Weingarten calculus and applications to Quantum Information Theory

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Talk outline

1. The Weingarten formula
2. Graphical Weingarten calculus
3. An application to QIT

The Weingarten formula

Computing expectation values

- Gaussian integrals: if $X \in \mathbb{C}^d$ is a centered random complex Gaussian vector, i.e. $\mathrm{d}\mathbb{P}/\mathrm{dLeb} \sim \exp(\langle x, Ax\rangle/2)$, then [Iss18]

$$\mathbb{E}[X_{i_1} \cdots X_{i_p} \bar{X}_{i'_1} \cdots \bar{X}_{i'_p}] = \prod_{\alpha \in \mathcal{S}_p} \prod_{k=1}^{p} \mathbb{E}[X_{i_k} \bar{X}_{i'_{\alpha(k)}}]$$

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- Spherical integrals: if Y is a uniform random point on the unit sphere of $\mathbb{C}^d$, then YN is a standard complex Gaussian in $\mathbb{C}^d$, where N is an independent χ^2 random variable. Thus one can use the Gaussian formula to compute the spherical integrals.

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- Unitary integrals?

The Weingarten formula

Theorem. [Wei78, Col03, CŚ06] Let d be a positive integer and $\mathbf{i} = (i_1, \ldots, i_p)$, $\mathbf{i}' = (i'_1, \ldots, i'_{p'})$, $\mathbf{j} = (j_1, \ldots, j_p)$, $\mathbf{j}' = (j'_1, \ldots, j'_{p'})$ be tuples of positive integers from $\{1, 2, \ldots, d\}$. Then, if $p \neq p'$

$$\int_{\mathcal{U}_d} U_{i_1 j_1} \cdots U_{i_p j_p} \bar{U}_{i'_1 j'_1} \cdots \bar{U}_{i'_{p'} j'_{p'}} \,\mathrm{d}U = 0.$$

If $p = p'$,

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$$\sum_{\alpha, \beta \in \mathcal{S}_p} \delta_{i_1 i'_{\alpha(1)}} \ldots \delta_{i_p i'_{\alpha(p)}} \delta_{j_1 j'_{\beta(1)}} \ldots \delta_{j_p j'_{\beta(p)}} \mathrm{Wg}(d, \alpha^{-1}\beta),$$

where Wg is a combinatorial weight, taking as parameters the dimension of the unitary group and a permutation.

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- Has found many applications (especially in RMT, e.g. [Col03]) and extensions (e.g. quantum groups [BC07])

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$$\mathrm{Wg}(d, \sigma) = \frac{1}{p!^2} \sum_{\lambda \vdash p, \ell(\lambda) \leqslant d} \frac{\chi^\lambda(e)^2}{s_{\lambda,d}(1)} \chi^\lambda(\sigma),$$

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- Example: $\int_{\mathcal{U}_d} U_{11} U_{22} U_{33} \bar{U}_{12} \bar{U}_{23} \bar{U}_{31} \,\mathrm{d}U = \mathrm{Wg}(d, (123)) = \frac{2}{d(d^2-1)(d^2-2)}$, since there is just one term in the sum, $\alpha = \mathrm{id}$ and $\beta = (123)$.

Schur-Weyl duality

Theorem. [Aub18] Consider the following two subalgebras of $M_{d^p}(\mathbb{C})$: $\mathcal{A} = \text{span}\{A^{\otimes p} \ : \ A \in M_d(\mathbb{C})\}$ and $\mathcal{B} = \text{span}\{P_\sigma \ : \ \sigma \in \mathcal{S}_p\}$, where P_σ permutes the tensor factors according to σ

$$P_\sigma x_1 \otimes \cdots \otimes x_p = x_{\sigma(1)} \otimes \cdots x_{\sigma(p)}.$$

Then $\mathcal{A}$ and $\mathcal{B}$ are the commutant of each other.

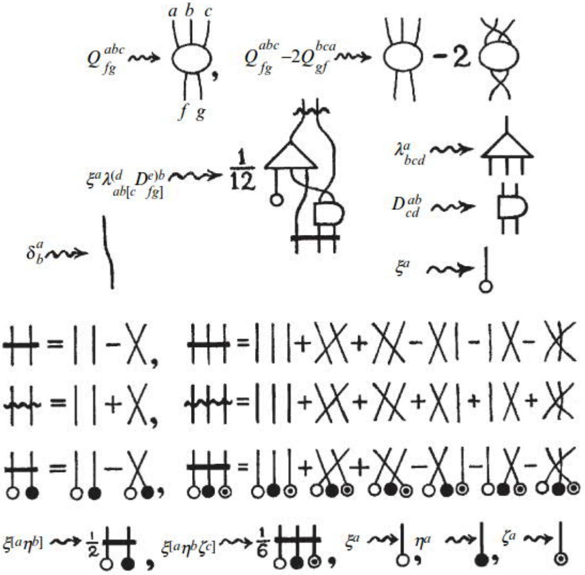
- We show $\mathcal{B}' \subseteq \mathcal{A}$. Let $X \in \mathcal{B}'$.
- $X = \frac{1}{p!} \sum_{\sigma \in S_p} P_\sigma X P_\sigma^{-1}$
- $M_{d^p}(\mathbb{C})$ is spanned by simple tensors, so it's enough to show $\sum_{\sigma \in S_p} P_\sigma X_1 \otimes \cdots \otimes X_p P_\sigma^{-1} \in \mathcal{A}$.
- We have, for i.i.d. ± 1 centered random variables ε_i

$$\sum_{\sigma \in S_p} P_\sigma X_1 \otimes \cdots \otimes X_p P_\sigma^{-1} = \sum_{\sigma \in S_p} X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(p)}$$
$$= \mathbb{E}\left[\left(\prod_{i=1}^p \varepsilon_i \right) \left(\sum_{j=1}^p \varepsilon_j X_j \right)^{\otimes p} \right].$$

- One can show $\mathcal{A} = \text{span}\{U^{\otimes p} \ : \ A \in \mathcal{U}_d\}$.

Graphical Weingarten calculus

Boxes & wires



[Pen05]

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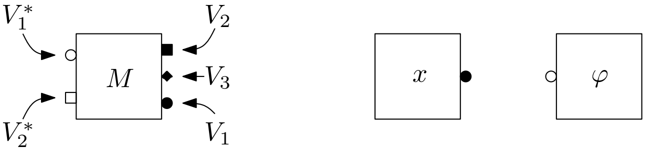

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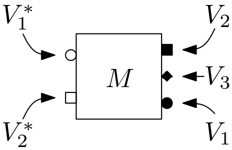

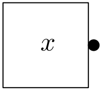

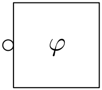

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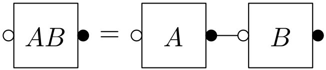
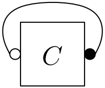
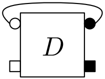
- Tensor contractions (or traces) $V \otimes V^* \to \mathbb{C}$ ⇝ wires.







$\mathrm{Tr(C)}$ $\mathrm{Tr_{V_1}(D)}$

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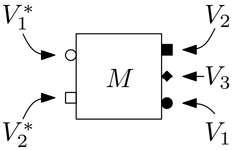

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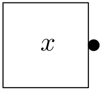

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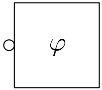

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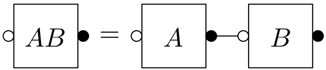

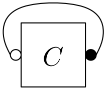

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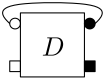

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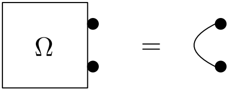
- Maximally entangled vector $\Omega := \sum_{i=1}^{\dim V_1} e_i \otimes e_i \in V_1 \otimes V_1$



"Graphical" Weingarten formula: main idea

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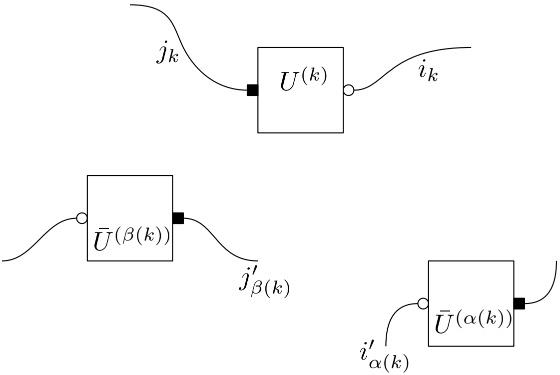

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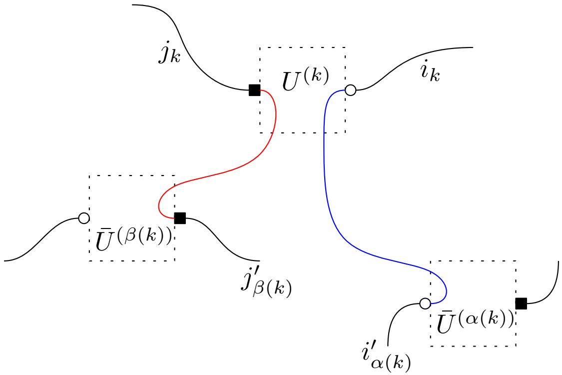

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4. For all $i = 1, \ldots, p$, add a wire between each white decoration of the i-th U box and the corresponding white decoration of the $\alpha(i)$-th $\bar{U}$ box. In a similar manner, use β to pair black decorations.

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5. Erase all U and $\bar{U}$ boxes. The resulting diagram is denoted by $\mathcal{D}_{(\alpha,\beta)}$.

Theorem.

$$\mathbb{E}\mathcal{D} = \sum_{\alpha,\beta \in \mathcal{S}_p} \mathcal{D}_{(\alpha,\beta)} \,\mathrm{Wg}(d, \alpha\beta^{-1}).$$

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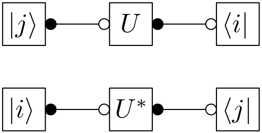

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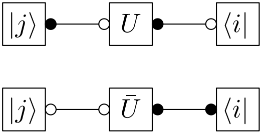

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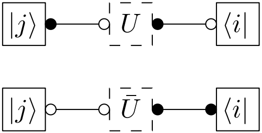

Figure: Erase U and $\bar{U}$ boxes.

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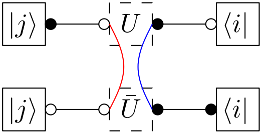

Figure: Pair white decorations (red wires) and black decorations (blue wires); only one possible pairing : $\alpha = (1)$ and $\beta = (1)$.

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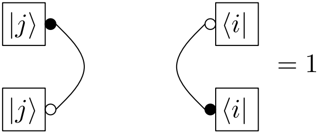

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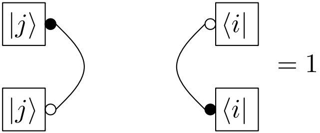

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- Conclusion :
  $\mathbb{E}|u_{ij}|^2 = \int |u_{ij}|^2 \mathrm{d}U = \mathcal{D}_{\alpha=(1),\beta=(1)} \cdot \mathrm{Wg}(N,(1)) = 1 \cdot 1/N = 1/N$.

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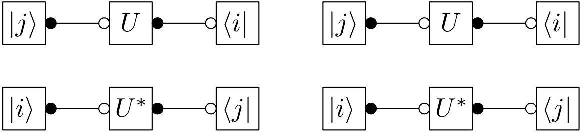

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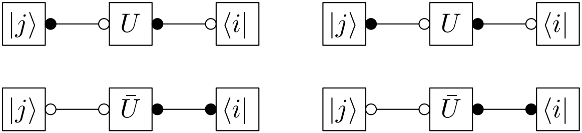

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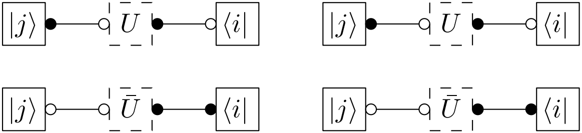

Figure: Erase U and $\bar{U}$ boxes.

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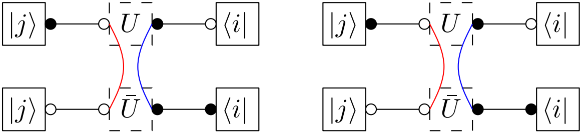

Figure: Pair white decorations (red wires) and black decorations (blue wires); first pairing : $\alpha = (1)(2)$ and $\beta = (1)(2)$.

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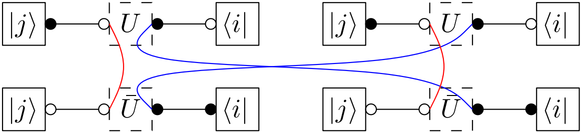

Figure: Second pairing : $\alpha = (1)(2)$ and $\beta = (12)$.

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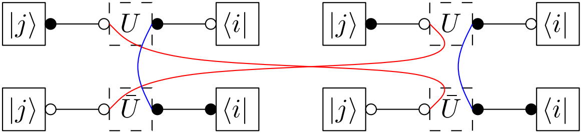

Figure: Third pairing : $\alpha = (12)$ and $\beta = (1)(2)$.

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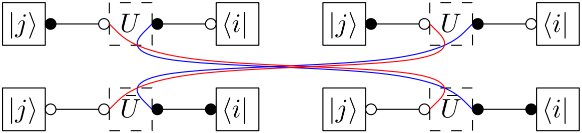

Figure: Fourth pairing : $\alpha = (12)$ and $\beta = (12)$.

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- Compute $\mathbb{E}|u_{ij}|^4 = \int_{\mathcal{U}(N)} |u_{ij}|^4 \mathrm{d}U$.
- Conclusion :

$$\mathbb{E}|u_{ij}|^4 = \int |u_{ij}|^4 \mathrm{d}U =$$

$$\begin{aligned}
&\mathcal{D}_{(1)(2),(1)(2)} \cdot \mathrm{Wg}(N,(1)(2)) + \\
&\mathcal{D}_{(1)(2),(12)} \cdot \mathrm{Wg}(N,(12)) + \\
&\mathcal{D}_{(12),(1)(2)} \cdot \mathrm{Wg}(N,(12)) + \\
&\mathcal{D}_{(12),(12)} \cdot \mathrm{Wg}(N,(1)(2)) \\
&= \mathrm{Wg}(N,(1)(2)) + \mathrm{Wg}(N,(12)) + \mathrm{Wg}(N,(12)) + \mathrm{Wg}(N,(1)(2)) \\
&= \frac{2}{N^2-1} - \frac{2}{N(N^2-1)} = \frac{2}{N(N+1)}.
\end{aligned}$$

Third example : twirling

- Consider a fixed matrix $A \in \mathcal{M}_N(\mathbb{C})$. Compute $\int_{\mathcal{U}(N)} U^* A U \mathrm{d}U$.

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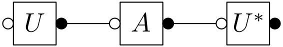

Figure: Diagram for $U^* A U$.

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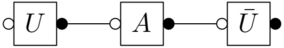

Figure: The U^* box replaced by an $\bar{U}$ box.

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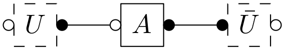

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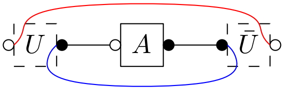

Figure: Pair white decorations (red wires) and black decorations (blue wires); only one possible pairing : $\alpha = (1)$ and $\beta = (1)$.

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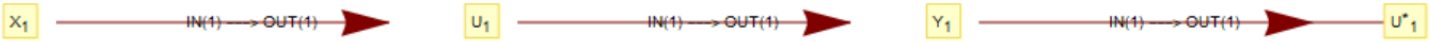
- Conclusion : $\int_{\mathcal{U}(N)} U^* A U \mathrm{d} U = \mathcal{D}_{\alpha=(1),\beta=(1)} \cdot \mathrm{Wg}(N,(1)) = \frac{\mathrm{Tr}(A)}{N} I_N$.

Random Tensor Network Integrator

- An implementation of the graphical Weingarten calculus in `Mathematica` and `python`

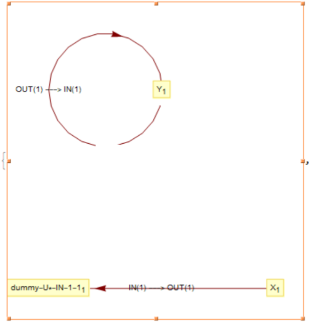
```
In[21]:= (* another example with dangling edges: XUYU^* *)
e1 = {{"X", 1, 0, 1}, {"U", 1, 1, 1}};
e2 = {{"U", 1, 0, 1}, {"Y", 1, 1, 1}};
e3 = {{"Y", 1, 0, 1}, {"U*", 1, 1, 1}};
g = {e1, e2, e3};
visualizeGraph[g]
integrateHaarUnitary[g, "U", {d}, {d}, d]
visualizeGraphExpansion[%]
```

Out[25]=




```
Out[26]= {{{{{Y, 1, 1, 1}, {Y, 1, 0, 1}}, {{X, 1, 0, 1}, {dummy-U*-IN-1-1, 1, 1, 1}}}, 1/d}}
```

Out[27]= {{

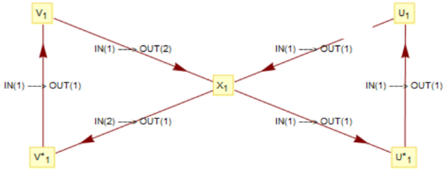


$, \frac{1}{d}\}\}$

Random Tensor Network Integrator

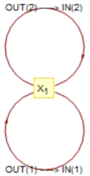
```
In[40]:= (* two independent unitary operators *)
e1 = {{"U", 1, 0, 1}, {"X", 1, 1, 1}};
e2 = {{"X", 1, 0, 1}, {"U*", 1, 1, 1}};
e3 = {{"U*", 1, 0, 1}, {"U", 1, 1, 1}};
e4 = {{"V", 1, 0, 1}, {"X", 1, 1, 2}};
e5 = {{"X", 1, 0, 2}, {"V*", 1, 1, 1}};
e6 = {{"V*", 1, 0, 1}, {"V", 1, 1, 1}};
g = {e1, e2, e3, e4, e5, e6};
visualizeGraph[g]
intU = integrateHaarUnitary[g, "U", {d1}, {d1}, d1];
intUV = integrateHaarUnitary[intU, "V", {d2}, {d2}, d2]
visualizeGraphExpansion[intUV]
```

Out[47]=




```
Out[49]= {{{{{X, 1, 1, 1}, {X, 1, 0, 1}}, {{X, 1, 1, 2}, {X, 1, 0, 2}}}, 1}}
```

Out[50]= {{



, 1}}

An application to QIT

Quantum information theory on one slide

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  - Completely positive: map PSD operators to PSD operators & stable by tensoring with the id: $[\Phi \otimes \mathrm{id}](\rho) \geqslant 0$
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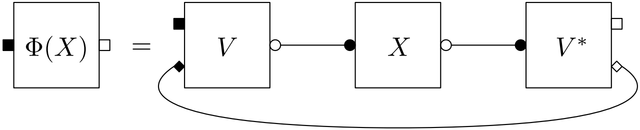
  - Trace preserving $\operatorname{Tr} \Phi(\rho) = \operatorname{Tr} \rho = 1$
- Stinespring dilation theorem: for any quantum channel Φ there exist an integer dimension n ($\rightsquigarrow$ size of the environment) and an isometry $V : \mathbb{C}^d \to \mathbb{C}^k \otimes \mathbb{C}^n$ such that

$$\Phi(\rho) = [\mathrm{id} \otimes \operatorname{Tr}](V \rho V^*)$$

Graphical representation of quantum channels

- A quantum channel $\Phi : M_d(\mathbb{C}) \to M_k(\mathbb{C})$



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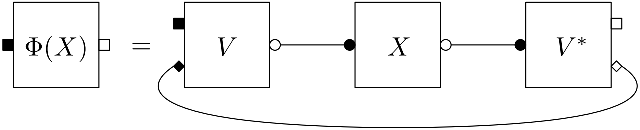

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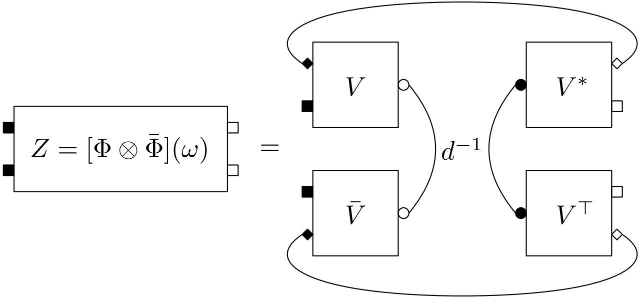
- Product of conjugate channels applied to the maximally entangled state $\omega = d^{-1}\Omega\Omega^*$



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- The MOE is not additive! [HW08, Has09]

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We have $H_{\min}(\Phi \otimes \bar{\Phi}) \leqslant H(\lambda)$.

- Previously known bound (deterministic, comes from linear algebra): for all t, n, k, the largest eigenvalue of Z_n is at least t.
- Two improvements:
  1. "better" largest eigenvalue,
  2. knowledge of the whole spectrum.

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- After doing the loop combinatorics, one is left with maximizing over $\mathcal{S}_{2p}^2$ quantities such as

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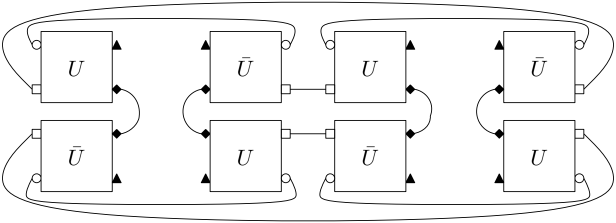
- Geodesic problems in symmetric groups $\Rightarrow$ non-crossing partitions $\Rightarrow$ free probability.
- Asymptotic for Weingarten weights:

$$\operatorname{Wg}(d,\sigma) = d^{-(p+|\sigma|)}(\operatorname{Mob}(\sigma) + O(d^{-2})).$$

Example: $\mathbb{E}\,\mathrm{Tr}(Z^2)$

- We have to compute a sum over all pairings of 4 "U" boxes with 4 "$\bar{U}$" boxes.
- Diagrams associated to pairings are indexed by 2 permutations $(\alpha, \beta) \in \mathcal{S}_4^2$. Consider the permutation $\delta = (1\ 4)\ (2\ 3) \in \mathcal{S}_4$.

The original diagram



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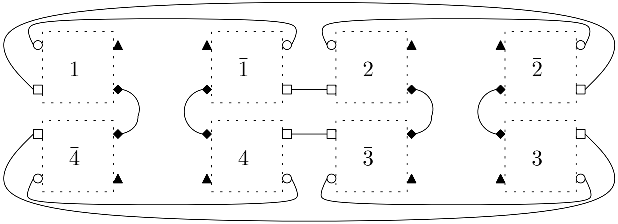
- We have to compute a sum over all pairings of 4 "U" boxes with 4 "$\bar{U}$" boxes.
- Diagrams associated to pairings are indexed by 2 permutations $({\color{red}\alpha}, {\color{blue}\beta}) \in \mathcal{S}_4^2$. Consider the permutation $\delta = (1\ 4)\ (2\ 3) \in \mathcal{S}_4$.

The diagram with the boxes removed



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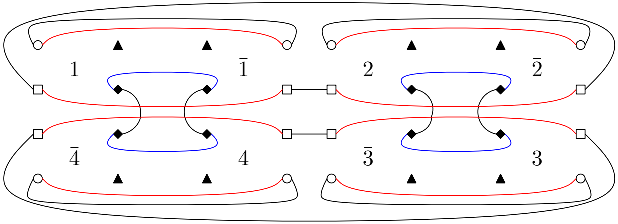
The wiring for $\alpha = \beta = \mathrm{id}$.



Contribution: $n^4 \cdot k^2 \cdot d^2 \cdot \mathrm{Wg}(\mathrm{id})$.

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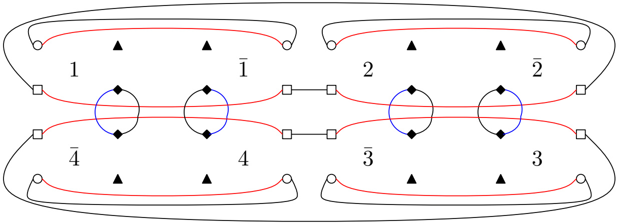

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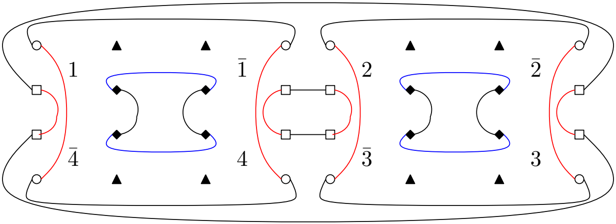

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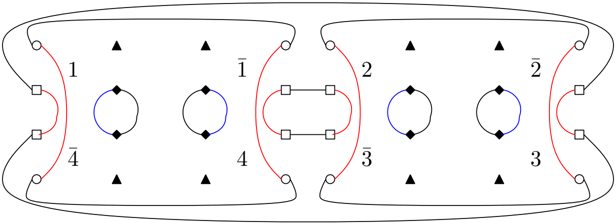

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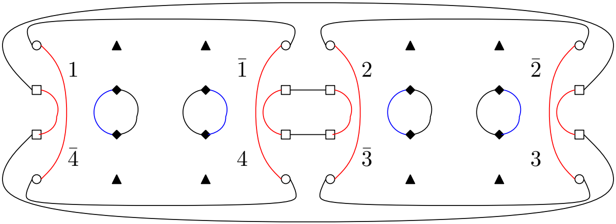


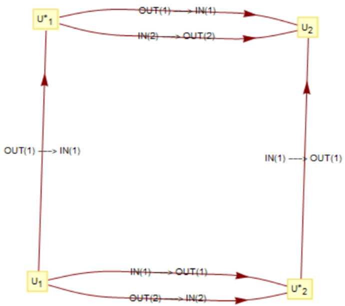
Contribution: $n^2 \cdot k^2 \cdot d^4 \cdot \mathrm{Wg}(\mathrm{id})$.

- Contributions of diagrams $\rightsquigarrow$ counting the loops $\rightsquigarrow$ statistics over permutations.

Random Tensor Network Integrator

```
In[61]:= (* Bell state as input in conjugate channels example; compute overlap of the output with another Bell state *)
e1 = {{"U", 1, 0, 1}, {"U*", 2, 1, 1}};
e2 = {{"U*", 1, 1, 1}, {"U", 2, 0, 1}};
e3 = {{"U", 1, 1, 1}, {"U*", 1, 0, 1}};
e4 = {{"U*", 2, 0, 1}, {"U", 2, 1, 1}};
e5 = {{"U", 1, 1, 2}, {"U*", 2, 0, 2}};
e6 = {{"U*", 1, 0, 2}, {"U", 2, 1, 2}};
g = {e1, e2, e3, e4, e5, e6};
visualizeGraph[g]
integrateHaarUnitary[g, "U", {d}, {n, k}, n k]
```

Out[68]=



Out[69]= $\left\{\left\{\{\}, \frac{d^2 k^2 n}{-1+k^2 n^2} + \frac{d k n^2}{-1+k^2 n^2} + \frac{d k^2 n}{k n - k^3 n^3} + \frac{d^2 k n^2}{k n - k^3 n^3}\right\}\right\}$

Thank you!

G. Aubrun.
A naive look at schur-weyl duality.
available on the author's website, 2018.

Teodor Banica and Benoit Collins.
Integration over compact quantum groups.
Publications of the Research Institute for Mathematical Sciences, 43:277–302, 2007.

Dan Bernstein.
The computational complexity of rules for the character table of sn.
Journal of Symbolic Computation, 37(6):727–748, 2004.

Benoît Collins and Ion Nechita.
Random quantum channels I: graphical calculus and the Bell state phenomenon.
Communications in Mathematical Physics, 297(2):345–370, 2010.

Benoît Collins.
Moments and cumulants of polynomial random variables on unitarygroups, the itzykson-zuber integral, and free probability.
International Mathematics Research Notices, 2003(17):953–982, 2003.

Benoît Collins and Piotr Śniady.
Integration with respect to the haar measure on unitary, orthogonal and symplectic group.
Communications in Mathematical Physics, 264(3):773–795, 2006.

Matthew B Hastings.
Superadditivity of communication capacity using entangled inputs.
Nature Physics, 5(4):255–257, 2009.

Patrick Hayden and Andreas Winter.
Counterexamples to the maximal p-norm multiplicativity conjecture for all $p > 1$.
Communications in mathematical physics, 284(1):263–280, 2008.

Leon Isserlis.
On a formula for the product-moment coefficient of any order of a normal frequency distribution in any number of variables.
Biometrika, 12(1/2):134–139, 1918.

Roger Penrose.
The road to reality.
Alfred A Knopf, New York, 2005.

Claude E. Shannon.

A mathematical theory of communication.
The Bell System Technical Journal, 27(3):379–423, 1948.

Peter W Shor.
Equivalence of additivity questions in quantum information theory.
Communications in Mathematical Physics, 246(3):453–472, 2004.

Don Weingarten.
Asymptotic behavior of group integrals in the limit of infinite rank.
Journal of Mathematical Physics, 19(5):999–1001, 1978.