R}$, which we denote by $f_\tau$ and call the linear part of $f$ on $\tau$.

The graph of $f$ in $\mathbb{R}^n \times \mathbb{R}$ is denoted by $\Gamma(X, f)$ or just $\Gamma(f)$. Obviously, $\Gamma(f)$ is a polyhedral set and in inherits multiplicities from $X$ by equipping the (generic) point $(x, f(x)) \in \Gamma(f)$ with the multiplicity of $x \in X$. Note that $\Gamma(f)$ is usually not balanced. Namely, over points where $f$ fails to be locally affine linear, $\Gamma(f)$ might violate the balancing condition. However, there is a canonical way to complete $\Gamma(f)$ to a balanced cycle (cf. remark 2.5.9). At each codimension one cell $\bar{\tau}$ of $\Gamma(f)$ where the balancing condition fails, we add the additional facet $\rho = \bar{\tau} + (\{0\} \times \mathbb{R}_{\geq 0})$ spanned by $-e_{n-1}$.

There is a unique choice of multiplicities such that this completed polyhedral set becomes balanced. To see this explicitly, let us consider a codimension one cell $\tau \subset X$ with adjacent facets $\sigma_1, \ldots, \sigma_n$. We may choose primitive generators $v_i = v_{\sigma_i \tau}$ such that $\sum_i \omega(\sigma_i)v_i = 0 \in \mathbb{Z}^n$. Let $\bar{\tau}, \bar{\sigma}_1, \ldots, \bar{\sigma}_n$ be the respective lifted cells in $\Gamma(f)$. Then the primitive

Figure 5.1: The modification of $\mathbb{R}^2$ along "\((-1)x^2 + 1xy + (-1)y^2 + x + y + 0\)"
5.1 Tropical modifications in $\mathbb{R}^n$

generator of $\bar{\sigma}_i$ modulo $\bar{\tau}$ is just given by
$$(v_i, f_{\sigma_i}(v_i)) \in \mathbb{Z}^n \times \mathbb{Z}.$$ 

Therefore, when we add up these vectors in order to check the balancing condition, we get zeros except for the last coordinate, where the term
$$\sum_{i=1}^n \omega(\sigma_i) f_{\sigma_i}(v_i)$$
remains. Hence we can restore the balancing condition by adding the facet $\rho = \bar{\tau} + (\{0\} \times \mathbb{R}_{\leq 0})$ with multiplicity $\omega(\rho) = \sum_i \omega(\sigma_i) f_{\sigma_i}(v_i)$.

**Definition 5.1.1**

Let $X \subseteq \mathbb{R}^n$ be a $k$-cycle and let $f$ be a piecewise-linear function. The modification of $X$ along $f$ is given by the union of the graph $\Gamma(X, f)$ with the facets $\rho = \bar{\tau} + (\{0\} \times \mathbb{R}_{\leq 0})$ for each codimension one cell $\bar{\tau}$ of $\Gamma(X, f)$ where the balancing condition fails. The multiplicity of $\rho$ is given by
$$\omega(\rho) = \sum_{i=1}^n \omega(\sigma_i) f_{\sigma_i}(v_i),$$
where the sum runs through the facets $\sigma_i \subset X$ which are adjacent to the projection of $\bar{\tau}$. We denote the resulting weighted polyhedral set by $\text{Mod}(X, f) \subset \mathbb{R}^n \times \mathbb{T}$.

By construction, $\text{Mod}(X, f)$ is balanced at all points in $\Gamma(X, f)$. However, by adding new facets, we also add new codimension one cells. It turns out that also these new codimension one cells satisfy the balancing condition and hence $\text{Mod}(X, f)$ is in fact a $k$-cycle. Before we prove this, let us introduce some additional terminology.

Let $\overline{\text{Mod}}(X, f)$ be the closure of $\text{Mod}(X, f)$ in $\mathbb{R}^n \times \mathbb{T}$. We are particularly interested in the intersection of $\overline{\text{Mod}}(X, f)$ with $H = \mathbb{R}^n \times \{-\infty\}$, which we might regard as the zero-set of $f$ on $X \cong X \times \{-\infty\}$. In our special case of modifications, we define $H \cdot \overline{\text{Mod}}(X, f)$ to be $H \cap \overline{\text{Mod}}(X, f)$ with multiplicities $\omega(\rho \cap H) = \omega(\rho)$ for all facets $\rho$ with boundary in $H$.

**Definition 5.1.2**

The divisor of $f$, denoted by $\text{div}(f)$, is defined to be the $k-1$-dimensional weighted polyhedral set supported on the projection of the codimension one skeleton of $\Gamma(X, f)$ and with multiplicities
$$\omega(\tau) = \sum_{i=1}^n \omega(\sigma_i) f_{\sigma_i}(v_i),$$

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where $\tau$ is a codimension one cell of $X$ and the sum runs through the adjacent facets $\sigma_i \subset X$. With the above notations and via the identification $X \cong X \times \{-\infty\}$, we have $\text{div}(f) = H \cdot \text{Mod}(X, f)$.

Let us make an important remark here. To obtain the divisor of $f$ we intersect the modification with $\mathbb{R}^n \times \{-\infty\}$ only (and not with $\mathbb{R}^n \times \{+\infty\}$). So it looks like we are only recording the “zeros” of $f$. However, this is compensated by the fact that we forced all the new facets of $\text{Mod}(X, f)$ to go downwards, in the direction of $-e_n$. Indeed, assume $f = \frac{1}{g} = -g$ is the reciprocal of a polynomial $g$. Then the concavity of $f$ together with our convention to add downward facets only will force us to put negative weights on the new facets. Hence the divisor of $f$ will carry negative weights, and in this sense we are recording the poles of $f$ also. The reader might ask why we do not drop our convention then and instead add facets in both directions, but with positive weights only. This question is even more justifiable as in the next step, when extending the above definitions to parts of higher sedentarity, we are forced to intersect with $+\infty$ anyway. The answer is twofold. If $f = \frac{b}{g}$ is the quotient of two polynomials, we might indeed use this variant of modification with facets in both direction. For example in the case of a floor-decomposed surface, a neighbourhood of the floor $F_i$ (e.g. the floor plus its two adjacent elevators) is just the modification of $\mathbb{R}^{n-1}$ by $\frac{f_i}{f_{i-1}} = f_i - f_{i-1}$ in this alternative sense. However, in this section we consider a general piecewise-linear function $f$ which does
not come with a (canonical) splitting as the quotient of two polynomials. Correspondingly, its divisor cannot be naturally splitted into a balanced part of zeros and a balanced part of poles. In particular, the alternative modification fails to be balanced in general.

By definition, \( \text{div}(f) = H \cdot \text{Mod}(X, f) \) is just the projection of \( \text{Mod}(X, f) \setminus \Gamma(X, f) \) along \( e_{n+1} \). In particular, the balancing condition of \( \text{Mod}(X, f) \) is equivalent to showing that \( \text{div}(f) \) is balanced. We will do this now.

**Proposition 5.1.3**

Both \( \text{div}(f) \) and \( \text{Mod}(X, f) \) are balanced. Therefore \( \text{div}(f) \) is a \( k-1 \)-cycle in \( X \) and \( \text{Mod}(X, f) \) is a \( k \)-cycle in \( \mathbb{R}^n \).

**Proof.** As explained before, it suffices to show that \( \text{div}(f) \) is balanced. So, let \( \rho \) be a codimension one cell of \( \text{div}(f) \) (i.e. a codimension two cell of \( X \)). First we note that adding a globally affine-linear function to \( f \) does not change \( \text{div}(f) \), so we may assume \( f|_\rho = 0 \). Recall that the balancing condition is a local condition and is formulated modulo \( R\rho \), so we may pass to the two-dimensional fan situation of \( \text{Star}_X(\rho) \) with the function induced by \( f \). So let us assume now that \( X \) is a two-dimensional fan and \( f \) is a piecewise linear function on \( X \) which is linear on each ray (in particular \( f(0) = 0 \)). We want to check the balancing condition at the origin \( \{0\} \).

Let us fix a polyhedral structure of \( X \). By refining sufficiently, we may assume that this structure is unimodular, i.e. for each 2-cell \( \sigma = \tau = \tau' \) spanned by two rays \( \tau, \tau' \), the primitive generators \( v_\tau, v_{\tau'} \) of \( \tau \) resp. \( \tau' \) also span \( \mathbb{Z}\sigma \). This implies that \( v_{\tau'} \) is a primitive generator of \( \sigma \) modulo \( \tau \). It follows that the multiplicity of \( \tau \) in \( \text{div}(f) \) is given by

\[
\omega(\tau) = (\sum_{\tau'}\omega(\tau + \tau') f(v_{\tau'})) - a_\tau f(v_\tau),
\]

where the sum runs through all rays \( \tau' \) such that \( \tau + \tau' \) is a 2-cell of \( X \). The coefficients \( a_\tau \in \mathbb{Z} \) are given by the balancing condition of \( X \) and

\[
\sum_{\tau'}\omega(\tau + \tau')v_{\tau'} = a_\tau v_\tau. \tag{5.1}
\]

Now, checking the balancing condition of \( \text{div}(f) \) at \( \{0\} \) we get

\[
\sum_{\tau} \omega(\tau) v_\tau = (\sum_{\tau, \tau'}\omega(\tau + \tau') f(v_{\tau'}) v_\tau) - \sum_{\tau} a_\tau f(v_\tau) v_\tau.
\]
Here the first sum runs through all pairs of rays such $\tau + \tau'$ is a 2-cell of $X$ and the second sum runs through all rays. By exchanging $\tau$ and $\tau'$ in the first sum, we get

$$\sum_\tau f(v_\tau)\left(\sum_{\tau'} \omega(\tau + \tau')v_{\tau'} - a_\tau f(v_\tau)\right).$$

The term in brackets vanishes due to the balancing condition (5.1) at $\tau$. Hence the whole sum vanishes, as required. \hfill \Box

Recall that we chapter 2 we defined the hypersurface $V(f) \subset \mathbb{R}^n$ of a given polynomial $f$. This definition agrees with the definition of the divisor of $f$ in the sense of this chapter (cf. remark 2.5.2).

**Proposition 5.1.4**

*Let $f$ be a tropical Laurent polynomial. Then $V(f) = \text{div}(f) \subset \mathbb{R}^n$.***

*Proof.* Both cycles have codimension one and are supported on the locus of non-linearity of $f$. Thus all we have to do is to compare the multiplicities of the facets. Let $\tau$ be such a facet. We know that in a neighbourhood of a generic point $x$ of $\tau$, $f$ can be replaced by two of its monomials, i.e. $f \equiv "a_1 x^{j_1} + a_2 x^{j_2}"$ near $x$. Recall that the multiplicity $\omega$ of $\tau$ in $V(f)$ was defined as the integer length of the segment spanned by the points $j_1, j_2 \in \mathbb{Z}^n$. In other words, let $l$ be a primitive generator of the ray $\mathbb{R}(j_2 - j_1)$, then $j_2 - j_1 = ml$. Let us now compute the multiplicity $\omega'$ of $\tau$ in $\text{div}(f)$. Let $\sigma$ and $\sigma'$ be the two full-dimensional cells adjacent to $\tau$. We choose representatives $v$ resp. $v'$ of the primitive generators $v_{\sigma/\tau}$ resp. $v_{\sigma'/\tau}$ such that $v + v' = 0$. Then, by definition of $\omega'$, we have

$$\omega' = j_1(v) + j_2(v') = j_1(v) - j_2(v) = \omega l(v) = \omega.$$ 

For the last equality we used the fact that, by choice of our orientations, $l(v) = +1$. So we see that the multiplicities agree and therefore $V(f) = \text{div}(f)$. \hfill \Box

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### 5.2 Modifications in toric varieties

Let $X$ be a $k$-cycle in $\mathbb{T}^n$ (or $\mathbb{T}^i \times \mathbb{R}^{n-i}$) of sedentary zero and set $X^\circ = X \cap \mathbb{R}^n$. Let $f$ be a piecewise affine-linear function on $X^\circ$. 

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Definition 5.2.1
We define the modification of \( X \subseteq T^n \) along \( f \), denoted by \( \text{Mod}(X, f) \), to be the closure of \( \text{Mod}(X^\circ, f) \) in \( T^n \times TP^1 \).

Again, we would like to define the divisor of zeros and poles of \( f \) in \( X \), this time taking into account also the behaviour at infinity. To do so, we intersect with \( H_+ = T^n \times \{ +\infty \} \) and \( H_- = T^n \times \{ -\infty \} \) and subtract the results. Note that this time the intersection with \( H_+ \) is really necessary as it adds "poles" at infinity whenever (the continuation of) \( f \) actually attains the value \(+\infty\). However, "poles" of \( f \) in the finite part still appear as negative multiplicities in the intersection with \( H_- \). In particular, for general piecewise affine-linear functions there is still no canonical way to split the divisor into a sum of the form "zeros minus poles".

Back to our problem, we must first define the intersection multiplicities when intersecting with \( H_- \) or \( H_+ \). More general, let \( X \) be a \( k \)-cycle in \( T^n \) of sedentarity zero, and let \( H \subset T^n \) be a coordinate hyperplane \( x_n = -\infty \). We will define a \( k-1 \)-cycle \( H \cdot X \) supported on \( H \cap X \). Let \( \sigma \) be a facet of \( X \) such that \( \tau = \sigma \cap H \) is \( k-1 \)-dimensional and thus a facet of \( H \cap X \). Assume \( \tau \) is of sedentarity type \( I \). Recall that this means \( I \subseteq [n] \) is the subset of coordinates which are \(-\infty\) for all points in \( \tau \), so in particular \( n \in I \). We may consider the projection \( \pi : \mathbb{R}^n \to \mathbb{R}^{[n] \setminus I} \) which forgets the coordinates in \( I \). From the definitions it follows \( \pi(\mathbb{R}\sigma) = \mathbb{R}\tau \), so \( \pi|_{\sigma} \) has a one-dimensional kernel. The primitive generator of this kernel with positive coordinate entries is called the primitive generator of \( \sigma \) modulo \( \tau \) and is denoted by \( v_{\sigma/\tau} \). Let \( v^n_{\sigma/\tau} \in \mathbb{N} \) be the \( n \)-th coordinate entry of \( v_{\sigma/\tau} \).

Definition 5.2.2
Let \( X \subseteq T^n \) be a \( k \)-cycle of sedentarity zero, and let \( H = V(x_n) \) be a coordinate hyperplane. Then we define \( H \cdot X \) to be the \( k-1 \)-dimensional polyhedral set \( H \cap X \) with multiplicities

\[
\omega(\tau) = \sum_{\sigma \supset \tau} \omega(\sigma) v^n_{\sigma/\tau}.
\]

Here \( \tau \) is a facet of \( H \cap X \), the sum runs through the facets \( \sigma \) of \( X \) containing \( \tau \), and \( v^n_{\sigma/\tau} \) denotes the \( n \)-th coordinate entry of the respective primitive generator.

Note that this definition can be generalized to any smooth toric variety immediately, i.e. cycles of sedentarity zero can always be intersected with boundary divisors by using affine charts. Given the fan of such a toric variety, this boils down to the following. Each negative primitive generator
$-v_{\sigma/\tau} \in \mathbb{Z}^n$ is contained in the relative interior of a cone of the fan, and we may write it as a sum (with positive coefficients) of the primitive generators of the spanning rays. The coefficient of the ray corresponding to the boundary divisor we want to intersect with is the one which enters the multiplicity formula as above.

Note also that in the case of a modification $\text{Mod}(X,f)$, $X \subseteq \mathbb{R}^n$, our previous definition $\text{div}(\sigma) = H \cdot \text{Mod}(X,f)$ agrees with the new one. In this situation, the sedentarility type of each facet $\tau$ is just \{n\} and therefore always $v_{\sigma/\tau} = e_{n+1}$.

However, in the general situation it remains to show that our intersection product provides a balanced polyhedral set.

**Lemma 5.2.3**

With the notations from above, $H \cdot X$ is a balanced polyhedral set, i.e. forms a $k-1$-cycle.

**Proof.** Let $\rho$ be a codimension one cell of $H \cdot X$, i.e. a codimension two cell of $X$. Assume $\rho$ is of sedentarility type $I$ and recall that for the balancing condition, we are only concerned about facets of $H \cdot X$ containing $\rho$ and of the same sedentarility type. Let $\tau$ be such a cell, and let $\sigma$ be a facet of $X$ such that $\tau = \sigma \cap H$. Then there exists a unique codimension one cell $\rho' \subset \sigma$ of sedimentary zero such that $\rho = \rho' \cap H$. This $\rho'$ is provided by $\sigma \cap \pi^{-1}(\rho)$, where $\pi : \mathbb{R}^n \to \mathbb{R}^{|n|}$ is the projection which forgets the coordinates in $I$.

Let us fix this $\rho'$ and let us consider the primitive generator $v_{\rho'/\rho} \in \mathbb{Z}^n$. The important observation is that for any facet $\sigma \supset \rho'$ such that $\tau = \sigma \cap H \neq \rho$ is of codimension one, the primitive generator (modulo $\tau$) is the same, i.e. $v_{\sigma/\tau} = v_{\rho'/\rho}$. In particular, we may denote the $n$-th coordinate of this vector by $k(\rho')$.

Moreover, note that $\tau = \sigma \cap H \neq \rho$ if and only if $\pi(v_{\sigma/\rho'}) \notin \mathbb{R}\rho$, and in this case $v_{\tau/\rho} = \pi(v_{\sigma/\rho'})$. Combining all these things, the balancing condition of $H \cdot X$ around $\rho$ reads as follows.

$$\sum_{\tau \supset \rho} \omega(\tau)v_{\tau/\rho} = \sum_{\rho' \supset \rho} k(\rho')\pi\left(\sum_{\sigma \supset \rho'} \omega(\sigma)v_{\sigma/\rho'}\right) = 0 \mod \mathbb{R}\rho.$$ 

In these various sums, as before, $\tau$ runs through all the facets of $H \cdot X$ of same sedentarility type as $\rho$, $\rho'$ denotes codimension one cells of $X$ of sedimentary zero and $\sigma$ denotes facets of $X$. We use that each $\sigma$-sum is contained in $\mathbb{R}\rho'$ because $X$ is balanced around $\rho'$, and $\pi(\mathbb{R}\rho') = \mathbb{R}\rho$. Note
also that a priori we might have contributions from facets $\sigma$ with either $\sigma \cap H = \rho$ or $\sigma \cap H$ has different sedentary type than $\rho$. But in both cases this implies $\pi(v_{\alpha/\rho}) \in R\rho$, so the equation remains correct. 

We are now ready to define the divisor of a function also at the boundary. So let again $X$ be a $k$-cycle in $T^n$ (or $T^n \times R^{n-l}$) of sedentary zero and set $X^0 = X \cap R^n$. Let $f$ be a piecewise affine-linear function on $X^0$.

**Definition 5.2.4**

The divisor of $f$, denoted by $\text{div}(f)$, is defined to be the $k-1$-cycle

$$\text{div}(f) = H_- \cdot \text{Mod}(X, f) - H_+ \cdot \text{Mod}(X, f),$$

where $H_- = T^n \times \{-\infty\}$ and $H_+ = T^n \times \{+\infty\}$ and we use the canonical identification $H_- \cong H_+ \cong T^n$.

Of course, in the sedentary zero part of $\text{div}(f)$ the multiplicities can still be computed according to the formula in definition 5.1.2, i.e. by comparing the slopes of $f$ around a given cell $\tau$. Actually, something similar is true at the boundary. Namely, let $\overline{\tau}$ be a codimension one cell of $\text{Mod}(X, f)$ with $\text{sed}(\tau) > 1$ and let $\overline{\sigma}$ be a facet of $\text{Mod}(X, f)$ containing $\overline{\tau}$. Then the pair $\overline{\tau} \subset \overline{\sigma}$ projects to a pair $\tau \subset \sigma \subset X$, where $\sigma$ is a facet of $X$ and $\tau$ is a codimension one cell with $\text{sed}(\tau) > 0$. Moreover, the primitive generators satisfy

$$v_{\overline{\sigma}/\overline{\tau}} = (v_{\sigma/\tau}, f_{\sigma}(v_{\sigma/\tau})) \in Z^n \times Z.$$

We get the following corollary.

**Corollary 5.2.5**

The multiplicities of $\text{div}(f)$ can be computed by the formula

$$\omega(\tau) = \sum_{i=1}^{l} \omega(\sigma_i) f_{\sigma_i}(v_i),$$

where $\tau$ is a codimension one cell of $X$ and the sum runs through the adjacent facets $\sigma_i \subset X$ (with primitive generators $v_i$). In particular, this formula also holds when $\text{sed}(\tau) > 0$.

**Example 5.2.6**

Consider the Laurent monomial $f = "x_2/x_1" = x_2 - x_1$ on $T^2$. Note that $f$ is not well-defined at $(-\infty, -\infty)$, but a perfectly nice linear function on
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\[ \text{Mod}(T^2, f) \]

Figure 5.3: The modification Mod(T^2, x_1 - x_2) with its 3 boundary lines

R^2, so we can try to compute \( \text{div}(f) \). In this case, Mod(R^2, f) is just the closure of the graph of \( f|_{R^2} \) in \( T^2 \times TP^1 \). It is easy to check that the additional points Mod(T^2, f) \( \setminus \) R^3 are the union of three lines

\[
\text{Mod}(T^2, f) \setminus R^3 = T \times \{-\infty\} \times \{-\infty\} \\
\cup \{-\infty\} \times T \times \{+\infty\} \\
\cup \{-\infty\} \times \{-\infty\} \times TP^1
\]

(cf. Figure 5.3). Whereas the last line just reflects the fact that \( f \) is not well-defined at \(( -\infty, -\infty )\) (in fact, we have blown up that point), the first two lines are given by \( H_- \cdot \text{Mod}(T^2, f) \) resp. \( H_+ \cdot \text{Mod}(T^2, f) \). Therefore, as expected we have \( \text{div}(f) = V(x_2) - V(x_1) \).

The next step is to consider modification in any smooth toric variety by homogeneous polynomials. To keep the exposition simple, we restrict to the most important case of projective space. However, as usual the definition can be easily extended to any smooth toric variety. The new aspect of homogeneous polynomials is that they do not really define functions on TP^n. However, in each affine chart they define a function and on overlaps the difference is “invertible”. This makes it possible to glue together the different modifications of each chart.

Let \( X \subseteq TP^n \) be a k-cycle of sedentarity zero and let \( f \) be a homogeneous polynomial in \( n+1 \) variables. Let \( U_i = \{ x \in TP^n : x_i \neq -\infty \} \) be the standard affine charts. We set \( f_i = f(x_i = 0) \) and consider \( f_i \) as a function on \( U_i \). We may therefore perform the modification of \( X \cap U_i \) along \( f_i \) and denote it by \( M_i := \text{Mod}(X \cap U_i, f_i) \subseteq U_i \times T \). We will glue together the various \( M_i \)
to the global modification \( \text{Mod}(X, f) \). To do so, note that on the overlap \( U_i \cap U_j \), the difference of the two functions is given by

\[
f_i(x) - f_j(x) = d(x_j - x_i) = a_{ij}(x).
\]

Important for us is that \( a_{ij} \) is a globally linear function on \( U_i \cap U_j \) which does not attain \( \pm \infty \) at the infinite points of \( U_i \cap U_j \), i.e. its hypersurface in \( U_i \cap U_j \) is \( \mathbb{V}(a_{ij}) = \emptyset \). Its inverse is given by \( a_{ji} \), and \( a_{ij} + a_{jk} = a_{ik} \). We can therefore glue the charts \( U_i \times \mathbb{T} \) along the maps

\[
\alpha_{ij} : (U_i \cap U_j) \times \mathbb{T} \to (U_i \cap U_j) \times \mathbb{T}
\]

\[
(x, w) \mapsto (x, w + a_{ij}(x)). \tag{5.2}
\]

As the glueing maps are linear maps, the result is a toric variety denoted by \( \mathbb{T}A \). By construction, it is also obvious that the glueing maps identify the various modifications \( M_i \).

**Definition 5.2.7**

The *modification* of \( X \subseteq \mathbb{P}^n \) along the homogeneous polynomial \( f \), denoted by \( \text{Mod}(X, f) \), defined to be the \( k \)-cycle obtained from glueing the various modifications \( M_i \) along the maps \( \alpha_{ij} \).

Let us describe \( \mathbb{T}A \) in more detail. Let \( \Delta^d_n \subset \mathbb{R}^n \) be the \( n \)-simplex of size \( d \), i.e. the convex hull of \((0, \ldots, 0), (d, 0, \ldots, 0), \ldots, (0, \ldots, 0, d)\). The polytope in \( \mathbb{R}^{n+1} \) obtained as the convex hull of \( \Delta^d_n \times \{0\} \) and \((0, \ldots, 0, 1)\) is denoted by \( \Delta' \). The associated toric variety \( \mathbb{T}A' \) is equal to weighted projective space \( \mathbb{P}(1, \ldots, 1, d) \). We have \( \mathbb{T}A = \mathbb{T}A' \setminus \{p\} \), where \( p \) is the (singular if \( d > 1 \)) torus fixed point corresponding to the vertex \((0, \ldots, 0, 1)\) of \( \Delta' \). For a general projective toric variety, the same construction works when we replace \( \Delta^d_n \) by the polytope corresponding to the degree of \( f \) (as long as this polytope is full-dimensional). Even more general, we may use the fan language to describe \( \mathbb{T}A \). Let \( F \) be the fan corresponding to the ambient toric variety we start with. Then the degree of \( f \) defines a piecewise-linear function on \( F \) (up to adding a globally linear function). Let us assume that this function is strictly convex on \( F \) (i.e. the corresponding divisor is ample). The graph of this function is subdivided into a fan in \( \mathbb{R}^{n+1} \) which projects to \( F \). We extend the "graph" fan by adding, for each cone \( \sigma \), the cone \( \sigma + \mathbb{R} e_{n+1} \). The resulting fan is the fan of \( \mathbb{T}A \), in which \( \text{Mod}(X, f) \) lives. As our function is strictly convex, the upper graph of it is a convex polyhedral cone as well. By adding this cone to the fan, we obtain \( \mathbb{T}A' \). Finally, let us remark that we may modify along
tropical quotients of homogeneous polynomials $f = g/h = g - h$. In this case, to construct $TA$, we just take the difference of the two piecewise linear functions on the fan given by the two degrees, and extend the “graph fan” towards both direction $\pm \epsilon_{n+1}$.

**Remark 5.2.8**

For any modification $\text{Mod}(X, f)$ of $X$ along $f$, we have a canonical projection map $\delta : \text{Mod}(X, f) \rightarrow X$ which is called a *contraction*. Note that this map can is induced by a projection of $TA$ to the base ambient toric variety.

**Example 5.2.9**

Modifications of $\mathbb{T}P^n$ along hyperplanes are hyperplanes in $\mathbb{T}P^{n+1}$. Let us consider two examples. In both cases, we modify $\mathbb{T}P^2$, but along two different lines. Let us first consider the standard line given by $f = "x_0 + x_1 + x_2"$. Then $\text{Mod}(\mathbb{T}P^2, f) = V("x_0 + x_1 + x_2 + x_3")$ is just the standard plane in $\mathbb{T}P^3$ (cf. Figure 5.3). If we take a degenerated line instead, for example given by $g = "x_1 + x_2"$, then $M = \text{Mod}(\mathbb{T}P^2, g)$ is still a plane in $\mathbb{T}P^3$, but degenerated as well. In particular, $M$ contains a special point of sedentarity 3, namely $(0 : -\infty : -\infty : -\infty)$. This is in direct correspondence to the degeneracy of $V(g)$ which manifests in $(0 : -\infty : -\infty) \in V(g)$. Both examples are illustrated in Figure 5.4

![Figure 5.4: Modifying $\mathbb{T}P^2$ along two different lines](image)

**Definition 5.2.10**


Two tropical \( k \)-cycles \( X, Y \) are called equivalent, denoted by \( X \equiv Y \), if there exists a chain of \( k \)-cycles \( X = X_0, X_1, \ldots, X_l = Y \) such for each consecutive pair \( X_i, X_{i+1} \) one cycle is a modification of the other one, i.e. either \( X_i = \text{Mod}(X_{i+1}) \) or \( X_{i+1} = \text{Mod}(X_i) \) for a suitable function.

5.3 Configurations of hyperplanes in projective space

A difference between classical and tropical varieties is that tropical varieties always are equipped with a divisor of special points, the points of higher sedentarity. The main meaning of modifications and contractions is to add or remove (sedentarity zero) hypersurfaces to this divisor if necessary.

As an example, let us regard tuples of three distinct points in \( \mathbb{TP}^1 \). For the classical projective line \( \mathbb{CP}^1 \), all such tuples are equivalent, as we can always find an automorphism of \( \mathbb{CP}^1 \) which maps the tuple to \( 0, 1, \infty \).

In contrast, tropical \( \mathbb{TP}^1 \) has two distinguished points \( \pm \infty \). They are topologically different from all finite points and therefore the classical statements can obviously not be translated directly. Indeed, as we will see later, the automorphisms of \( \mathbb{TP}^1 \) are translations of the finite part by a real constant (which keeps the infinite points fixed) and reflections at any real number (which exchanges the infinite points). We see that automorphisms do not change the sedentarity of points. Instead, we are forced to use modifications here. Let us study this in more details.

First, we may fix 3 points in \( \mathbb{TP}^1 \) as reference points (like \( 0, 1, \infty \) for \( \mathbb{CP}^1 \)). A natural choice is \( -\infty, 0, \infty \). After what we just said above, it is even more natural to modify \( \mathbb{TP}^1 \) along 0 to get the line \( L = V(\"x_0 + x_1 + x_2\") \) in \( \mathbb{TP}^2 \). Now all our three reference points \( p_0, p_1, p_2 \) are infinite and are symmetrically given by \( x_0 = -\infty \), \( x_1 = -\infty \) resp. \( x_2 = -\infty \). Now let \( q_0, q_1, q_2 \in L \) be any other configuration of three distinct points. Then we can formulate the following statement.

**Lemma 5.3.1**

*There exists a series of modifications and contractions of \( L \) to an isomorphic line \( L' \) which transforms the points \( q_0, q_1, q_2 \) to the infinite points \( p_0, p_1, p_2 \).*

*Proof.* We modify along the points \( q_0, q_1, q_2 \), obtaining a line in \( \mathbb{TP}^5 \) with coordinates \( x_0, x_1, x_2, y_0, y_1, y_2 \). We now contract three times, namely we...
forget the original coordinates $x_0, x_1, x_2$ of $\text{TP}^2$. We end up with a line $L'$ in $\text{TP}^2$ whose infinite points $q'_0, q'_1, q'_2$ correspond to $q_0, q_1, q_2$. $L'$ is non-degenerated as the points $q_0, q_1, q_2$ are distinct, hence also $q'_0, q'_1, q'_2$. Hence $L$ and $L'$ are isomorphic and the identification $p_i = q'_i$ satisfies the required properties.

Basically the same is true for higher dimensions. Again as a reference we may fix the standard hyperplane $H \in \text{TP}^{n+1}$ with its $n+2$ planes $P_0, \ldots, P_{n+1}$ at infinity. Let $Q_0, \ldots, Q_{n+1}$ be another collection of hyperplanes in $H$. We assume that this collection is generic, i.e. the intersection of any choice of $n+1$ of these hyperplanes is empty. Then the following is true.

**Lemma 5.3.2**

There exists a sequence of modifications and contractions of $H$ to an isomorphic hyperplane $H'$ which transforms the chosen hyperplanes $Q_0, \ldots, Q_{n+1}$ to the infinite hyperplanes the infinite hyperplanes $P_0, \ldots, P_{n+1}$.

Under the assumption that each $Q_i$ is obtained as the stable intersection $H_i$ of $H$ with another hyperplane $H_i$ of $\text{TP}^{n+1}$, the proof of this statement is completely analogous to the one-dimensional case. The assumption is not necessarily satisfied, but these issues will be addressed more thoroughly in the following chapters.

Instead of hyperplanes, we may also consider points in $\text{TP}^n$. Let us consider the case $n = 2$. Let $p_0, p_1 \in \text{TP}^2$ be two distinct points. Then there exists a line $L \subset \text{TP}^2$ containing $p_0$ and $p_1$. The modification along $L$ gives a hyperplane $H \subset \text{TP}^3$, and we identify $L$ with the line $\overline{L} = H \cap \{x_3 = -\infty\}$ of sedentarity (at least) 1. Consistently we identify $p_0$ and $p_1$ with the points $\overline{p_0}, \overline{p_1} \in \overline{L}$ which are mapped to $p_0$ resp. $p_1$ by the contraction map. For any point $p \in \text{TP}^2 \setminus L$, there is a unique lift to $H$, denoted by $\overline{p}$. Note
that \( H \) can be contracted in any of the 4 standard directions \(-e_0, \ldots, -e_3\) to projective space \( TP^2 \). For a point configuration \( p_0, p_1, p_2, \ldots, p_n \) and a chosen contraction of \( H \), the images of \( \overline{p}_0, \overline{p}_1, \overline{p}_2, \ldots, \overline{p}_n \) give a new point configuration in \( TP^2 \). Two point configurations are called \textit{projectively equivalent} if they can be connected by a series of the above construction and automorphisms of \( TP^2 \) (generated by translations and permutations of the coordinates).

**Lemma 5.3.3**

Let \( p_0, \ldots, p_3 \) be 4 generic points in \( TP^2 \) \textit{(i.e. no three of them are contained in a line)}. Then this point configuration is tropically equivalent to the configuration \((-\infty : 0 : 0), (0 : -\infty : 0), (0 : 0 : -\infty), (0 : 0 : 0)\).

**Proof.** Let \( s = \text{sed}(p_0) + \text{sed}(p_1) + \text{sed}(p_2) \) be the sum of the sedentaries of the first three points. We are done if \( s = 6 \), because this implies \( \{p_0, p_1, p_2\} = \{(-\infty : 0 : 0), (0 : -\infty : 0), (0 : 0 : -\infty)\} \) and, as the \( p_i \) are generic, \( p_3 \) must be a finite point in \( R^2 \). Thus, we can use an automorphism of \( TP^2 \) to reorder the first 3 points correctly and to move \( p_3 \) to \((0, 0, 0)\).

It remains to show that, if \( s < 6 \), there is always a choice of modification-contraction as above such that the obtained tropically equivalent point configuration is still generic and has sedentarity \( s' > s \). Then the claim follows. So let us assume \( s < 6 \). Let \( L_0, L_1, L_2 \) be the three lines passing through \( \{p_1, p_2\} \) \textit{resp.} \( \{p_0, p_1\} \). Note that \( s < 6 \) implies

\[
\{L_0, L_1, L_2\} \neq \{V(x_0), V(x_1), V(x_2)\}.
\]

In other words, one of the lines, say \( L_0 \), must be of sedentarity zero. Moreover, there exists at least one coordinate lines, say \( V(x_0) \), which contains at most one of the points \( p_0, p_1, p_2 \). We modify along \( L_0 \) and then contract \( V(x_0) \) (to be precise, we contract by forgetting the coordinate \( x_0 \)). Let \( H = \text{Mod}(TP^2, L_0) \) be the modified plane in \( TP^3 \). By construction we have \( \text{sed}(\overline{p}_0) = \text{sed}(p_0) + 1 \), \( \text{sed}(\overline{p}_1) = \text{sed}(p_1) + 1 \) and \( \text{sed}(\overline{p}_2) = \text{sed}(p_2) \). When contracting (let us call the contraction \( \pi \)), the sedentarity of a point drops (by one) if and only if it is contained in the contracted line at infinity \( \overline{V(x_0)} = \text{Mod}(V(x_0), L_0 \cap V(x_0)) \). By our choice, \( \overline{V(x_0)} \) contains at most one of the points \( \overline{p}_0, \overline{p}_1, \overline{p}_2 \). It follows

\[
\text{sed}(\pi(\overline{p}_0)) + \text{sed}(\pi(\overline{p}_1)) + \text{sed}(\pi(\overline{p}_2)) \geq s + 2 - 1 > s,
\]

so indeed, we increased the sedentarity of the new point configuration.
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![Figure 5.6: Two examples of the modification-contraction process](image)

It remains to check that \( \pi(\vec{p}_0), \ldots, \pi(\vec{p}_3) \) are still generic. By construction, \( \pi(\vec{p}_0), \pi(\vec{p}_1), \pi(\vec{p}_2) \notin \pi(\overline{L}_\infty) \). Let \( M \) be a line passing through at least 3 of the points \( \pi(\vec{p}_0), \ldots, \pi(\vec{p}_3) \). For any \( x \in M \cap \pi(\overline{L}_\infty) \), we can choose the “height” of \( \text{Mod}(M, x) \) such that \( \text{Mod}(M, x) \subset H \). We choose \( x = \pi(\vec{p}_3) \) if \( \pi(\vec{p}_3) \in M \), or any other \( x \in M \cap \pi(\overline{L}_\infty) \) otherwise. Then \( \text{Mod}(M, x) \) is a line in \( H \) which contains at least 3 of the points \( \vec{p}_0, \ldots, \vec{p}_3 \). Contracting along the original modification gives a line containing at least 3 of the points \( p_0, \ldots, p_3 \). This contradicts to the assumption that the \( p_0, \ldots, p_3 \) are generic.

Similar arguments also work in higher dimensions.

### 5.4 Equivalence of rational curves

Let us anticipate the definition of an abstract tropical variety in its easiest case, namely for (rational) tropical curves. An abstract tropical smooth curve is a tuple \( C = (\Gamma, d) \), where \( \Gamma \) is a graph (i.e. a topological space homeomorphic to a one-dimensional simplicial complex) and \( d \) is a complete inner metric on the “finite part” \( \Gamma^\circ := \Gamma \setminus \partial \Gamma \). Here \( \partial \Gamma \) denotes the set of 1-valent vertices of \( \Gamma \) (which is clearly independent of the chosen simplicial structure). It follows that, for a chosen simplicial structure, we call the edges containing an 1-vertex ends of \( C \). Other edges are called bounded edges. A bounded edge is homeomorphic to an interval \([0, l] \subset \mathbb{R}, l > 0\),
and we call \(l\) the *length* of the edge. An end is homeomorphic to \([-\infty, 0]\), where \(-\infty\) is identified with the 1-valent vertex. Therefore, \(C\) is completely described by its combinatorial graph \(\Gamma\) and by a positive real length \(l(E)\) for any bounded edge \(E\). Finally, \(C\) is called *rational* if \(\Gamma\) is a tree.

The reader might be surprised that the definition of an abstract curve does not require any balancing condition. In fact, as we will later see, given the valence of a point, there is only one smooth local model in dimension 1. For a point of valence \(n + 1\), this local model is given by the standard line in \(\text{TTP}^n\) (i.e. the 1-dimensional fan with \(n + 1\) rays pointing to the standard directions \(-e_0, \ldots, -e_n\)) or \(T\), if \(n = 0\). Hence the tropical structure of a smooth curve is completely determined by its graph and the balancing condition at each (finite) point is somewhat hidden by the uniqueness of local models.

An *isomorphism* of two abstract tropical curves \(C, D\) is a continuous map \(C \to D\) which restricts to an isometry \(C^o \to D^o\).

**Lemma 5.4.1**

Any smooth rational tropical curve \(C\) is isomorphic to a sequence of modification of \(\text{TTP}^1\) along single points.

*Proof.* Assume that \(C = (\Gamma, d)\) has more than two 1-valent vertices. Then we can contract one of them, i.e. we just remove the vertex and the interior of the adjacent edge \(E\) to obtain a new abstract curve \(C' = C \smallsetminus E\). Let \(x \in C'\) be the point where \(E\) was attached to \(C'\). When we modify \(C'\) along \(x\), we just glue an interval \([-\infty, 0]\) to \(C'\) (identifying 0 to \(x\)). Hence \(\text{Mod}(C', x)\) is isomorphic to \(C\). Now we repeat this process until we end up with a curve with only two ends. Clearly its graph must be linear and hence the curve is isomorphic to \(\text{TTP}^1\), which proves the claim. \(\square\)

### 5.5 Equivalence of linear spaces

In classical algebraic geometry, a linear space of dimension \(n\) (embedded in some big \(\mathbb{CP}^N\)) is always isomorphic to \(\mathbb{CP}^n\). In tropical geometry, linear spaces of a fixed dimension \(n\), i.e. positive tropical \(n\)-cycles of degree 1 in some \(\text{TTP}^N\), can look quite differently. This is already happens for lines where, aside from the most natural "model" \(\text{TTP}^1\), we may consider lines in \(\text{TTP}^N\) with a bigger number of infinite points and more complicated (though rational) graphs. Again, tropical equivalence generated by tropical modifications is the right notion to identify all these different models/embeddings of tropical lines and linear spaces.
Lemma 5.5.1

Let $L \subseteq \mathbb{TP}^N$ be a tropical linear space of dimension $n$. Then $L$ is a multiple modification of $\mathbb{TP}^n$. In particular, all linear spaces of dimension $n$ are tropically equivalent to each other.

Proof. We choose a torus fixed point $p \in \mathbb{TP}^N$ such that $p \notin L$ and project along $p$. More precisely, we consider the projection map $\pi : \mathbb{TP}^N \setminus \{p\} \to \mathbb{TP}^{N-1}$ which in each affine chart not containing $p$ is just given by $\mathcal{T}^N \to \mathcal{T}^{N-1}$ forgetting one coordinate (say, after reordering, the last one). Let $L' = \pi(L)$ be the image of $L$. For each point $x \in L'$, the preimage $\pi^{-1}(x) \cap L$ is either a single point or an interval of the form $[-\infty, a]$. In any other case, (e.g. an interval) one can the existence of a nearby fiber which contains at least two isolated points which contradicts the assumption that $L$ is of degree 1. It follows that $L$, equipped with trivial weights, is a linear space again. Moreover, a generic fiber $\pi^{-1}(x) \cap L$ consists of only one point, say $\hat{x}$, and we may define, in each affine chart, a piecewise linear function $f$ on $L'$ whose value at a generic $x$ is given by the last coordinate of $\hat{x}$. Let us now modify $L'$ along $f$ (in each chart). By construction, the graph of $f$ is contained in $L$ and ... Therefore $L$ is in fact equal to Mod($L', f$). If $L' \notin \mathbb{TP}^{N-1}$, we continue until we finally end up with $\mathbb{TP}^n$. \hfill $\square$
6 Smooth varieties

6.1 Tropical spaces

Let us give some definitions/notation:

- Our local building blocks for tropical spaces will be open subsets of polyhedral sets in $\mathbb{T}^n$. We call such sets open polyhedral sets (in $\mathbb{T}^n$).

- As usual polyhedral sets, an open polyhedral set $X$ contains an open dense subset of generic points $X^\text{gen}$. Remember that we call a point generic if it has sedimentarity zero and if a neighbourhood of the point in $X$ looks like an affine plane. We can turn $X$ into a weighted open polyhedral subset by specifying an integer weight for all connected components of $X^\text{gen}$. $X$ is called effective if all weights are strictly positive.

- As the balancing condition is a local condition, it carries over to open polyhedral sets directly. More precisely, for any point $p \in X$ we define the star of $p$ $\text{Star}_X(p)$ as before to be the fan in $\mathbb{R}^{n-\text{sed}(p)}$ containing all direction vectors $v$ for which $p + \epsilon v \in X$ holds for small $\epsilon > 0$. As $\text{Star}_X(p)$ is a usual (non-open) fan, we call $X$ balanced if and only if $\text{Star}_X(p)$ is balanced for all $p \in X$ of minimal sedimentarity. In other words, after choosing a polyhedral structure for $X$ (or at least for $\text{Star}_X(p)$), we may check the balancing condition as in Equation (4.1).

Recall from chapter 3 that in tropical arithmetics, monomial maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ correspond simply to linear maps which map $\mathbb{Z}^n$ to $\mathbb{Z}^m$. We call such maps $\mathbb{Z}$-linear maps. In chapter 2 we discussed that the tropical structure of $\mathbb{R}^n$ manifests through the data of integer tangent vectors $T_x^\mathbb{Z} \subset T_x \mathbb{R}^n$ at each point $x \in \mathbb{R}^n$ (cf. Definition 2.1.1). Obviously, $\mathbb{Z}$-linear maps preserve this tropical structure. However, as we only fix the lattices of integer tangent vectors and not the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ itself, we may combine the monomial maps with arbitrary translations with translation vector $v \in \mathbb{R}^m$. Still, the tropical structure of integer tangent vectors is preserved. Such maps are called affine $\mathbb{Z}$-linear maps. Conversely, any smooth map $F : \mathbb{R}^n \to \mathbb{R}^m$ whose differential is $\mathbb{Z}$-linear at each point is affine $\mathbb{Z}$-linear, as $\text{Mat}(n,m,\mathbb{Z})$ is discrete (cf. Lemma 2.3.1).
Definition 6.1.1
A map $F : \mathbb{R}^n \to \mathbb{R}^m$ is called affine $\mathbb{Z}$-linear if it is the sum of a $\mathbb{Z}$-linear map $\mathbb{R}^n \to \mathbb{R}^m$ and a translation by an arbitrary vector $v \in \mathbb{R}^m$.

Let $U \subseteq \mathbb{T}^n$ be open. A map $f : U \to \mathbb{T}^m$ is called affine $\mathbb{Z}$-linear if it is continuous and if $f|_{U \cap \mathbb{R}^n}$ is the restriction of an affine $\mathbb{Z}$-linear map $F : \mathbb{R}^n \to \mathbb{R}^m$, where $\mathbb{R}^m \subset \mathbb{T}^m$ is a suitable stratum of $\mathbb{T}^m$.

Let $X \subseteq \mathbb{T}^n$ and $Y \subseteq \mathbb{T}^m$ be two open balanced polyhedral sets. A map $f : X \to Y$ is called a tropical morphism if it is locally affine $\mathbb{Z}$-linear. This means that for each point $p \in X$ there exists an open neighbourhood $U$ of $p$ in $\mathbb{T}^n$ and an affine $\mathbb{Z}$-linear map $F : U \to \mathbb{T}^n$ which coincides with $f$ on $U \cap X$.

$f$ is called a tropical isomorphism if there exists an inverse tropical morphism $g : Y \to X$ and, moreover, if

$$\omega(x) = \omega(f(x))$$

for all $x \in X^{\text{gen}}$. Note that the existence of $g$ implies that $f(x)$ is a generic point of $Y$.

Example 6.1.2
The group of tropical automorphisms of $\mathbb{R}^n$ is the semidirect product

$$\text{Aut}(\mathbb{R}^n) = \text{GL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n,$$

where the first factor represents $\mathbb{Z}$-invertible linear maps and the second factor parameterizes translations.

We are now ready to introduce the first notion of general tropical spaces obtained from glueing balanced polyhedral sets. Later on, we will add more and more additional requirements to these spaces to make them more manageable.

Definition 6.1.3
Let $X$ be a topological space. A tropical atlas or tropical structure on $X$ is a collection of tuples $(U_i, \psi_i, V_i)_i$ subject to the following constraints:

- $X = \bigcup_i U_i$ is an open covering of $X$.
- For each $i$, $V_i \subset \mathbb{T}^N$ (for suitable $N$) is an open effective balanced polyhedral set.
- For each $i$, $\psi_i : U_i \to V_i$ is a homeomorphism.
• For each pair $i, j$ with $U_i \cap U_j \neq \emptyset$, the composition map

$$\psi_i \circ \psi_j^{-1} : \psi_j(U_i \cap U_j) \to \psi_i(U_i \cap U_j)$$

is a tropical morphism with $\omega(x) = \omega(\psi_i \circ \psi_j^{-1}(x))$ whenever $x$ and $\psi_i \circ \psi_j^{-1}(x)$ are generic. In other words, $\psi_i \circ \psi_j^{-1}$ is a tropical isomorphism with inverse $\psi_j \circ \psi_i^{-1}$.

Two atlases on $X$ are called equivalent if their union also forms an atlas. A tropical space is a topological space $X$ together with the choice of an equivalence class of atlases.

If all $V_i$ are of pure dimension $n$, we say that $X$ is of dimension $n$.

Let us list some obvious properties of a tropical space $X$.

• $X$ contains an open dense subset of generic points $X^{\text{gen}}$.

• The weights on the various charts are compatible and therefore glue to give weights on each connected component of $X^{\text{gen}}$.

• For each point $p \in X$, the codimension $\text{codim}_X(p)$ is well-defined as the codimension of $\psi_i(p)$ in $V_i$ for any chart containing $p$. In particular, the $k$-skeleton

$$X^{(k)} := \{x \in X | \text{codim}_X(p) \geq n - k\}$$

is a polyhedral subset of $X$ (i.e. its restriction to any chart is a polyhedral subset of $V_i$). In particular, $X^{(k)}$ is closed. A point $p \in X$ is generic if and only if $\text{codim}_X(p) = 0$.

• For each point $p \in X$, we define the sedentarity of $p$ in $X$ to be

$$\text{sed}_X(p) := \min\{\text{sed}_{TN}(\psi_i(p)) - \text{sed}_{TN}(\psi_i(p'))\}, \quad (6.1)$$

where $(U_i, \psi_i, V_i)$ is a chart containing $p$ and $p' \in U_i$ is a generic point. This is well-defined, but might be strictly greater than the codimension of $p$. In any case, in the full generality of tropical spaces, the notion of sedentarity requires some caution.

• For each point $p \in X$, we can consider $\text{Star}_X(p) \in \mathbb{R}^N$ (for suitable $N$). If $X$ is of dimension $n$ and $\text{sed}_X(p) = 0$, $\text{Star}_X(p)$ is a balanced fan of dimension $n$. It is well-defined only up to tropical isomorphisms of fans given by $\mathbb{Z}$-linear maps in $\text{Mat}(N, N', \mathbb{Z})$. 

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6.2 Regularity at Infinity

In most situations, we would like to impose some kind of finite type condition on our tropical spaces. This regards the local polyhedral sets $V_i$ (a priori, we might allow polyhedral sets which have an infinite number of cells), the number of charts needed to give an atlas of $X$ and, finally, the completeness of $X$ (open polyhedral sets can just stop at some bounded distance — this is usually undesired for tropical spaces). We will address all these issues in the following definition.

**Definition 6.1.4**

A tropical space $X$ is called of finite type if it admits a tropical atlas $\{(U_i, \psi_i, V_i)\}_i$ subject to the following conditions.

(a) The number of charts is finite, i.e. $i \in I$ is taken from a finite index set $I$.

(b) Each $V_i$ is the open subset of a finite polyhedral set, i.e. the union of finitely many polyhedra. This was implicit in the general definition, but we want to emphasize it here.

(c) Each chart can be extended, i.e. there exists a chart $(U'_i, \psi'_i, V'_i)$ such that $V'_i \subseteq V_i$ for all $i$. The closure is taken in $\mathbb{T}^N$.

In the following, a tropical space of pure dimension and of finite type is called a tropical variety.

**Example 6.1.5**

A (finite) balanced effective polyhedral set in $\mathbb{T}^n$ is a tropical space of finite type. More general, tropical cycles, considered as polyhedral sets, in any toric variety are examples of tropical spaces of finite type. Note however that in order to satisfy the third condition from above, it might be necessary to split a chart in $\mathbb{T}^n$ into several smaller ones. For example, tropical cycles in $\mathbb{R}^n$ are of finite type, but in general we have to use more than one embedding $\mathbb{R}^n \subseteq \mathbb{T}^n$ to satisfy the third condition.

**Example 6.1.6**

A simple example of a tropical space which is not of finite type is the unit interval $(0, 1) \subseteq \mathbb{T}$.

6.2 Regularity at Infinity

As we know from chapter 5, tropical varieties come naturally in classes given by the process of modification. This reflects the fact that, when
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comparing tropical to classical varieties, they correspond not only to a variety but to a tuple \((X, D)\) where \(X\) is a variety and \(D \subset X\) is a distinguished divisor. In the tropical world, this divisor is given by points of higher sedentarity and can be changed by modifications. Unlike in the classical world, we cannot completely forget this additional choice of divisor. As a consequence, when we want to define a good class of tropical spaces, we are forced to think about requirements imposed on the divisor at infinity. Probably the most important property to impose is regularity at infinity. It should be compared to the normal crossing divisor property when considering a tuple \((X, D)\) in the classical world.

**Definition 6.2.1**

Let \(P \subseteq \mathbb{T}^N\) be an open polyhedral set of pure dimension \(n\). Then \(P\) is called *regular at infinity* if one of the following equivalent conditions hold:

(a) For each stratum \(T_I \subset \mathbb{T}^N\) such that \(P \cap T_I \neq \emptyset\), we have

\[
\dim(P \cap T_I) \leq n - |I|.
\]

(b) For each open stratum \(R_I \subset \mathbb{T}^N\) such that \(P \cap R_I \neq \emptyset\), we have

\[
\dim(P \cap R_I) = n - |I|.
\]

(c) For each point \(p \in P\), we have

\[
sed_{\mathbb{T}^N}(p) \leq \text{codim}_P(p).
\]

A tropical space \(X\) is called *regular at infinity* if it admits a tropical atlas \(\{(U_i, \psi_i, V_i)\}_i\) where all \(V_i\) are regular at infinity.

**Remark 6.2.2**

According to our definition, if \(P \in \mathbb{T}^N\) is regular at infinity, then \(P\) is necessarily of sedentarity zero (in the sense that \(P \cap R^N\) is dense in \(P\)). So \(P\) cannot have components sitting in the boundary of \(\mathbb{T}^N\). In particular, the sedentarity of a point \(p \in P\) defined as in equation (6.1) is equal to the usual sedentarity \(sed_{\mathbb{T}^N}(p)\). Thus we can just write \(sed(p)\) without causing confusion.

The main feature of regularity at infinity is we can easily describe neighbourhoods of points of higher sedentarity. Before we formulate this, let
us fix some notations. Let $X$ be a tropical space. We denote the subset of points of sedentarity greater or equal than $k$ by $X^{(k)}_{\infty}$. We also use the abbreviation $X_{\infty} := X^{(1)}_{\infty}$. Obviously, these sets are polyhedral subsets of $X$.

**Theorem 6.2.3**
Let $X$ be a tropical space which is regular at infinity. Then each point $p \in X$ of sedentarity $k$ has a neighbourhood isomorphic to $U \times V \subseteq X^{(k)}_{\infty} \times \mathbb{T}^k$, where $U$ and $V$ are neighbourhoods of $p$ resp. $(-\infty, \ldots, -\infty)$ in $X^{(k)}_{\infty}$ resp. $\mathbb{T}^k$.

*Proof.* Let us first consider the case of a usual polyhedral set $P \subseteq \mathbb{T}^N$ which is regular at infinity. Choose $I$ such that $P \cap \mathbb{R}_I \neq \emptyset$ and consider $(P \cap \mathbb{R}_I) \times \mathbb{T}^I \subset \mathbb{T}_I \times \mathbb{T}^I = \mathbb{T}^N$. Regularity at infinity implies that this is a polyhedral set of dimension $n$. We know from that any cell $\sigma \in P$ with $\sigma \cap \mathbb{R}_I \neq \emptyset$ must be contained in $(P \cap \mathbb{R}_I) \times \mathbb{T}^I$. Hence the union $W$ of the relative interiors of all such cells forms an open neighbourhood of $P \cap \mathbb{R}_I$ in both $P$ and $(P \cap \mathbb{R}_I) \times \mathbb{T}^I$. The topology of $(P \cap \mathbb{R}_I) \times \mathbb{T}^I$ is given by the product topology of the two factors. Hence, for each $p \in P \cap \mathbb{R}_I$, we find $p \in U \times V \subset W$ as promised by the theorem.

For a general tropical space $X$, the assertion follows easily from the local case considered before. Indeed, all we have to check is that given a local chart $(U, \psi, V)$ with $V \in \mathbb{T}^N$ regular at infinity, then $\text{sed}_X(p) = \text{sed}_X^{\psi}(\psi(x))$ for all $p \in U$. This is a consequence of the definitions. \hfill $\square$

**Corollary 6.2.4**
If we equip with generic points of $X^{(k)}_{\infty}$ with the weights of nearby generic points of $X$, this is well-defined and turns $X^{(k)}_{\infty}$ into a tropical space which is of pure dimension $n - k$ and regular at infinity.

**Remark 6.2.5**
Theorem 6.2.3 can be refined as follows. Each point $p \in X$ in a tropical space regular at infinity has a neighbourhood isomorphic to a neighbourhood $U \times V \subseteq \text{Star}_X(p) \times \mathbb{T}^k$ of $(0, -\infty)$. Furthermore, we can assume that $U = \tilde{U} \cap \text{Star}_X(p)$, where $\tilde{U} \subseteq \mathbb{R}^n$ is a convex neighbourhood of 0, and $V = [-\infty, 0)^k$. Such a neighbourhood is called a *standard neighbourhood* of $p$.

Codimension and Sedentarity are examples of local properties of points in polyhedral sets which are invariant under isomorphism as defined in 6.1.1 Therefore they give to rise to well-defined notions for abstract
tropical spaces. We will now identify more of such local and invariant data in the case of $P \subseteq \mathbb{T}^N$ being regular at infinity.

**Definition 6.2.6**

Let $P \subseteq \mathbb{T}^N$ be an open polyhedral set which is regular at infinity. Let $p \in P$ be a point with sedentarity $k = \text{sed}(p)$.

We define *germs of affine $\mathbb{Z}$-linear functions at $p$* to be elements of the set

$$\mathcal{A}ff_p := \mathcal{O}^\ast_{P,p} := \{ f : U \to \mathbb{T} \mid p \in U \subseteq P \text{ open}, \ f \text{ tropical morphism } \}/ \sim,$$

(6.2)

where $f \sim g$ when the two functions agree on a neighbourhood of $p$. Note that by definition of tropical morphism, we can assume that $f$ is the restriction (and extension to infinity) of an affine $\mathbb{Z}$-linear function on $\mathbb{R}^N$.

Obviously, tropical multiplication on $\mathbb{T}$ (i.e. usual addition of functions) turns $\mathcal{A}ff_p$ into a semigroup with identity $-\infty$. A couple of interesting objects can be derived from this semigroup.

The *cotangent space lattice of $P$ at $p$* is defined to be

$$\mathbb{Z}T^\ast_p := \{ f \in \mathcal{A}ff_p \mid f(p) = 0 \}.$$

(6.3)

Note that $\mathbb{Z}T^\ast_p$ is an honest group (all elements are invertible). Moreover, theorem [6.2.3] shows that $\mathbb{Z}T^\ast_p$ is free abelian and finitely generated. The *cotangent space of $P$ at $p$* is defined to be the $\mathbb{R}$-vectorspace

$$T^\ast_p := \mathbb{Z}T^\ast_p \otimes \mathbb{R}.$$

(6.4)

By dualizing, we obtain the *tangent space (lattice) $\mathbb{Z}T^\ast_p \subset T^\ast_p$*. Note that an isomorphism of open balanced polyhedral sets induces isomorphisms of the above groups/vector spaces given by pulling back functions. Hence, all the objects are well-defined as abstract groups/vector spaces for any tropical space $X$. Note also that $\text{Star}_X(p)$ is canonically embedded in $T^\ast_p$ and spans the vector space (i.e. is not contained in a smaller subspace).

From now on, we will always think of $\text{Star}_X(p)$ as sitting in $T^\ast_p$. Hence, a standard neighbourhood of $p$ is a open polyhedral set in $T^\ast_p \times \mathbb{T}^k$.

$\mathbb{Z}T^\ast_p$ pretty much describes the invertible elements of $\mathcal{A}ff_p$. Indeed, we have

$$\mathcal{A}ff^\ast_p \cong \mathbb{Z}T^\ast_p \times \mathbb{R},$$

(6.5)

where the $\mathbb{R}$-factor parameterizes $f(p)$. The remaining data of $\mathcal{A}ff_X$ is captured by the *slope semigroup of $P$ at $p$*

$$S_p := \mathcal{A}ff_p / \mathcal{A}ff^\ast_p.$$

(6.6)
6.3 Tropical morphisms

Clearly, the slope semigroup is isomorphic to \( Z_{\geq 0}^k \). If we restrict \( f \) to a standard neighbourhood of \( p \), the isomorphism is given by the slopes of \( f \) in \( e_i \)-direction (where \( e_i \) is the standard basis vector in \( \mathbb{R}^k \subseteq \mathbb{T}^k \)). Again, \( S_p \) is a well-defined semigroup for general tropical spaces as well.

6.3 Tropical morphisms

We start with the definition of a tropical morphism.

**Definition 6.3.1**

Let \( X \) and \( Y \) be two tropical spaces. A continuous map \( f : X \to Y \) is called a tropical morphism if for each pair of charts \((U, \psi, V)\) for \( X \) and \((U', \psi', V')\) for \( Y \) the composition map

\[
\psi' \circ f \circ \psi^{-1} : \psi(f^{-1}(U') \cap U) \to V'
\]

is a tropical morphism in the sense of definition 6.1.1.

6.3.1 Maps between tropical spaces

**Definition 6.3.2**

A tropical map \( h : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \) between two tropical spaces is a continuous map \( h : X \to Y \) such that for any open convex set \( U \) and any regular function \( f \in \mathcal{O}_Y(U) \) the composition \( f \circ h \) is a regular function on \( h^{-1}(U) \) (recall that in terms of open convex sets this means that for any convex open set \( V \subset h^{-1}(U) \) we have \( f \circ h|_V \in \mathcal{O}_X(V) \)). We denote the pull-back map \( f \mapsto f \circ h \) with \( h^* \). For each open convex set \( U \in Y \) the map \( h^* \) gives \( h^*_U : \mathcal{O}_Y(U) \to \mathcal{O}_X(h^{-1}(U)) \). Note that \( h^* \) induces a map between \( \text{Aff}(U) \) and \( \text{Aff}(h^{-1}(U)) \) and that this map is affine. Note also that \( h^* \) induces a linear map \( T^*_f(x) Y \to T^*_x X \) and a map \( T^*_f(x) Y \to T^*_x Z \) on the lattices there. When the sheaves \( \mathcal{O}_X \) and \( \mathcal{O}_Y \) are clear from the context we may denote a tropical map just as \( h : X \to Y \).

**Remark 6.3.3**

Note that the sedentarity of \( h(X) \subset Y \) can be positive (e.g., we may consider a constant map to a point of non-zero sedentarity). Nevertheless, we may pass to the case of zero-sedentarity for \( h(X) \subset Y \) by the following trick. Suppose that the sedentarity of \( h(x) \) in \( Y \) is \( k > 0 \). Consider the closure of the connected component of points of \( Y \) of sedentarity \( k \) that contains points of \( h(X) \) (note that there is only one such component as the set of points of sedentarity 0 in \( X \) is connected).
**Definition 6.3.4**

A tropical affine map \( H : \mathbb{R}^M \to \mathbb{R}^N \) is a map that can be obtained as a composition of a linear map \( \mathbb{R}^M \to \mathbb{R}^N \) defined over \( \mathbb{Z} \) and an arbitrary translation in \( \mathbb{R}^N \). We say that \( H \) extends to \( x \in TM \) if for any sequence of points \( \{x_j\}_{j=1}^{\infty} \) of \( \mathbb{R}^N \) converging to \( x \) the sequence \( \{H(x_j)\}_{j=1}^{\infty} \) converges in \( \mathbb{T}^N \) to a point that only depends on \( x \) and not on the choice of \( \{x_j\} \). We can extend \( H \) to such \( x \) by setting \( H(x) \) to be equal to this limit.

A (partially-defined) tropical affine map \( \bar{H} : T^M \to T^N \) is a map obtained by extending an affine-linear map \( H : \mathbb{R}^M \to \mathbb{R}^N \) to all such points. We have \( T^M \supset D \supset \mathbb{R}^M \) for its domain of definition \( D \). Furthermore, it is easy to compute \( D \) explicitly in terms of the integer \( M \times N \) matrix defining the linear part of \( H : \mathbb{R}^M \to \mathbb{R}^N \). Extendability of \( H \) to a point \( x \in T^M \) depends only on the refined sedentariness of \( x \). If \( j_1, \ldots, j_k \) are the coordinates turning to \(-\infty \) at \( x \) then clearly \( H \) is extendable to \( x \) if and only if the \( j_1, \ldots, j_k \)th columns of the \( M \times N \) matrix do not contain negative numbers.

Alternatively, as the next proposition shows, one can define tropical maps as such maps \( h : X \to Y \) that become affine-linear when viewed in local charts.

**Proposition 6.3.5**

A continuous map \( h : X \to Y \) is tropical if and only if for every \( x \in X \) there is a chart \( U \ni x, \phi : U \to T^M \), in \( X \) and a chart \( V \ni h(x), \psi : V \to T^N \), in \( Y \) and a tropical affine map \( \bar{H} : T^M \to T^N \) with the domain of definition containing \( \phi(U) \) such that we have \( h(z) = \psi^{-1} \circ \bar{H} \circ \phi \) for any \( z \in U \).

**Proof.** Suppose that \( h \) satisfies to Definition 6.3.2. Let us take the star \( \text{Star}(E_{h(x)}) \) of the open stratum \( E_{f(x)} \subset Y \) of \( f(x) \) for \( V \) and \( \text{Star}(E_x) \cap h^{-1}(V) \) for \( U \). By Corollary ?? our map in these charts can be presented as a partially-defined map between

Suppose that \( h \) is tropically affine in local charts. In such case pull-backs of affine functions (Laurent monomials) are also affine. Thus the same is true for tropical Laurent polynomials. \( \square \)

**Definition 6.3.6**

A tropical automorphism of a tropical space \((X, \mathcal{O}_X)\) is a self-homeomorphism of \( X \) preserving the structure sheaf \( \mathcal{O}_X \). In other words it is a tropical map that admits a tropical inverse map.

Clearly, all automorphisms of a given space form a group.
Lemma 6.3.7
Suppose that $h : X \to X$ is a tropical automorphism such that $h(x) = x$ and $h^* : T^*_x \to T^*_x$ is identity. Then $h|_{E_x}$ is identity. Furthermore, if the sedentarity of $x$ is 0 then $h|_{\text{Star}(E_x)}$ is identity as well.

Proof. Note that after dualization $h^*$ determines the affine map corresponding to $h^*$ in a local chart of $x$, except for those coordinates that turn to $-\infty$. If $h^*$ is identity then this map is identity as well. □

Definition 6.3.8
Given a tropical map $h : X \to Y$ and a point $x \in X$ we define a linear map

$$(dh)_x : T^*_x(X) \to T^*_y(Y)$$

as follows. We identify $T^*_x(X)$ and $T^*_y(Y)$ with the affine spans of $\text{Star}(E_x)$ and $\text{Star}(E_{h(x)})$ in local charts with the help of Corollary ???. By Proposition 6.3.5 we get a linear map.

Taking the dual we can restate Lemma 6.3.7 in terms of $dh$ as follows.

Lemma 6.3.9
Suppose that $h : X \to X$ is a tropical automorphism such that $h(x) = x$ and $(dh)_x : T^*_x(X) \to T^*_x(X)$ is identity. Then $h|_{\text{Star}(E_x)}$ is identity. Furthermore, if the sedentarity of $x$ is 0 then $h|_{\text{Star}(E_x)}$ is identity as well.

Lemma 6.3.10
The group of tropical automorphisms of $\mathbb{R}^N$ is generated by $\text{GL}_N(\mathbb{Z})$ acting by linear transformations and $\mathbb{R}^N$ acting by translations. Here we view $\mathbb{R}^N$ as $(\mathbb{T}^*)^N$ i.e. the tropical structure sheaf is constant on all open convex sets and is given by tropical Laurent polynomials in $N$ variables.

Proof. The lemma follows from Lemma 6.3.9 after applying a suitable tropical affine transformation. □

Lemma 6.3.11
The group of tropical automorphisms of $\mathbb{T}^n$ is generated by the symmetric group $S_n$ interchanging the $n$ coordinates and $\mathbb{R}^n$ acting on $\mathbb{T}^n$ by translations.

Proof. Clearly a tropical automorphism of $\mathbb{T}^n$ induces a tropical automorphism of $S^*_0\mathbb{T}^n$, where $0^*_\mathbb{T} = \{-\infty, \ldots, -\infty\}$ is the origin in $\mathbb{T}^n$. By Proposition ?? the group of automorphisms of $S^*_0\mathbb{T}^n$ is the symmetric group $S_n$, 130
corresponding to permutations of coordinates. Suppose that the cor-
responding automorphism of $S_{0 \times n}^*$ is identity. By Proposition \[6.3.5\] near $x$ our automorphism of $\mathbb{T}^n$ must be given by a tropical affine map, but since the coordinates are not exchanged the linear part of this map is identity. Thus near $0_T$ such automorphism must be a translation. By Lemma \[6.3.9\] it is a translation globally. 

**Proposition 6.3.12**

A sufficiently small neighborhood $U$ of a point $x$ in a tropical space $X$ regular at infinity comes with an embedding $\Phi: U \subset \mathbb{T}^s \times \mathbb{R}^{N_x}$ that is well-defined up to tropical isomorphisms of $\mathbb{T}^s \times \mathbb{R}^{N_x}$. Here $s$ is the sedentarity of $x$ and $N_x = \dim \text{Aff}_x$. For the purpose of this statement $U$ is sufficiently small if $U \subset \text{Star}(E_x)$. This chart serves as a canonical chart near $x$.

**Proof.** The proposition follows from Corollary ?? immediately if $\text{sed}(x) = 0$. If $\text{sed}(x) > 0$ we apply Lemma ?? to reduce the proposition to the case of a point of sedentarity 0 in an $(n - k)$-dimensional tropical space $X'$ by local decomposition $X = X' \times \mathbb{T}^{\text{sed}(x)}$.

\[6.3.2\] **Tropical manifolds**

Tropical manifolds are special case of tropical spaces. They are regular at infinity and non-singular at every point. To define what it means to be non-singular we need to introduce the *multiplicity* at every point of a tropical space that takes positive integer values.

For that we have to consider the tropical intersection number of balanced $\mathbb{Z}$-polyhedral complexes in $\mathbb{R}^N$. Let $\{L_j\}, j = 1, \ldots, k$ be a collection of transverse affine spaces with $\sum_{j=1}^k (N - \dim(L_k)) = N$ such that each $L_j$ is parallel to a linear space in $\mathbb{R}^N$ defined over $\mathbb{Z}$. Then the sublattices $\Lambda_j \subset \mathbb{Z}^N$ composed by integer vectors parallel to $L_j$ generate a subgroup of a finite index $\iota$ in $\mathbb{Z}^N$ (since $L_j$ are transverse). We define the tropical intersection number

$$\#(L_1, \ldots, L_k) = \iota \in \mathbb{N}.$$

Let $\{Y_j\}, j = 1, \ldots, k$ be a collection of $\mathbb{Z}$-polyhedral complexes $Y_j \subset \mathbb{R}^N$ with $\sum_{j=1}^k (N - \dim(Y_k)) = N$ and $x \in \bigcap_{j=1}^k Y_j$. For an open dense set $\Omega \in (\mathbb{R}^N)^k$ (it is easy to see that $\Omega$ can be presented as a union of affine hyperplanes) the intersection $\bigcap_{j=1}^k(Y_j + \tau_j), (\tau_1, \ldots, \tau_k) \in \Omega$, is a finite set

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composed of points contained in the relative interior of a facet in each \( Y_j \). In other words, after translation of \( Y_j \) by vectors \( \tau_j \) our polyhedral complexes locally coincide with affine spaces near each intersection point. Note that since the intersection becomes a finite set after translation, the corresponding affine spaces are transverse.

**Proposition 6.3.13** (cf. [PRGST])

For each \( x \in \bigcap_{j=1}^k Y_j \) and a sufficiently small open neighborhood \( U \) of the origin in \( \mathbb{R}^N \) the sum

\[
\sum_y t_y(Y_1 + \tau_1, \ldots, Y_k + \tau_k)
\]

(6.7)

over all points \( y \in U \cap \bigcap_{j=1}^k (Y_j + \tau_j) \) does not depend on the choice of \( (\tau_1, \ldots, \tau_k) \in \Omega \cap (\frac{1}{N}U)^k \). Here by \( \frac{1}{N}U \) we mean \( U \) scaled by \( \frac{1}{N} \).

**Definition 6.3.14**

The sum (6.7) is called the stable tropical intersection number of \( \{Y_j\} \) at \( x \) and denoted with \( \iota_x(Y_1, \ldots, Y_k) \).

**Proof of Proposition 6.3.13** The proposition follows from the balancing property of \( Y_j \). Since the complement of a union of affine subspaces of codimension of at least two is connected we can join any two choices of \( \tau \) with a path \( \tau(s), s \in [0,1] \), so that the only values of \( s \) with \( \tau(s) \notin \Omega \) correspond to the situations when in addition to points of transverse intersections of relative interiors of facets of \( Y_j \) we have points of \( \bigcap_{j=1}^{k-1} (Y_j + \tau_j(s)) \) which are contained in the relative interiors of facets of all but one complexes \( Y_j \), which intersect transversely, and in the relative interior of a face \( E \subset Y_l \) of codimension 1 of the remaining one.

Since near \( E \) the other complexes \( Y_j \) intersect transversely by interior of facets, their intersection locally coincides with an affine space \( L \subset \mathbb{R}^N \) of dimension complimentary to that of \( Y_l \) and such that \( L \cap E = \emptyset \). For values of \( \tau \in \Omega \) close to \( \tau_s \) we have \( L^\tau \cap E = \emptyset \), but instead \( L^\tau \), which is a translate of \( L \) by a small vector, intersects interiors of some of the facets adjacent to \( E \). Independence of the sum of the tropical intersection numbers at the resulting intersection points in the interior of the faces follows from the balancing of \( Y_l \) at \( E \). (Furthermore, the balancing condition is equivalent to such invariance for all possible integer directions of \( L \) with \( \dim L + \dim Y_l = N \).) \( \square \)
**Definition 6.3.15**
Let \( e'_j \in \mathbb{Z}^N, \ j = 1, \ldots, N, \) be a collection of vectors generating the lattice \( \mathbb{Z}^N. \) We may augment these collection by

\[
e'_0 = - \sum_{j=1}^{N} e'_j \in \mathbb{Z}^N.
\]

This gives us a collection \( e' = \{e'_j\}_{j=0}^N \) with \( \sum_{j=0}^N e'_j = 0 \) called the *generating circuit* of \( \mathbb{Z}^N. \) Clearly we have a countable number of different generating circuits in \( \mathbb{R}^N. \)

To a generating circuit \( e' \) we may associate a \( k \)-dimensional balanced \( \mathbb{Z} \)-polyhedral complex

\[
Z^k_{e'} \subset \mathbb{R}^N
\]

for every integer number \( k \) between 1 and \( N - 1 \) as follows.

We take rays \( R_v \subset \mathbb{R}^N \) emanating from the origin in the direction of \( v \in \mathbb{Z}^N \) for vectors \( e_j \) from \( e'. \) Note that the union of these \( N + 1 \) rays is a balanced 1-dimensional \( \mathbb{Z} \)-polyhedral complex in \( \mathbb{R}^N. \) The complex \( Z^k_{e'} \) is defined as the union of the cones spanned by all possible \( k \)-tuples of these rays. Clearly, \( Z^k_{e'} \) is \( k \)-dimensional and balanced.

**Definition 6.3.16**
Let \( Y \subset \mathbb{R}^N \) be a balanced \( \mathbb{Z} \)-polyhedral complex and \( x \in Y \) be a point. Let \( L \subset \mathbb{R}^N \) be the affine span of \( \text{Star}_Y(x), \) i.e. the union of all faces of \( Y \) adjacent to \( x. \) Clearly, the lattice \( \Lambda \subset \mathbb{Z}^N \) composed of the integer vectors parallel to \( L \) has the same dimension as \( L. \) We define

\[
\mu_x(Y) = \min_{e'} \mu_x(Y, Z^k_{e'} + x),
\]

where the minimum is taken over all possible generating circuits of \( \mathbb{R}^N \) and \( k = \text{dim } L - \text{dim } Y. \) The number \( \mu(x) \) is called the *multiplicity* of \( Y \) at \( x. \) We say that \( Y \) is *simply balanced* if its multiplicity at every point is 1.

Similarly we may define multiplicity of a balanced \( \mathbb{Z} \)-polyhedral complex \( P \) in \( \mathbb{T}^N \) regular at infinity. If \( x \in P \) has sedentarity 0 we use the definition above for \( \mathbb{R}^N \cap P. \) If \( \text{sed}(x) > 0 \) we apply Lemma ?? and take the multiplicity of the corresponding \((n - \text{sed}(x))\)-dimensional complex \( P'. \)

**Proposition 6.3.17**
Let \( X \) be a tropical space and \( x \in X \) be its point. The multiplicity of \( P \) at \( \phi(x) \) does not depend on the choice of the choice of a chart \( \phi: U \to P \) (see Definition ??).
6.3 Tropical morphisms

Proof. The proposition follows from Corollary ??.

Thus we may speak of multiplicities of points of tropical spaces.

Definition 6.3.18
A tropical space is called tropical manifold if it is regular at infinity and all its points have multiplicity 1.

Note that there may exist several generating circuits $c'$ with $\epsilon_x(Y, Z_k^c) = 1$. Once we choose such a circuit $c'$ it is convenient to compactify $\mathbb{R}^N$ according to $c'$. Note that the set $\epsilon_l'$ of $N$ vectors in the collection $c'$ obtained by removing $e_l$ generates the lattice $\mathbb{Z}^N$ for any $l = 0, \ldots, N$. Thus we may find an invertible linear map $\alpha : \mathbb{R}^N \to \mathbb{R}^N$, defined over $\mathbb{Z}$, such that the $\alpha(-e'_l)$ is one of the standard basis vectors of $\mathbb{R}^N \subset \mathbb{T}^N$. With the help of the map $\alpha$ we can consider $\mathbb{T}^N$ as a partial compactification $\mathbb{T}^N_{\epsilon'_l} \supset \mathbb{R}^N$ associated to $\epsilon_l$. Alternatively, we may define $\mathbb{T}^N_{\epsilon'_l} \supset \mathbb{R}^N$ in coordinates provided by the basis $\epsilon'_l$ by allowing the coordinates to take $-\infty$ as a value. Clearly, $\mathbb{T}^N_{\epsilon'_l}$ is isomorphic to $\mathbb{T}^N$.

Definition 6.3.19
We define the $c'$-projectivization $\mathbb{T}^N_{\epsilon'_l} \supset \mathbb{R}^N$ by gluing together $\mathbb{T}^N_{\epsilon'_l}$ for all $l = 0, \ldots, N$. It can be thought of as the $N$-dimensional tropical projective space with homogeneous coordinates given by the circuit $c'$.

Recall that in Definition ?? we have defined the tropical fan $\text{Cone}_x(X)$ of a point $x$ in a tropical space $X$ and that this fan gives a complete local description of $X$ near $x$ by Theorem ??$. In particular to compute the multiplicity of $X$ at $x$ it is sufficient to look at $\text{Cone}_x(X)$. We may also consider tropical fans on their own.

Definition 6.3.20
A balanced $\mathbb{Z}$-polyhedral complex $Y \subset \mathbb{R}^N$ is called tropical fan if $\mathbb{R}^N$ is a conical set, i.e. for every $y \in Y$ the ray $R_y$ emanating from the origin and passing through $y$ is contained in $Y$. Clearly, if $Y$ is a tropical fan then each face of $Y$ is a convex cone.

Tropical fan is called non-singular if its multiplicity is 1.

Definition 6.3.21 (cf. e.g. [matroid])
Recall that a finite matroid is a finite set $S$ enhanced with a function

$$\text{rk} : 2^E \to \mathbb{Z}_{\geq 0}$$
subject to the following properties.

- For any subset \( A \subset S \) we have \( \text{rk}(S') \leq \#(S') \).
- For any pair of subsets \( A, B \subset S \) we have \( \text{rk}(A) + \text{rk}(B) \geq \text{rk}(A \cup B) - \text{rk}(A \cap B) \).
- If \( A \subset B \) then \( \text{rk}(A) \leq \text{rk}(B) \).

Theorem 6.3.22 (cf. [Ardila-Klivans])

There is a natural 1-1 correspondence between pairs \((Y, \epsilon')\) made of a non-singular \(n\)-dimensional tropical fans \(Y \subset \mathbb{R}^N\) and a generating circuit \(\epsilon'\) in \(\mathbb{Z}^N\) such that \(t_0(Y, Z_{\epsilon'}^{N-n}) = 1\), and matroids on \(\epsilon'\) considered as a set in \(N + 1\) elements.

Proof. Given a pair \((Y, \epsilon')\) we consider the topological closure of \(Y \subset \mathbb{R}^N \subset TP_{\epsilon'}^N\). Orbits under the action of \(\mathbb{R}^n\) are naturally labelled by subsets of \(E\). Consistently, the degree of a tropical cycle in \(\mathbb{R}^n\) is computed with respect to \(E\) here.

6.4 A description of smooth fans using matroids

In this section, we fix the circuit \(E = \{-e_0, -e_1, \ldots, -e_n\}\) of \(\mathbb{R}^n\), where \(e_1, \ldots, e_n\) is the standard basis of \(\mathbb{R}^n\) and \(e_0 = e_1 + \ldots + e_n = (1, \ldots, 1)\). We consider the corresponding standard compactification \(\mathbb{R}^n \subset TP^n\). Orbits under the action of \(\mathbb{R}^n\) are naturally labelled by subsets of \(E\). Consistently, the degree of a tropical cycle in \(\mathbb{R}^n\) is computed with respect to \(E\) here.

For any subset \(S \subset E\), we define the vector \(e_S = \sum_{i \in S} e_i\). In particular, \(e_E = 0\). To any chain of subsets \(S = (\emptyset \neq S_1 \neq \ldots \neq S_l \neq E)\), we assign the cone \(\sigma_S = R_{\geq 0} e_{S_1} + \ldots + R_{\geq 0} e_{S_l}\). Here, \(l = \dim \sigma_S\) is called the length of \(S\).

The collection of \(\sigma_S\) for all possible chains of subsets of \(E\) forms a unimodular polyhedral fan covering \(\mathbb{R}^n\). It is called the fine subdivision of \(\mathbb{R}^n\) and denoted by \(FS\).

Definition matroid

We will now show that there is a one-to-one correspondence between degree 1 fans of \(\mathbb{R}^n\) (with respect to the chosen circuit \(E\)) and loopfree matroids defined on the set \(E\). A matroid is loopfree if and only if the empty set is closed. The following definition explains how matroids and degree 1 fans are related.

Definition 6.4.1

Let \(M\) be a loopfree matroid on \(E\). The associated matroid fan \(B(M)\)
6.4 A description of smooth fans using matroids

consists of the collection of cones \( \sigma_F \), where \( \mathcal{F} = (\emptyset \subseteq F_1 \subseteq \ldots \subseteq F_l \subseteq E) \) is a chain of flats of \( M \). In particular, \( B(M) \) is a subfan of the fine subdivision of dimension \( \text{rank}(M) - 1 \) (which is the maximal length of chains of flats in \( M \)).

Now, let \( X \) be a tropical fan in \( \mathbb{R}^n \) of degree 1. We assign a matroid \( M(X) \) to it by describing its (proper) flats. A subset \( F \subseteq E \) is closed in \( M(X) \) if the property

\[
\overline{X} \cap \mathcal{O}_F \neq \emptyset
\]

is satisfied. Here \( \overline{X} \) denotes the closure of \( X \) in \( TP^n \) and \( \mathcal{O}_F \) denotes the orbit in \( TP^n \) where the coordinates in \( F \) are infinite.

**Theorem 6.4.2**

There is a one-to-one correspondence between degree 1 fans in \( \mathbb{R}^n \) and loopfree matroids on \( E \) given by

\[
\{\text{degree 1 fans in } \mathbb{R}^n\} \leftrightarrow \{\text{loopfree matroids}\},
\]

\[
X \leftrightarrow M(X),
\]

\[
B(M) \leftrightarrow M.
\]

**Remark 6.4.3**

Note that the map \( X \mapsto M(X) \) has a completely analogous counterpart in classical algebraic geometry. An embedding of a linear space \( L \) in \( \mathbb{C}P^n \) is basically equivalent (up to translating \( TP^n \) by the torus action) to a hyperplane arrangement on \( L \). The hyperplanes are given by the intersection of \( L \) with the coordinate hyperplanes of \( \mathbb{C}P^n \). In this situation, it is easy to check that the above definition of \( M(L) \) is equivalent to the standard definition of the matroid of the hyperplane arrangement.

In fact, it is useful to consider a slight generalization of our definitions and statements.

A degree 1 fan \( X \) in \( TP^n \) is the closure of a degree 1 fan in an orbit \( \mathcal{O}_F \), \( F \subset E \). We extend our definition of \( M(X) \) to this situation unchanged. This obviously leads to matroids which are not loopfree, as \( F \) is the minimal flat here.

Correspondingly, let \( M \) be a (not necessarily loopfree) matroid \( M \) and let \( F = \emptyset \) be its minimal flat. Then \( M \setminus F \) is a loopfree matroid and we define

\[
B(M) := B(M \setminus F) \subset \mathcal{O}_F.
\]

The philosophy here is that loops in the matroid correspond to fans contained in the corresponding coordinate hyperplane at infinity.
With these generalizations, we wish to establish a one-to-one correspondence between degree 1 fans in $\mathbb{T}P^n$ and matroids on $E$.

\[ \{ \text{degree 1 fans in } \mathbb{T}P^n \} \leftrightarrow \{ \text{matroids on } E \} \]

To prove the theorem, we have to show that the maps in consideration are well-defined and inverse to each other. In particular, we will prove the following steps:

(a) The map $X \mapsto M(X)$ is well-defined, i.e. the sets which we declared to be closed satisfy the axioms of flats of a matroid.

(b) The map $M \mapsto B(M)$ is well-defined, i.e. $B(M)$ forms a balanced fan (with trivial weights) and is of degree 1.

(c) For any degree 1 fan $X$, we have $B(M(X)) = X$.

(d) For any matroid $M$, we have $M(B(M)) = M$.

The following lemma will be used repeatedly.

**Lemma 6.4.4**

Let $X \subset \mathbb{R}^n$ be a degree 1 fan. Then for any index set $I \subset E$, the (set-theoretic) intersection with the corresponding coordinate plane $X \cap H_I$ is a degree 1 again.

Let us now prove the four steps announced above.

**Proof of (a).** A collection of subsets of $E$ forms the flats of a matroid if and only if the following conditions hold.

(A) $E$ itself is closed.

(B) If $F$ and $G$ are closed, then $F \cap G$ is also closed.

(C) Let $F$ be closed and let $G_1, \ldots, G_l$ be all flats which cover $F$ (i.e. $F \subset G_i$ and $F \subset H \subset G_i$ implies $F = H$ or $F = G_i$). Then the sets $G_i \setminus F$ form a partition of $E \setminus F$.

In our case, $E$ is closed by definition. For (B), we consider the degree 1 fan $X' = X \cap H_{F \cap G}$. By definition, the sedentarity $\text{sed}(X')$ is closed in $M(X)$ and $F \cap G \subset \text{sed}(X')$. On the other hand, $F$ and $G$ are also flats for $X'$, and therefore $\text{sed}(X') \subset F \cap G$. Hence, $F \cap G = \text{sed}(X')$ is closed.
For (C), we first note that by replacing $X$ by $X \cap H_F$ we can assume $F = \emptyset$. By lemma 6.4.4 and dimension reasons, it turns out that the covering flats are just the subsets $G_i$ of $E$ such that $X \cap O_{G_i}$ is of dimension $\dim(X) - 1$. Given $i \in E$, the unique such subset containing $i$ is given by $\text{sed}(X \cap H_i)$.

Proof of (b) By our definitions we have to show that for any loopfree matroid $M$, the fan $B(M)$ is balanced and of degree 1. The balancing condition has to be checked on codimension 1 cones of $B(M)$. These are given by chains of flats where exactly one rank step, say $l \leq r := \text{rank}(M)$, is missing. Let us fix such a chain $\mathcal{F}$, and let us assume that the two flats surrounding the gap are $F \not\subset G$. The facets adjacent to $\sigma_{\mathcal{F}}$ are given by all possibilities to fill in the gap of $F$ in order to obtain a maximal chain of flats for $M$. In other words, they correspond to all flats $H_i$ with $F \not\subset H_i \not\subset G$. For each such facet, $e_{H_i}$ is an admissible primitive vector (modulo $\sigma_{\mathcal{F}}$). Now, as $M$ is a matroid, the sets $H_i \setminus F$ form a partition of $G \setminus F$. In terms of the corresponding vectors, this translates to

$$\sum_{i=1}^{k} e_{H_i} = e_G + (k-1)e_F,$$

where $k$ is the number of adjacent facets. As the right hand side is obviously a vector in $\sigma_{\mathcal{F}}$, the balancing condition is satisfied.

So $B(M)$ forms indeed a tropical fan in $\mathbb{R}^n$ and it remains to compute its degree. Our approach is as follows. Let $H \subset \mathbb{R}^n$ be the standard hyperplane. We will show that the stable intersection $H \cdot B(M)$ is the fan of a loopfree matroid again. Therefore, by induction it suffices to prove the claim for the single loopfree matroid of rank 1 on $E$. Its only flats are $\emptyset$ and $E$ itself. Therefore $B(M)$ is just the origin which is obviously of degree 1.

So, let us now prove that when $\text{rank}(M) > 1$, $H \cdot B(M)$ is a matroid fan again. Indeed, we will show $H \cdot B(M) = B(M')$ where $M'$ is the matroid obtained from $M$ by removing all flats of rank $\text{rank}(M) - 1$ (which can be easily checked to be an admissible collection of flats again). To do so, we recall that $H$ is given by the homogeneous linear tropical polynomial $l = \max\{x_0, \ldots, x_n\}$, where the $x_i$ are the homogeneous coordinates on $\text{TP}^n$. Therefore $H \cdot B(M)$ can be computed as the divisor of $l$ on $B(M)$. As $l$ is obviously linear on all cones of the fine subdivision, $H \cdot B(M)$ is supported on the codimension one skeleton of $B(M)$. Using the notations from the first part of the proof again, the weight of a codimension one
cone $\sigma_F$ of $B(M)$ in $H \cdot B(M)$ is given by

$$\omega(\sigma_F) = \sum_{i=1}^{k} l(e_{H_i}) - l(e_G) - (k-1)l(e_F).$$

To be precise, in this expression $l$ should be replaced by an arbitrary dehomogenization of $l$. Alternatively, we may set

$$l(e_S) = \begin{cases} -1 & \text{if } S = E, \\ 0 & \text{otherwise.} \end{cases}$$

In any case, it is easy to compute

$$\omega(\sigma_F) = \begin{cases} 1 & \text{if } G = E, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $H \cdot B(M)$ consists of all the cones $\sigma_F F$ where $F$ is a chain of flats in $M$ not containing a flat of rank $\text{rank}(M) - 1$. But this agrees with the definition of $B(M')$.

Proof of (c). First, we prove the simplest case when $X$ is a classically linear space (in addition to being of degree 1). We will deal with this case by induction on $\dim(X) = k$. If $k = 0$, $X$ just consists of the origin with weight 1. Therefore the only flats of $M(X)$ are $\emptyset$ and $E$, and obviously $B(M(X)) = \{0\} = X$. So we may assume $k > 0$ now. Let us pick a flat $F$ of $M(X)$ of rank 1, which is equivalent to saying that $X' = X \cap F$ is a degree 1 fan of dimension $k - 1$. Of course, $X'$ is also (classically) linear in $F$, and hence the induction hypothesis implies $X' = B(M(X'))$. In particular, $X'$ is the linear span of a cone $\sigma_{F'}$, where

$$F' = (\emptyset \subsetneq F'_2 \subsetneq \ldots \subsetneq F'_k \subsetneq E \setminus F)$$

is a chain of subsets of $E \setminus F$ of length $k - 1$. Note that $X$ and the cone of directions in $\mathbb{R}^n$ ending up in $F$ intersect in a single ray (otherwise $X'$ would be of smaller dimension or empty). This ray is generated by a primitive vector of the form

$$v = \sum_{i \in F} a_i e_i \text{ with } a_i \in \mathbb{N}.$$
let us remove this collection of elements from $E$ to obtain a set $B$ with $|B| = n - k$. Let $V_B$ the linear space by all the standard vectors contained in $B$. The stable intersection $V_B \cdot X$ is is the origin with weight equal to the lattice index $[\mathbb{Z}^n : \mathbb{Z}V_B + \mathbb{Z}X]$. By our choice of $B$ we can easily compute this lattice index as $$[\mathbb{Z}^n : \mathbb{Z}V_B + \mathbb{Z}X] = a_i \cdot [\mathbb{Z}O_F : \mathbb{Z}V_{B \setminus F} + \mathbb{Z}X] = a_i,$$
where $i$ is the element we removed from $F$. On the other hand, it follows from $\deg(X) = 1$ that $\deg(V_B \cdot X) = 1$ (as we may move the standard plane $H^{n-k}$ such that the (interior of the) face spanned by $e_j, j \in B$ intersects $X$). It follows
$$a_i = \deg(V_B \cdot X) = 1,$$
for all $i \in F$. To summarize, we proved that $e_F \in X$ for any flat of rank 1. Let $F_1, \ldots, F_l$ be the collection of all flats. From matroid theory, we know that this collection forms a partition of $E$ and that $l \geq k + 1$ (as picking a single element of each rank 1 flat produces a generating set of the matroid and therefore must have cardinality at least the rank of the matroid). As the span of the vectors $e_{F_1}, \ldots, e_{F_l}$ is of dimension $l - 1$, we get $l \leq k + 1$ and thus $l = k + 1$. This implies that the matroid $M(X)$ is a sum of the $k + 1$ matroids
$$M(X) = \bigoplus_{i=1}^{k+1} F_i.$$
In particular, all flats of $M(X)$ are just unions of some of the rank 1 flats $F_1, \ldots, F_l$. It follows that $B(M(X))$ is equal to the span of $e_{F_1}, \ldots, e_{F_l}$, as well as $X$, and hence $X = B(M(X))$.

We now continue with the general case. It turns out that after a small trick, this is actually not much harder. First note that for any $x \in X$, the local fan $\text{Star}_x(X)$ is of degree 1 again (by the locality of stable intersection). In particular, choosing $x$ in the interior of a facet of $X$, we see that the linear span of this facet, denoted by $S$, is of degree 1 and thus $S = B(M(S))$. Let $k = \dim(X) = \dim(S)$. Then, by construction, $B(M(S))$ is supported on the $k$-skeleton $FS^{(k)}$ of the fine subdivision of $\mathbb{R}^n$. As our original facet is contained $S$, we may conclude that $X$ itself is supported on $FS^{(k)}$. So all that remains to check is whether $X$ and $B(M(X))$ contain the same cones of $FS^{(k)}$. This amounts to check that for any chain $\mathcal{F}$ of subsets in $E$ of length $k$, we have
$$\sigma_{\mathcal{F}} \subset X \iff \mathcal{F} \text{ is a chain of flats in } M(X).$$
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The direction $\Rightarrow$ is clear. Namely, the ray spanned by $e_F$ has its limit point at infinity in $O_F$. Thus $e_F \in X$ implies that $F$ is closed in $M(X)$. For the other direction, let us assume that $\mathcal{F} = (\emptyset \not\subset F_1 \subset \cdots \not\subset F_k \not\subset E)$ is a chain of flats in $M(X)$. Consider $X' = X \cap O_{F_1}$. Obviously, for every subset $S \subset E \setminus F_1$, we have $X' \cap O_S = X \cap O_{S \cup F_1}$. Therefore the chain

$$\mathcal{F}' = (\emptyset \not\subset F_2 \setminus F_1 \not\subset \cdots \not\subset F_k \setminus F_1 \not\subset E \setminus F_1)$$

is a chain of flats in $M(X')$. By induction, this implies $\sigma_{\mathcal{F}'} \subset X'$. But the only cone $\sigma$ in $FS^{(k)}$ satisfying $\sigma \cap O_{F_1} = \sigma_{\mathcal{F}'}$ is $\sigma_{\mathcal{F}}$, and thus $\sigma_{\mathcal{F}} \subset X$. □

Proof of (d). We will show that the flats of $M$ and $M(B(M))$ agree. Let $F$ be a flat of $M$. Then $e_F \in B(M)$ and thus $F$ is also closed in $M(B(M))$.

Now let $G$ be a flat of $M(B(M))$. Therefore $B(M)$ contains a vector which ends up in $O_G$. A cone $\sigma_{\mathcal{F}} \in FS$ contains such a vector if and only if $G$ appears as one of the elements in $\mathcal{F}$. Thus $B(M)$ contains such a cone, and consequently $G$ is a flat of $M$ as well. □

Theorem 6.4.5

Let $Y \subset \mathbb{R}^n$ be a matroid fan of dimension $k$ and let $V \subset Y$ be a balanced polyhedral complex of dimension $k - 1$ contained in $Y$. Then there exist tropical polynomials $f, g$ such

$$V = \text{div}(\frac{f}{g}|_Y).$$

Proof. Use François Rau and Alexander Esterov: We can find $X' \subset \mathbb{R}^n$ of dimension $n - 1$ such that $X = Y \cdot X'$ (stable intersection). Use the previous theorem to find $X' = V(\frac{f}{g})$ and use compatibility of stable intersection and Cartier divisors. □

Theorem 6.4.6

In the above two theorems, the function $\varphi = \frac{f}{g}$ is unique up to adding an affine linear function.

Proof. In the $\mathbb{R}^n$-case, it follows from the proof that once we fix $f$ on one connected component, it is uniquely determined everywhere else. For a general matroid fan, we can use the fact that it is a (multiple) modification of $\mathbb{R}^k$ along matroid divisors and show that the property survives such a modification. So let us assume $VV$ is a matroid modification of $V$ and $f \cdot VV = 0$ for some rational function $f$. We want to show that $f$ is affine linear. We prove first that $f$ is the pull-back of a function $g$ on $X$. Then
the claim follows as the function $g$ has to be affine linear by induction assumption.

We check the claim by proving that $f$ is affine linear on every half-ray $VV \subset \{p\} \times \mathbb{R}$, where $p$ is a generic point of $\text{divisor}$. Assume there is a point $W$.

We write $f = \delta^*g + h$, where $g$ is a function on $X$ and $h = 0$ away from $\text{divisor} \times \mathbb{R}$. It follows $\delta_*h \cdot VV \subset \text{divisor}$, and, as $\text{divisor}$ is irreducible, $\delta_*h \cdot VV = a \cdot \text{divisor}$. This implies $g \cdot V = -a\text{divisor}$ and therefore $g = -a\varphi$, where $\varphi$ is the modification function, by uniqueness in $X$. From this follows $f = -ay$. \hfill \square
7 Tropical cycles and the Chow group

7.1 Tropical cycles

We defined tropical cycles in toric varieties in section 4.1. The main idea was that tropical cycles should be polyhedral sets (with rational slopes) whose generic points carry multiplicities such that around each codimension one cell the balancing condition holds. As all these requirements are of a local nature, we have no trouble in extending the definition to arbitrary tropical varieties.

Let \( V \) be a tropical variety. A subset \( X \subseteq V \) is called a polyhedral set if it is (finite closed) polyhedral in any chart. Recall that in charts with points of higher sedentarity, we define a polyhedron to be the closure of a usual polyhedron in \( \mathbb{R}^n \). Let \( x \in X \) be a point. We define the speciality of \( x \), denoted by \( \text{spec}_X(x) \) or just \( \text{spec}(x) \), to be the minimal codimension of the polyhedra \( P \) such that \( x \in \text{ReInt}(P) \subseteq X \). Points with speciality 0 are called generic points. Let \( X \) be of pure dimension \( m \). Then the closure of all points of speciality \( m - k \) is called the \( k \)-skeleton of \( X \), denoted by \( X^{(k)} \).

\[
X^{(k)} := \{ x \in X : \text{spec}(x) = m - k \} = \{ x \in X : \text{spec}(x) \geq m - k \}.
\]

A polyhedral set is called weighted if it is equipped with a locally constant function \( \text{mult} : X^{\text{gen}} \to \mathbb{Z} \setminus \{0\} \), where \( X^{\text{gen}} \) denotes the set of generic points of \( X \). If \( P \in X \) is a polyhedron of maximal dimension \( m \), then we also write \( \text{mult}(P) \) for the number \( \text{mult}(x) \) for any \( x \in \text{ReInt}(P) \).

**Definition 7.1.1**

Let \( V \) be a tropical variety. A tropical \( k \)-cycle \( X \) of \( V \) is a weighted polyhedral set \( X \subseteq V \) of pure dimension \( k \) such that for any chart \( U \in V \), any polyhedral structure of \( X \cap U \) and any codimension one cell \( \tau \subseteq X \cap U \) the balancing condition

\[
\sum_{\tau \subset \sigma \text{ face}, \text{sed}(\tau) = \text{sed}(\sigma)} \text{mult}(\sigma)v_{\sigma \tau} = 0 \mod \mathbb{R}^\tau
\]

is satisfied.
7.2 Push-forwards of tropical cycles

$X$ is called of pure sedentariness if all its generic points have the same sedentariness (in $V$).

$X$ is called effective if all its weights are positive, i.e. $\text{mult} : X^\text{gen} \to \mathbb{N}$.

**Remark 7.1.2**

Note that an effective tropical cycle $X \subseteq V$ is a tropical space itself by restricting the charts of $V$ to $X$. In other words, in our terminology effective tropical cycles are just the closed tropical subvarieties of $V$. In particular, $V$ satisfies the requirement of a cycle itself. Hence $V$ is the fundamental cycle of itself.

Given two tropical cycles $X_1$ and $X_2$, we can form the sum $X_1 + X_2$. We just take the union $X_1 \cup X_2$ (which is again a polyhedral set) and add weights. This means for a generic point of $x$ of $X_1 \cup X_2$ we set $\text{mult}_{X_1+X_2}(x) = \text{mult}_{X_1}(x) + \text{mult}_{X_2}(x)$ (where $\text{mult}_{X}(x) = 0$ if $x \notin X$). If this sum turns out to be zero, we just remove the point. Thus, in general, $X_1 + X_2$ is only supported on a subset of $X_1 \cup X_2$. It is straightforward to check that $X_1 + X_2$ still satisfies the balancing condition. So the set of all cycles in $V$, denoted by $Z_k(V) = \bigoplus Z_k(W)$, forms a group under addition with neutral element the empty cycle $0 := \emptyset$.

One further remark: As above, we will always ignore points of weight zero. This is to say, whenever a construction (like summing two cycles) produces zero weights, we just discard these points.

**7.2 Push-forwards of tropical cycles**

Let $V$ and $W$ be tropical varieties and let $f : X \to Y$ be a tropical morphism. Given a tropical cycle $X \in Z_k(V)$, we define its push-forward $f_*(X) \in Z_k(W)$ as follows. Let $y \in f(X)$ be a point such that $X_y := f^{-1}(y) \cap X$ is isolated and generic in $X$ (in particular, $y$ is generic in $f(X)$ and $X_y$ is finite). Fix $x \in X_y$ and choose charts around $x$ and $y$. Then $f$ induces a map of lattices $df^Z_x : T^Z_x X \to T^Z_y f(X)$, and we define

\[
\omega_{f_*(X)}(y) := \sum_{x \in X_y} [T^Z_y f(X) : \text{Im}(df^Z_x)] \cdot \omega_X(x).
\]

**Definition and Proposition 7.2.1**

In the situation above, there is a unique $k$-cycle supported on $f(X)$ whose weight function agrees with (7.1) for sufficiently generic points. This cycle is called the push-forward of $X$, denoted by $f_*(X) \in Z_k(W)$. 

Proof. \( f(X) \) is a polyhedral set in \( W \). If \( \dim(f(X)) < k \), then \( f_*(X) = 0 \). If \( \dim(f(X)) = k \), let \( Y \) denote its \( k \)-dimensional part. Then equation (7.2) defines a locally constant weight function on an open polyhedral dense subset \( U \in Y \). We have to show that this weight function satisfies the balancing condition.

Choose \( y \in Y \) and let \( S \) denote the part of \( \text{Star}_Y(y) \) given by points of the sedentarity (in \( Y \)). We have to show that \( S \) with weight function (7.1) is balanced. We use the following locality statement. Let \( X_y := f^{-1}(y) \cap X \) and let \( Z \subseteq X_y \) be the set of vertices of \( X_y \). For each \( z \in Z \), let \( S_z \) denote the sedentarity part of \( \text{Star}_X(z) \) as above. Then, for \( y' \in S \), we have

\[
\omega_{f_*(X)}(y') = \sum_{z \in Z} \omega_{f_*(S_z)}(y').
\]

This follows from the fact that when we let \( y' \) converge to \( y \), then the preimage points in \( X_{y'} \) have to converge to points in \( Z \). Using this equation, we can assume that \( X \) is a fan, and \( f \) is integer linear. The balancing condition is a condition for the ridges of \( f(X) \). Hence, by applying locality one more time and using the fact that dividing by a linearity space is compatible with the lattice index showing up in (7.1), we can assume that \( X \) is one-dimensional. We denote by \( u_\rho \) the primitive generator of a ray \( \rho \). For any ray \( \rho' \) of \( Y \) we have

\[
\omega_{f_*(X)}(\rho') = \sum_{\rho \in X \atop f(\rho) = \rho'} [T^Z_{\rho'} f(X) : \text{Im}(df^Z_{\rho'})] \omega_X(\rho).
\]

Note that the primitive generators are related by

\[
f(u_\rho) = [T^Z_{\rho'} f(X) : \text{Im}(df^Z_{\rho'})] u_{\rho'}.
\]

Hence the balancing condition for \( Y \) follows from

\[
\sum_{\rho' \in Y} \omega_{f_*(X)}(\rho') u_{\rho'} = \sum_{\rho' \in Y} \sum_{\rho \in X \atop f(\rho) = \rho'} [T^Z_{\rho'} f(X) : \text{Im}(df^Z_{\rho'})] \omega_X(\rho) u_{\rho'}
\]

\[
= \sum_{\rho \in X} \omega_X(\rho) f(u_\rho)
\]

\[
= f \left( \sum_{\rho \in X} \omega_X(\rho) u_\rho \right) = f(0) = 0.
\]

\( \square \)
7.3 Linear equivalence of cycles

Let $V$ be a tropical space of pure dimension $n$ and consider the variety $V \times \mathbb{P}^1$. Let $Z$ be a cycle in $V \times \mathbb{P}^1$ which is the closure of a cycle in $V \times \mathbb{R}$. Then $Z$ can be intersected with $V_{-\infty} := V \times \{-\infty\}$. Namely, in every chart $U \times T$ we can use Definition 5.2.2. It is easy to check that the results agree on the overlaps and therefore can be glued together to give the cycle $Z_{-\infty} := Z \cdot V_{-\infty}$. As $V \times \{-\infty\} = V$, we think of $Z_{-\infty}$ as a cycle in $V$ (of dimension $\dim(Z) - 1$). In the same way, we can construct $Z_{+\infty} := Z \cdot Z_{+\infty}$. This construction suffices to translate the classical definition of linear equivalence to the tropical world.

Definition 7.3.1

Let $X_1, X_2$ be two $k$-cycles in the tropical space $V$. Then $X_1$ and $X_2$ are called linearly equivalent, denoted by $X_1 \sim X_2$, if there exists a $(k+1)$-cycle $Z \subseteq V \times \mathbb{P}^1$ such that

1. $Z$ is the closure of a cycle in $V \times \mathbb{R}$,
2. $X_1 = Z_{-\infty}$, and
3. $X_2 = Z_{+\infty}$.

Lemma 7.3.2

The relation $\sim$ defined in the previous definition is an equivalence relation. Furthermore, we have

$$X_1 \sim X_2, Y_1 \sim Y_2 \Rightarrow X_1 + Y_1 \sim X_2 + Y_2.$$ 

Proof. To show $X \sim X$, we take $Z = X \times \mathbb{P}^1$. The symmetry of $\sim$ follows from the symmetry $\mathbb{P}^1 \rightarrow \mathbb{P}^1 : x \mapsto -x$. To show the compatibility with sums, let $Z$ be the cycle in $V \times \mathbb{P}^1$ showing $X_1 \sim X_2$ (and, analogously, $Z'$ for $Y_1 \sim Y_2$). Then the sums $Z + Z'$ shows $X_1 + Y_1 \sim X_2 + Y_2$. Finally, using the additivity twice, we find

$$X_1 \sim X_2, X_2 \sim X_3 \Rightarrow X_1 + X_2 \sim X_2 + X_3 \Rightarrow X_1 \sim X_3,$$

which finishes the proof.

7.4 Algebraic equivalence of cycles

As in classical algebraic geometry, we may replace $\mathbb{P}^1$ by any other smooth tropical curve to obtain another, weaker equivalence relation for
tropical cycles. Let $C$ be a smooth tropical curve, and let $V$ be a tropical space. For a cycle $Z \subseteq V \times C$, we have to make sense of

$$Z_p = Z \cdot (V \times \{p\})$$

for any $p \in C$. If $\text{sed}(p) = 1$, $C$ looks like $\mathbb{T}$ near $p$ and we can use Definition 5.2.2 again (assuming that $Z$ is the closure of a cycle in $V \times C \setminus \{p\}$). If $\text{sed}(p) = 0$, then $C$ locally near $p$ looks like the line $L \subseteq \mathbb{R}^m$ with single vertex sitting at $0$. So we can pull-back the function $\max\{x_1, \ldots, x_m, 0\}$ to $V \times U$, where $U$ is a neighbourhood of $p$. Let us denote this pull-back by $\varphi_p$. Then we define $Z_p = \text{div}(\varphi_p|_{Z \cap V \times U})$.

**Definition 7.4.1**

Let $X_1, X_2$ be two $k$-cycles in the tropical space $V$. Then $X_1$ and $X_2$ are called *algebraically equivalent*, denoted by $X_1 \sim_{\text{alg}} X_2$, if there exists a tropical curve $C$, two points $p_1, p_2 \in C$ and a $(k+1)$-cycle $Z \subseteq V \times C$ such that

- $Z$ is the closure of a cycle in $V \times C \setminus \{p_1, p_2\}$,
- $X_1 = Z_{p_1}$, and
- $X_2 = Z_{p_2}$.

**Lemma 7.4.2**

*The relation $\sim_{\text{alg}}$ defined in the previous definition is an equivalence relation. Furthermore, we have*

$$X_1 \sim_{\text{alg}} X_2, Y_1 \sim_{\text{alg}} Y_2 \Rightarrow X_1 + Y_1 \sim_{\text{alg}} X_2 + Y_2.$$ 

*Proof.* It suffices to show that the cycles $X$ such that $X \sim_{\text{alg}} 0$ form a subgroup of $Z_*(V)$. So let $X_1, X_2$ be two tropical cycles which are algebraically equivalent to zero. We want to show that $X_1 - X_2 \sim_{\text{alg}} 0$. According to our definition, there exists a curve $C_1$, two points $p, p'$ and a cycle $Z_1 \subseteq V \times C_1$ such that $(Z_1)_p = X_1$ and $(Z_1)_{p'} = 0$. Analogously, we find $C_2, q, q' \in C_2$ and $Z_1 \subseteq V \times C_1$ such that $(Z_2)_q = X_2$ and $(Z_2)_{q'} = 0$. If $p', q'$ are finite, we can glue $C_1$ and $C_2$ by modifying at $p', q'$ and gluing the new legs at a finite length. If the new curve is called $D$, then $Z_1$ and $Z_2$ can be naturally embedded in $V \times D$. We set $Z := Z_1 + Z_2 = X_1 + X_2$.


Let $V$ be a tropical space of pure dimension $n$. Then a (Weil-)divisor $D \subset V$ is a subcycle of dimension $n-1$. Therefore, the group of all divisors is $\mathbb{Z}_{n-1}(V)$.

**Definition 7.5.1**

Let $V$ be a tropical variety and let $D \subset V$ be a divisor. Then the set of all effective linearly equivalent divisors

$$L(D) = |D| := \{ D' \geq 0 : D' \sim D \}$$

is called the *complete linear system* of $D$. 
Bibliography


