Essential bases, semigroups and toric degenerations

Annual meeting SFB/TRR 195
27.09.2018, Tübingen

Ghislain Fourier

LBfM Lehrstuhl B für Mathematik | RWTH AACHEN UNIVERSITY

Throughout the talk, we consider the **flag variety**

$$\mathcal{F}_n = \{(U_1, \ldots, U_{n-1}) \mid U_i \subset U_{i+1}, \dim U_i = i\}.$$

We want to see this embedded into

$$\mathcal{F}_n \subset \mathbb{P}(\mathbb{C}^n) \times \mathbb{P}(\Lambda^2 \mathbb{C}^n) \times \ldots \times \mathbb{P}(\Lambda^{n-1} \mathbb{C}^n).$$

By fixing coordinates for each $\mathbb{P}(\Lambda^i \mathbb{C}^n)$, $X_{j_1,\ldots,j_i}$, the image is described by the **Plücker relations**, for example

$$X_{12}X_{34} - X_{13}X_{24} + X_{14}X_{23} = 0.$$

So the homogeneous (all Plücker coordinates have degree 1) coordinate ring of the flag variety is

$$\mathbb{C}[\mathcal{F}_n] = \mathbb{C}[X_J \mid 1 \leq |J| \leq n-1]/\mathcal{I}.$$

Let us **degenerate** the flag variety!

We want to construct **a family** $\mathcal{X}_t$ such that

$$X_t \cong \mathcal{F}_n \text{ for } t \neq 0, \;\; X_0 \text{ being interesting}$$

There are various tools, one idea is to associate a degree/weight to each Plücker coordinate and consider the initial ideal.

Describe the possible degree vector, such that the initial ideal is monomial free?

$\longrightarrow$ **tropical flag variety**.

First steps by Bossinger-Lamboglia-Mincheva-Mohammadi, but this is quite hard, even for "easier" varieties such as the Grassmannian of planes.

We need more tools ... use Representation Theory of the $\mathfrak{sl}_n(\mathbb{C})$.

Recall:
$\Lambda^i\mathbb{C}^n$ is a simple module for the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, so each Plücker coordinate is the **dual element of a weight vector**.

Moreover,

$$\Lambda^i\mathbb{C}^n \cong U(\mathfrak{sl}_n)/I \cong U(\mathfrak{n}^-)/I.$$

Here: $e_1 \wedge \ldots \wedge e_i$ is mapped to 1 and $U(\mathfrak{n}^-)$ is spanned by monomials in a basis of $\mathfrak{n}^-$

$$\longrightarrow \text{monomials in } f_{i,j}\,,\ i > j$$

We set

$$\deg f_{i,j} = (i-j)(n-j+1),$$

and consider the associated graded algebra and module.

This is actually a **good choice**, as the vanishing ideal of the associated graded module is monomial!

$$e_{k_1} \wedge \ldots \wedge e_{k_\ell} \mapsto \prod_{i>j} f_{i,j}^{m_{i,j}} \leftarrow \textbf{ essential monomial}$$

and we set

$$S(\omega_\ell) := \{\underline{m} \mid \underline{m} \text{ essential } \} \subset \mathbb{R}^{\binom{n-1}{2}}.$$

$\rightarrow$ Lattice points in a convex polytope $P(\omega_i)$.

More general:
Let $\lambda = m_1\omega_1 + \ldots + m_{n-1}\omega_{n-1}$ and $V(\lambda)$ be the simple $\mathfrak{sl}_n$-module.

(Feigin-F-Littelmann, '11) The essential monomials for $V(\lambda)$ satisfy

$$P(\lambda) = \sum m_i P(\omega_i) \text{ and } S(\lambda) = \sum m_i S(\omega_i).$$

The semigroup of essential monomials is finitely generated

$$\bigcup_{\lambda \in P^+} (S(\lambda) \times \lambda) \subset \mathbb{Z}^N \times P^+.$$

Short excursion

Stanley: Two poset polytopes, ’86

For a given finite poset, say $(P, \geq)$, Stanley introduced two polytopes, the **order polytope**

$$\mathcal{O}_P = \{(x_q) \in \mathbb{R}^{|P|} \mid x_q \geq x_p \text{ if } q \geq p \text{ and } 0 \leq x_q \leq 1\}$$

and the **chain polytope**

$$\mathcal{C}_P = \{(x_q) \in \mathbb{R}^{|P|} \mid x_q \geq 0 \text{ and } \sum_{q \in \text{ chain}} x_q \leq 1\}$$

- Ideals vs. anti-chains.
- There is a piecewise linear transfer map, the polytopes are Ehrhart equivalent (Stanley).
- The two polytopes are unimodular equivalent if and only if there is no star-subposet (Hibi-Li ’16).

Using Polymake, we can compute the **f**-vector of the polytopes:

$(20, 122, 376, 690, 807, 615, 302, 91, 15)$ vs $(20, 122, 372, 670, 766, 571, 276, 83, 14)$.

Conjecture (Hibi-Li)

The difference of the **f***-vectors is non-negative.*

- Our polytope is a chain polytope, Gelfand-Tsetlin is an order polytope.
- Inspired by our work: Marked chain and marked order polytopes (Ardila-Bliem-Salazar).
- $P(\lambda)$ is a marked chain polytope, the Gelfand-Tsetlin polytope is a marked order polytope.
- Two such polytopes are Ehrhart equivalent but not unimodular equivalent (in general).
- Much more dualities between the two polytopes for our most favorite poset.
- More general, marked poset polytopes and conjecture on the **f**-vector $\longrightarrow$ Polymake, OSCAR?

Back to Plückers for the flag variety: we obtain t-deformed Plücker relations, for example the relation

$$t^0 X_{12} t^5 X_{34} - t^2 X_{13} t^3 X_{24} + t^2 X_{14} t^4 X_{23} = 0.$$

Theorem (Feigin-F-Littelmann)

This defines our family $\mathcal{X}_t$, and X_0 is an irreducible toric variety (defined by binomials) with moment polytope $P(\lambda)$.

This result does not depend on the precise degree but on the cone defined by

$$(a)\ \ a_{i+1,i} + a_{i+2,i+1} \geq a_{i+2,i} \text{ for } 1 \leq i \leq n-2$$

and

$$(b)\ \ a_{j,i} + a_{j+1,i+1} \geq a_{j+1,i} + a_{j,i+1} \text{ for } 1 \leq i < j \leq n-2.$$

What about the faces of the cone?

$$(a)\ \ a_{i+1,i} + a_{i+2,i+1} \geq a_{i+2,i} \text{ for } 1 \leq i \leq n-2$$

In $U(\mathfrak{n}^-)$ we have: $x \otimes y - y \otimes x = [x,y]$, which implies with strict inequalities

$$\operatorname{gr} U(\mathfrak{n}^-) \cong S(\mathfrak{n}^-).$$

Short excursion

Universal linear degenerate flag variety [Cerulli Irelli-Fang-Feigin-F-Reineke]

$$\pi : \{(U_1, \ldots, U_{n-1}, f_1, \ldots, f_{n-2}) \mid \dim U_i = i, f_i U_i \subset U_{i+1}\} \longrightarrow \operatorname{End}(\mathbb{C}^n)^{n-2}$$

- $\pi^{-1}(\mathrm{id}, \ldots, \mathrm{id}) \cong \mathcal{F}_n$.
- $\pi^{-1}(0, \ldots, 0) \cong \prod \mathrm{Gr}(k, n)$.
- Irreducible or normal or flat fibres are described.
- The PBW fibres correspond to some (a) inequalities being strict.

Translating back to Plücker coordinates:
We define $\mathcal{C} \subset \mathbb{R}^{2^n-2}$ by the equalities and inequalities

1. $s_{1,\ldots,k} = 0$ für $1 \leq k \leq n-1$
2. For any $1 \leq i < j \leq n$ and $i \leq k < \ell < j$:
   $s_{1,\ldots,i-1,i+1,\ldots,k,j} = s_{1,\ldots,i-1,i+1,\ldots,l,j}$.
3. For a given I, there are precise subsets J_1 and J_2 with $s_I = s_{J_1} + s_{J_2}$.
4. $s_{1,\ldots,i-1,i+1} + s_{1,\ldots,i,i+2} \geq s_{1,\ldots,i-1,i+2}$ for $1 \leq i \leq n-2$
5. $s_{1,\ldots,i-1,j} + s_{1,\ldots,i,j+1} \geq s_{1,\ldots,i-1,j+1} + s_{1,\ldots,i,j}$ for $1 \leq i < j-1 \leq n-2$

Theorem (Fang-Feigin-F-Makhlin)

$\mathcal{C}$ is a maximal cone in the tropical flag variety.

Remark

Recently, Makhlin described another maximal cone of the tropical flag variety, providing the Gelfand-Tsetlin degeneration.

General setup:
We consider G/B, a generalized flag variety, then

$$\mathbb{C}[G/B] \cong \bigoplus_{\lambda \in P^+} V(\lambda)^*.$$

Let $(\beta_1, \ldots, \beta_N)$ a sequence of positive roots and U^- a maximal unipotent subgroup.
We call the sequence **birational** if

$$U^-_{\beta_1} \times \ldots \times U^-_{\beta_N} \longrightarrow U^-$$

is birational. Then

$$U(\mathfrak{n}^-) = \langle f_{\beta_1}^{\ell_1} \cdots f_{\beta_N}^{\ell_N} \mid \ell_i \geq 0 \rangle_{\mathbb{C}}.$$

Question: Is this if and only if? Does proper imply birational in this setup?

Remark

We can play the same game for Grassmann varieties, G/P, spherical varieties...

The set $\{f_{\beta_1}^{\ell_1} \cdots f_{\beta_N}^{\ell_N} \mid \ell_i \geq 0\}$ is not necessarily a basis of $U(\mathfrak{n}^-)$:

- If all β_i are pairwise distinct, then this is a basis (PBW Theorem).
- Let $\underline{w_0} = s_{i_1} \ldots s_{i_N} \in W$, then $\{f_{\alpha_1}^{\ell_1} \cdots f_{\alpha_N}^{\ell_N} \mid \ell_i \geq 0\}$ is not linearly independent.

We fix a lexicographic order $\geq$ on $\mathbb{Z}_{\geq 0}^N$, to obtain a basis (of **essential monomials**).

- In the first case, the basis is parametrized by lattice points in the positive orthant.
- In the second case and choosing the opposite lexicographic order, the basis is parametrized by lattice points in the **string cone** $\mathcal{C}_{\underline{w_0}}$ (Berenstein-Zelevinsky, Littelmann). There is an iterative and an explicit description of the string cone.
- In the general case, ... ?

Let $\lambda \in P^+$ and $V(\lambda)$ the corresponding simple G-module. Then

$$V(\lambda) = U(\mathfrak{n}^-).v_\lambda,$$

our chosen birational sequence and lexicographic order on $\mathbb{Z}_{\geq 0}^N$ induce a monomial basis for $V(\lambda)$. We denote

$$S(\lambda) = \{\underline{m} \in \mathbb{Z}_{\geq 0}^N \mid f^{\underline{m}} \text{ is essential for } V(\lambda)\}.$$

Remark

How to compute this? Use GAP and canonical bases of quantum groups, to compute the essential monomials.

Since $V(\lambda + \mu) \subset V(\lambda) \otimes V(\mu)$, we obtain a semigroup

$$S(G, \beta_1, \ldots, \beta_N, \geq) = \bigcup_{\lambda \in P^+} (S(\lambda) \times \lambda) \quad \subset \mathbb{Z}^N \times P^+.$$

Conjecture

For any choice of birational sequence and lexicographic order, $S(G, \beta_1, \ldots, \beta_N, \geq)$ is finitely generated.

Conjecture

For any choice of birational sequence and lexicographic order, $S(G, \beta_1, \ldots, \beta_N, \geq)$ is finitely generated.

Remark

1. *Combine the results from GAP with Polymake $\longrightarrow$ OSCAR?.*
2. *This conjecture is true for our previous examples, especially the string cone and the Lusztig cone.*
3. *Gornitskii proposed a local criterium to check that the semigroup is finitely generated.*

Conjecture

The semigroup is generated by all essential monomial for $\lambda \leq \rho$.

Remark

By considering for every $f^{\underline{m}}.v_\lambda$ the dual element $\zeta_{\underline{m},\lambda}$, we obtain a basis of $\mathbb{C}[G/B] \longrightarrow$ Standard-Monomial-Theory.

Example: (back to beginning)
The set of all Plücker coordinates of length j is a basis for $V(\omega_j)^*$. The basis of $\mathbb{C}[G/B]$ is then given by semi-standard Young tableaux.

$$\mathcal{C} := \overline{\mathbb{R}_{\geq 0} S(G, \beta_1, \ldots, \beta_N, \geq)} \subset \mathbb{R}^N \times \mathbb{R}P^+.$$

Suppose $S(G, \beta_1, \ldots, \beta_N, \geq)$ is finitely generated and saturated, then $P(\lambda) \cap \mathbb{Z}^N = S(\lambda)$.

Theorem (Alexeev-Brion)

There is a toric degeneration of G/B, such that the moment polytope of the special fibre is $P(\lambda)$.

Given G/B, then there are sequences and orders such that the semigroup is finitely generated and saturated

$$\longrightarrow \text{ String polytopes, Lusztig polytopes}$$

Question:
Is there a choice such that the semigroup is generated by degree 1?

- Plücker coordinates in type A.
- Known for type C, G.
- String polytopes do not work for type B, D, E, F, G.

Question:
For given G/B, is there a homogeneous order such that the semigroup is finitely generated and saturated?

- String and Lusztig polytopes do not work.
- Type A, C, G are solved.

Proposition

Suppose there exists such an order, then the PBW degenerate variety G/B^a is a flat degeneration of G/B.

$\longrightarrow$ framework of PBW degenerations, so far only in type A, C

Thank you!