An explicit matrix factorization of cubic hypersurfaces of small dimension

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Matrix factorization

Question

How to decompose a polynomial $f = xy - zw$?

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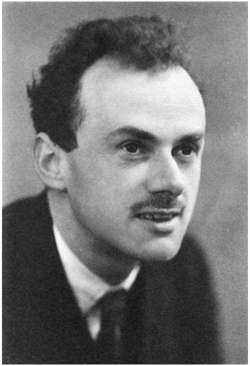
Answer

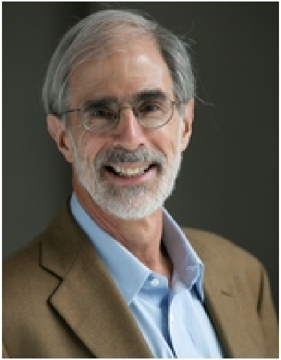
Replace f *by a* 2×2 *matrix*

$$
\begin{aligned}
f\mathrm{I} &= \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} \\
&= \begin{pmatrix} x & z \\ w & y \end{pmatrix} \begin{pmatrix} y & -z \\ -w & x \end{pmatrix}
\end{aligned}
$$

Matrix factorization



Paul Dirac



David Eisenbud

Matrix factorization

Definition

Let $f \in S = k[x_0, \cdots, x_n]$ be a homogeneous form. A matrix factorization of f is a pair of matrices (A, B) such that

$$AB = BA = f \cdot \mathrm{I}.$$

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Proposition (Eisenbud, 1980)

There is a bijection between

1. *linear equivalence classes of matrix factorizations* (A, B) *of* f*;*
2. *isomorphism classes of maximal Cohen-Macaulay* $S/(f)$*-modules*

via $(A, B) \mapsto \operatorname{coker} A$*.*

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Remark

When A appears in a matrix factorization (A, B) of f, then A uniquely determines B. Hence, we will sometimes say: "A is a matrix factorization of f".

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1. *When* $N = 3$ *or* $N = 4$, f *is linearly determinantal. In particular, there is a rank 1 Ulrich module.*
2. *When* $N = 5$, f *is not linearly determinantal but linearly Pfaffian. In particular, there is a rank* 2 *Ulrich module.*

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3. *When f is a cubic in 6 variables, then f is linearly Pfaffian if and only if $V(f)$ contains a del Pezzo surface of degree 5 (Beauville, 2000).*

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4. *When* $N \leq 9$, *there is a rank* 9 *Ulrich module (Manivel, 2018).*

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Want : Describe Manivel's result explicitly by computing a matrix factorization.

Shamash's construction

$$\begin{array}{ccc} X = V(f) \quad \text{degree } d \text{ hypersurface} & \Leftrightarrow & R = S_X \\ \cup & & \downarrow \\ Z & & S_Z \end{array}$$

Let $F_\bullet : \cdots \to F_1 \to F_0 = S \to S_Z \to 0$ be the minimal S-free resolution of Z.

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$$F_1 \otimes_S R \to F_0 \otimes_S R = R \to S_Z \to 0,$$

we have an exact sequence of free R-modules

$$\cdots \to G_4 \oplus G_2(-d) \oplus G_0(-2d) \to G_3 \oplus G_1(-d) \to G_2 \oplus G_0(-d) \to G_1 \to G_0 \to S_Z \to 0$$

where $G_i := F_i \otimes_S R$.

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where $G_i := F_i \otimes_S R$. The resolution becomes 2-periodic after a finite step; say A and B, both are annihilated by f. In particular, A provides a matrix factorization of f.

Shamash's construction

Example

1. $Z \subset X \subset \mathbb{P}^3$ a twisted cubic on a cubic surface; we have the Betti table

$$\begin{matrix} 1 & - & - \\ - & 3 & 2 \end{matrix}$$

and hence Shamash's construction will provide a 3×3 linear MF of X.

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1. $Z \subset X \subset \mathbb{P}^3$ a twisted cubic on a cubic surface; we have the Betti table

1	–	–
–	3	2

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2. $Z \subset X \subset \mathbb{P}^4$ an elliptic normal curve on a cubic threefold; we have the Betti table

1	–	–	–
–	5	5	–
–	–	–	1

   and hence Shamash's construction will provide a 6×6 linear MF of X. In fact, since X is linearly Pfaffian but not determinantal, one can obtain a skew-symmetric MF after a certain linear coordinate change.

Shamash's construction

Today, we will begin with the table

1	–	–	–	–	–
–	10	16	–	–	–
–	–	–	16	10	–
–	–	–	–	–	1

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Answer

Yes, we can find Z as a prime Fano threefold of index 1 and genus 7.

Digestion: toward the Cartan cubic

Both approaches have strong connections between the Cartan cubic.

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Recall

The Lie group E_6 acts on a 27-dimensional vector space V_{27}, *i.e.*, on $\mathbb{P}^{26}$ with three orbits:

1. Cayley plane $\mathbb{OP}^2$, the Severi variety of dimension 16;
2. $X \setminus \mathbb{OP}^2$, where $X = \mathrm{Sec}(\mathbb{OP}^2) = V(F_{\mathcal{C}})$ the Cartan cubic hypersurface;
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$\rightsquigarrow$ Recover the Cartan cubic from the previous Betti table.

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The previous Betti table comes from the spinor tenfold $\mathcal{S}_{10}(= OG(5,10)) \subset \mathbb{P}^{15}$. It is natural to consider the "universal cubic" containing $\mathcal{S}_{10}$.

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$$\mathcal{S}_{10} \subset \mathbb{P}^{15} = \mathbb{P}(\wedge^0\mathbb{C}^5 \oplus \wedge^2\mathbb{C}^5 \oplus \wedge^4\mathbb{C}^5)$$

$$N = \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ -x_{12} & 0 & x_{23} & x_{24} & x_{25} \\ -x_{13} & -x_{23} & 0 & x_{34} & x_{35} \\ -x_{14} & -x_{24} & -x_{34} & 0 & x_{45} \\ -x_{15} & -x_{25} & -x_{35} & -x_{45} & 0 \end{pmatrix}$$

10 generators of $\mathcal{S}_{10}$ are quadrics $q_1, \cdots, q_5, q'_1, \cdots, q'_5$:

$$\begin{cases} q_i &= x_0 y_i + (-1)^i \operatorname{Pf}(N, i) \\ q'_i &= i\text{-th entry of } (y_1 \cdots y_5)N. \end{cases}$$

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Put 10 extra variables $a_1, \cdots, a_5, b_1, \cdots, b_5$ of degree 1 corresponding to q_i, q_i', and take $F := \sum(q_i a_i + q_i' b_i)$. Shamash's construction yields:

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$$
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & y_1 & y_2 & y_3 & y_4 & y_5 & \cdot & a_5 & a_4 & a_3 & a_2 & a_1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & y_1 & y_2 & y_3 & y_4 & \cdot & \cdot & \cdot & \cdot & \cdot & a_5 & \cdot & x_{45} & x_{35} & x_{25} & x_{15} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & y_1 & y_2 & y_3 & \cdot & \cdot & \cdot & -y_5 & \cdot & \cdot & \cdot & \cdot & \cdot & a_4 & -x_{45} & \cdot & x_{34} & x_{24} & x_{14} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & y_1 & y_2 & \cdot & \cdot & -y_4 & \cdot & \cdot & -y_5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_3 & -x_{35} & -x_{34} & \cdot & x_{23} & x_{13} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & y_1 & \cdot & -y_3 & \cdot & -y_4 & \cdot & \cdot & -y_5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_2 & -x_{25} & -x_{24} & -x_{23} & \cdot & x_{12} \\
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Proposition (K.-Schreyer)

$F_{\mathcal{C}}$ is the Cartan cubic form.

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\left(\begin{array}{ccccccccccccccccccccccccccc}
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & y_1 & y_2 & y_3 & y_4 & \cdot & \cdot & \cdot & \cdot & w & a_5 & \cdot & x_{45} & x_{35} & x_{25} & x_{15} \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -y_2 & -y_3 & \cdot & -y_4 & \cdot & \cdot & -y_5 & \cdot & \cdot & \cdot & w & \cdot & \cdot & \cdot & \cdot & a_1 & -x_{15} & -x_{14} & -x_{13} & -x_{12} & \cdot \\
\cdot & \cdot & \cdot & \cdot & y_1 & -y_2 & \cdot & \cdot & \cdot & \cdot & \cdot & -a_5 & \cdot & \cdot & a_4 & -a_3 & \cdot & \cdot & -x_{45} & x_{35} & -x_{34} & \cdot & \cdot & \cdot & \cdot & -b_1 & b_2 \\
\cdot & \cdot & \cdot & y_1 & \cdot & -y_3 & \cdot & \cdot & \cdot & \cdot & a_5 & \cdot & \cdot & -a_4 & \cdot & a_2 & \cdot & x_{45} & \cdot & -x_{25} & x_{24} & \cdot & \cdot & \cdot & -b_1 & \cdot & b_3 \\
\cdot & \cdot & \cdot & y_2 & -y_3 & \cdot & \cdot & \cdot & \cdot & -a_5 & \cdot & \cdot & a_4 & \cdot & \cdot & -a_1 & -x_{45} & \cdot & \cdot & x_{15} & -x_{14} & \cdot & \cdot & \cdot & -b_2 & b_3 & \cdot \\
\cdot & \cdot & y_1 & \cdot & \cdot & -y_4 & \cdot & \cdot & -a_5 & \cdot & \cdot & \cdot & \cdot & a_3 & -a_2 & \cdot & \cdot & -x_{35} & x_{25} & \cdot & -x_{23} & \cdot & \cdot & -b_1 & \cdot & \cdot & b_4 \\
\cdot & \cdot & y_2 & \cdot & -y_4 & \cdot & \cdot & a_5 & \cdot & \cdot & \cdot & \cdot & -a_3 & \cdot & a_1 & \cdot & x_{35} & \cdot & -x_{15} & \cdot & x_{13} & \cdot & \cdot & -b_2 & \cdot & b_4 & \cdot \\
\cdot & \cdot & y_3 & -y_4 & \cdot & \cdot & -a_5 & \cdot & \cdot & \cdot & \cdot & \cdot & a_2 & -a_1 & \cdot & \cdot & -x_{25} & x_{15} & \cdot & \cdot & -x_{12} & \cdot & \cdot & -b_3 & b_4 & \cdot & \cdot \\
\cdot & y_1 & \cdot & \cdot & \cdot & -y_5 & \cdot & \cdot & a_4 & \cdot & -a_3 & a_2 & \cdot & \cdot & \cdot & \cdot & \cdot & x_{34} & -x_{24} & x_{23} & \cdot & \cdot & -b_1 & \cdot & \cdot & \cdot & b_5 \\
\cdot & y_2 & \cdot & \cdot & -y_5 & \cdot & \cdot & -a_4 & \cdot & a_3 & \cdot & -a_1 & \cdot & \cdot & \cdot & \cdot & -x_{34} & \cdot & x_{14} & -x_{13} & \cdot & \cdot & -b_2 & \cdot & \cdot & b_5 & \cdot \\
\cdot & y_3 & \cdot & -y_5 & \cdot & \cdot & a_4 & \cdot & \cdot & -a_2 & a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & x_{24} & -x_{14} & \cdot & x_{12} & \cdot & \cdot & -b_3 & \cdot & b_5 & \cdot & \cdot \\
\cdot & y_4 & -y_5 & \cdot & \cdot & \cdot & -a_3 & a_2 & -a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -x_{23} & x_{13} & -x_{12} & \cdot & \cdot & \cdot & -b_4 & b_5 & \cdot & \cdot & \cdot \\
y_1 & \cdot & \cdot & \cdot & \cdot & w & \cdot & \cdot & -x_{45} & \cdot & x_{35} & -x_{25} & \cdot & -x_{34} & x_{24} & -x_{23} & \cdot & \cdot & \cdot & \cdot & \cdot & b_1 & \cdot & \cdot & \cdot & \cdot & x_0 \\
y_2 & \cdot & \cdot & \cdot & w & \cdot & \cdot & x_{45} & \cdot & -x_{35} & \cdot & x_{15} & x_{34} & \cdot & -x_{14} & x_{13} & \cdot & \cdot & \cdot & \cdot & \cdot & b_2 & \cdot & \cdot & \cdot & x_0 & \cdot \\
y_3 & \cdot & \cdot & w & \cdot & \cdot & -x_{45} & \cdot & \cdot & x_{25} & -x_{15} & \cdot & -x_{24} & x_{14} & \cdot & -x_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & b_3 & \cdot & \cdot & x_0 & \cdot & \cdot \\
y_4 & \cdot & w & \cdot & \cdot & \cdot & x_{35} & -x_{25} & x_{15} & \cdot & \cdot & \cdot & x_{23} & -x_{13} & x_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_4 & \cdot & x_0 & \cdot & \cdot & \cdot \\
y_5 & w & \cdot & \cdot & \cdot & \cdot & -x_{34} & x_{24} & -x_{14} & -x_{23} & x_{13} & -x_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_5 & x_0 & \cdot & \cdot & \cdot & \cdot \\
\cdot & a_5 & a_4 & a_3 & a_2 & a_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b_1 & b_2 & b_3 & b_4 & b_5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
a_5 & \cdot & -x_{45} & -x_{35} & -x_{25} & -x_{15} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -b_1 & -b_2 & -b_3 & -b_4 & \cdot & \cdot & \cdot & \cdot & x_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
a_4 & x_{45} & \cdot & -x_{34} & -x_{24} & -x_{14} & \cdot & \cdot & \cdot & -b_1 & -b_2 & -b_3 & \cdot & \cdot & \cdot & b_5 & \cdot & \cdot & \cdot & x_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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a_2 & x_{25} & x_{24} & x_{23} & \cdot & -x_{12} & -b_1 & \cdot & b_3 & \cdot & b_4 & \cdot & \cdot & b_5 & \cdot & \cdot & \cdot & x_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
a_1 & x_{15} & x_{14} & x_{13} & x_{12} & \cdot & b_2 & b_3 & \cdot & b_4 & \cdot & \cdot & b_5 & \cdot & \cdot & \cdot & x_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)
$$

which is the Hessian matrix $\mathcal{H}(F_{\mathcal{C}})$.

Cubics whose Hessian give matrix factorizations

Theorem (K.-Schreyer)

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- $f = x_0^3$;
- $f = x_0^2 x_1$;
- $f = x_0(x_1^2 + \cdots + x_n^2)$;
- $f =$ *the equation of the secant variety of the one of* 4 *Severi varieties:* $v_2(\mathbb{P}^2), \mathbb{P}^2 \times \mathbb{P}^2, Gr(2,6), \mathbb{OP}^2$.

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Key idea

XJC-correspondence [Pirio-Russo, 2014].

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In the case, the partial derivatives of such an f induces a $(2,2)$-Cremona transformation, and the Hessian matrix $\mathcal{H}(f)$ induces a cubic Jordan algebra structure.

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Collect more examples and do the experiments for higher degrees.

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Example

Let g be the hyperdeterminant of a generic $2 \times 2 \times 2$ hypermatrix (in 8 variables). Then g is a quartic homogeneous form, invariant under the $SL(2)^3$-action. A result of Ein and Shepherd-Barron implies that the partial derivatives of g induces a $(3,3)$-Cremona transformation. We observed that the Hessian matrix $\mathcal{H}(g)$ does not give a matrix factorization of g but of g^2.