K3 polytopes and their quartic surfaces

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Tropical hypersurfaces

K3 polytopes

Stablity of quartic surfaces

Tropical polynomials

We work over the tropical semiring $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$.
Tropical operations are defined as follows:

$$a \oplus b = \min\{a, b\} \text{ and } a \odot b = a + b.$$

Let $x_1, x_2, \ldots, x_n$ be variables representing elements in the tropical semiring. A monomial is any product of variables:

$$x_1^{i_1} \odot x_2^{i_2} \odot \cdots \odot x_n^{i_n}.$$

A polynomial is a finite linear combinations of monomials:

$$f(x_1, x_2, \ldots, x_n) = a_i \odot x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \oplus a_j \odot x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \oplus \cdots .$$

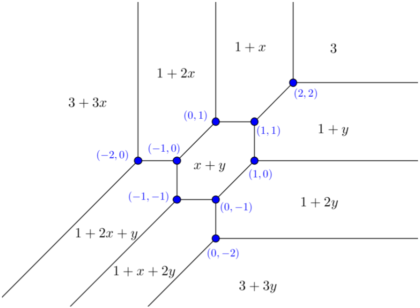
Tropical hypersurfaces

Given a polynomial f, we define the hypersurface $T(f)$ of f as the set of points $x \in \mathbb{R}^n$ at which *the minimum is attained at least twice*.

Example:

$f = 3x^3 \oplus 1x^2y \oplus 1xy^2 \oplus 3y^3 \oplus 1x^2 \oplus xy \oplus 1y^2 \oplus 1x \oplus 1y \oplus 3$

$\min\{3+3x, 1+2x+y, 1+x+2y, 3+3y, 1+2x, x+y, 1+2y, 1+x, 1+y, 3\}$



Tropical and classical hypersurfaces

Let K be an algebraically closed field with non-trivial non-Archimedean valuation (e.g. $K = \mathbb{C}\{\{t\}\}$). Let f be an Laurent polynomial

$$f = \sum_{u=(u_1,\ldots,u_n)\in\mathbb{Z}^n} c_u x_1^{u_1}\cdots x_n^{u_n}, \text{with } c_u \in K.$$

We define its tropicalization $\operatorname{trop}(f)$ as

$$\operatorname{trop}(f) = \min_{u\in\mathbb{Z}^n} \Big\{ \operatorname{val}(c_u) + \sum_{i\leq n} u_i\, x_i \Big\}.$$

Theorem (Kapranov's theorem)

The following sets coincide:

1. $\{w \in \mathbb{R}^n \,|\, \textit{the min in } trop(f)(w) \textit{ is attained at least twice}\}$;
2. *the closure of* $\{(val(y_1), \ldots, val(y_n)) \,|\, (y_1, \ldots, y_n) \in V(f)\}$.

Tropical hypersurfaces

Theorem (Structure theorem)

The tropical hypersurface $T(f)$ is the support of a pure rational polyhedral complex of dimension $n-1$.

The closure of the connected components of the complement of a tropical hypersurface $T(f)$ are called regions of $T(f)$. They are convex polyhedra.

Let $T(f)$ be a smooth tropical surface of degree 4 in $\mathbb{R}^3$. The surface $T(f)$ cuts at most one bounded region out.

Definition

A 3-dimensional polytope $\mathcal{P}$ is a K3 polytope if it arises as the closure of the bounded region in the complement of a smooth tropical surface of degree 4.

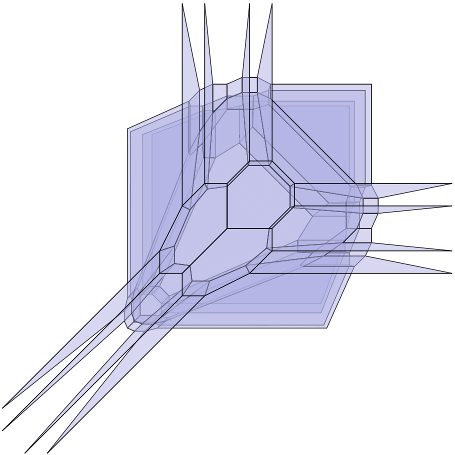
Example

Consider the tropical quartic surface defined by the polynomial:

$$
\begin{aligned}
f = {} & 5(x^4 \oplus y^4 \oplus z^4) \oplus 3(x^3y \oplus x^3z \oplus xy^3 \oplus y^3z \oplus xz^3 \oplus yz^3) \\
& \oplus 2(x^2y^2 \oplus x^2z^2 \oplus y^2z^2) \oplus 0(x^2yz \oplus xy^2z \oplus xyz^2) \oplus 3(x^3 \oplus y^3 \oplus z^3) \\
& \oplus 0(x^2y \oplus x^2z \oplus xy^2 \oplus y^2z \oplus xz^2 \oplus yz^2) \oplus 2(x^2 \oplus y^2 \oplus z^2) \\
& \oplus 0(xy \oplus xz \oplus yz) \oplus 3(x \oplus y \oplus z) \oplus (-9xyz) \oplus 5.
\end{aligned}
$$

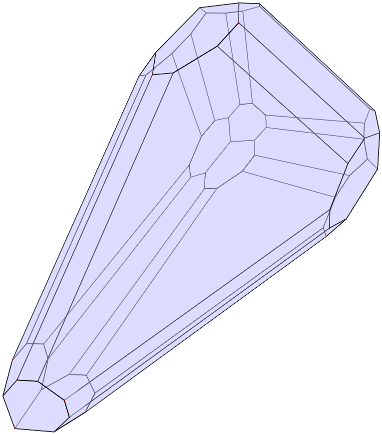
Let's look at the K3 polytope defined by $T(f)$

This is the smooth tropical quartic surface $T(f)$:



Let's look at the K3 polytope defined by $T(f)$

This is the K3 polytope:



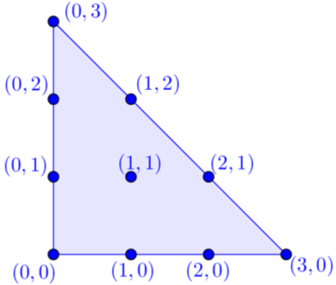
Its f-vector is $(64, 96, 34)$.

Newton polytopes

Given a tropical polynomial $f = \bigoplus_{v \in \mathbb{Z}^n} a_v x^v$, we define the Newton polytope $\mathrm{Newt}(f)$ as the polytope

$$\mathrm{Newt}(f) = \mathrm{conv}(v \ : \ a_v \neq \infty).$$

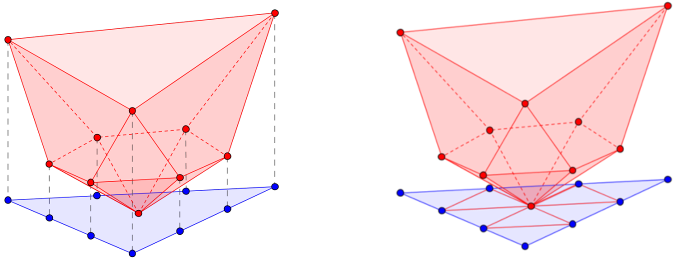
Example: $f = 3x^3 \oplus 1x^2y \oplus 1xy^2 \oplus 3y^3 \oplus 1x^2 \oplus xy \oplus 1y^2 \oplus 1x \oplus 1y \oplus 3$



Newton polytopes

We consider the convex hull in $\mathbb{R}^{n+1}$ of the points (v, a_v). The projection of the lower faces on $\mathrm{Newt}(f)$ induces a subdivision of the Newton polytope.

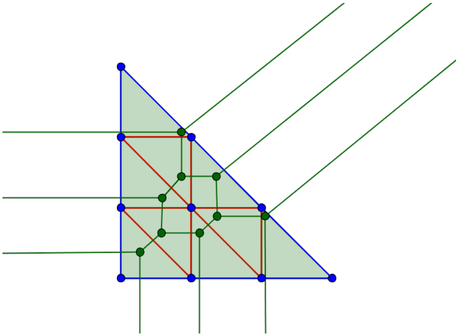
Example: $f = 3x^3 \oplus 1x^2y \oplus 1xy^2 \oplus 3y^3 \oplus 1x^2 \oplus xy \oplus 1y^2 \oplus 1x \oplus 1y \oplus 3$



Newton polytopes and tropical hypersurfaces

Tropical hypersurfaces are dual to the regular subdivision of their Newton polytopes induced by the coefficients.

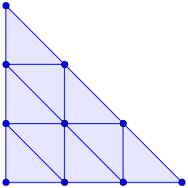
Example: $f = 3x^3 \oplus 1x^2y \oplus 1xy^2 \oplus 3y^3 \oplus 1x^2 \oplus xy \oplus 1y^2 \oplus 1x \oplus 1y \oplus 3$



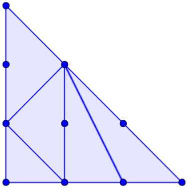
Smooth tropical hypersurfaces

A tropical hypersurface is smooth if the regular subdivision induced by its coefficients is a unimodular triangulation, i.e., cells in the subdivision are simplices of minimal volume $\frac{1}{n!}$.

Examples:



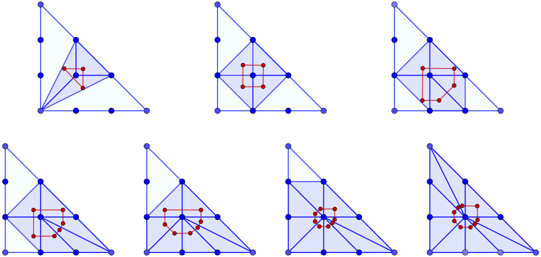
Unimodular



Not unimodular

Regions of tropical cubic curves

Exercise 13, Section 1.3 of Maclagan–Sturmfels: show that the unique bounded region of a smooth cubic curve in the plane is and m-gon with $m \in \{3,4,5,6,7,8,9\}$.



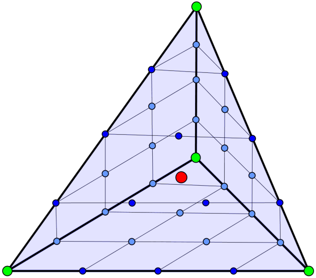
The boundary of the m-gon carries the group structure of the tropical elliptic curve (Vigeland '09) and its lattice length is the tropical j-invariant (Katz-Markwig-Markwig '08).

Regions of a tropical quartic surface

Let $4\Delta_3$ be the 4-th dilatation of the standard simplex,

$$4\Delta_3 = \mathrm{conv}\Big((0,0,0),(4,0,0),(0,4,0),(0,0,4)\Big)$$

The point $\mathbf{p} = (1,1,1)$ is the unique interior lattice point of $4\Delta_3$.



The hunt for K3 polytopes

The Newton polytope of a quartic surface is contained in $4\Delta_3$. If the Newton polytope contains p in its relative interior, then a smooth surface will determine a K3 polytopes.

We switch our attention to regular unimodular triangulations of polytopes contained in $4\Delta_3$ containing $\mathbf{p} = (1, 1, 1)$ in their relative interior. They are canonical polytopes.

The hunt for K3 polytopes

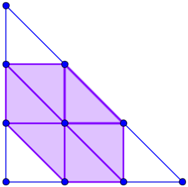
Warning: we are talking about a lot of triangulations!

- $2\Delta_3$ has 10 lattice points and 15 regular triangulations,
- $3\Delta_3$ has 20 lattice points and 21 125 102 regular triangulations.

Triangulations of $3\Delta_3$ were computed by Jordan, Joswig and Kastner with `MPTOPCOM`.

Central triangulations

We define the central part of a triangulation $\mathcal{T}$ as the subset of $\mathcal{T}$ given by the simplices of $\mathcal{T}$ containing $\mathbf{p}$. If $\mathcal{T}$ coincides with its central part, then we say that $\mathcal{T}$ is central.



The K3 polytope is uniquely determined by the central part of the triangulation $\mathcal{T}$.

It is enough to consider central triangulations!

Central triangulations of canonical polytopes

How can we construct central triangulations of a canonical polytope?

$$\{\mathcal{T} \text{ central triangulation of } P\} \leftrightarrow \{\mathcal{T} \text{ triangulation of } \partial P\}$$

We are interested in polytopes P which admit at least one unimodular central triangulation.

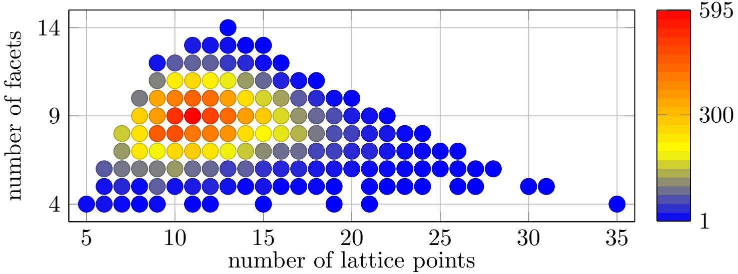
A polytope P is reflexive if $A\mathbf{p} - c = 1$, where $Ax \geq c$ are the equations defining P.
If P is reflexive,

$$\{\mathcal{T} \text{ central unimod triang of } P\} \leftrightarrow \{\mathcal{T} \text{ is a unimod triang of } \partial P\}$$

A three dimensional canonical lattice polytope P is reflexive if and only if every central fine triangulation of P is unimodular. We need to consider reflexive polytopes!

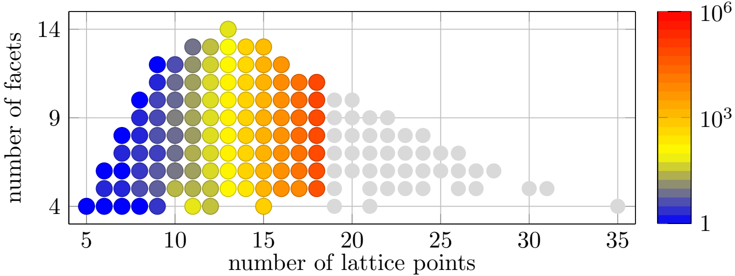
Theorem (Balletti-P-Sturmfels)

Up to symmetry there are 15139 possible reflexive polytopes contained in $4\Delta_3$.



Theorem (Balletti-P-Sturmfels)

The reflexive polytopes of volume ≤ 30 *in the theorem above admit a total of* 36 297 333 *regular unimodular central triangulations. Every K3 polytope with* ≤ 30 *vertices arises from one of these.*



f-vectors of K3 polytopes

Theorem (Balletti-P-Sturmfels)

Let $\mathcal{P}$ be a K3 polytope obtained from a polytope P. Then $\mathcal{P}$ is a simple polytope. The entries of its f-vector are

$$v = vol(P), \quad e = \frac{3vol(P)}{2}, \quad f = |P \cap \mathbb{Z}^2| - 1.$$

f-vect	# of P	f-vect	# of P	f-vect	# of P
(4, 6, 4)	9	(22, 33, 13)	1248	(40, 60, 22)	27
(6, 9, 5)	102	(24, 36, 14)	922	(42, 63, 23)	18
(8, 12, 6)	412	(26, 39, 15)	628	(44, 66, 24)	7
(10, 15, 7)	959	(28, 42, 16)	465	(46, 69, 25)	9
(12, 18, 8)	1642	(30, 45, 17)	295	(48, 72, 26)	2
(14, 21, 9)	2083	(32, 48, 18)	203	(50, 75, 27)	2
(16, 24, 10)	2194	(34, 51, 19)	128	(54, 81, 29)	1
(18, 27, 11)	1997	(36, 54, 20)	85	(56, 84, 30)	1
(20, 30, 12)	1646	(38, 57, 21)	53	(64, 96, 34)	1

Moduli space of quartic surfaces

A quartic surface is the variety in $\mathbb{P}^3$ defined by a homogeneous polynomial of degree 4 in $\mathbb{C}[x, y, z, w]$,

$$f(x, y, z, w) \quad = \sum_{i+j+k \leq 4} c_{ijk} x^i y^j z^k w^{4-i-j-k}.$$

The 35 coefficients c_{ijk} parameterize all the quartic surfaces. So we consider the 34-dimesional projective space $\mathbb{HS}_{4,3} = \mathbb{P}(\mathsf{HS}_{4,3})$ of quartic surfaces. This gives us a "moduli space of quartic surfaces".

More precisely, the special linear group SL(4) acts on $\mathbb{HS}_{4,3}$, and on the associated polynomial ring $\mathbb{C}[\mathbb{HS}_{4,3}]$, generated by c_{ijk}. The moduli space of quartic surfaces in $\mathbb{P}^3$ is the projective variety determined by $\mathrm{Proj}\big(\mathbb{C}[\mathbb{HS}_{4,3}]^{\mathsf{SL}(4)}\big)$.

Stable elements

In the context of Geometric Invariant Theory, we want to restric to a "nicer" set inside the moduli space. This is the set of stable elements.

More precisely, f is stable if the orbit $O(f)^{\mathrm{SL}(4)}$ is closed and the stabilizer $\mathrm{stab}(f)$ is finite.

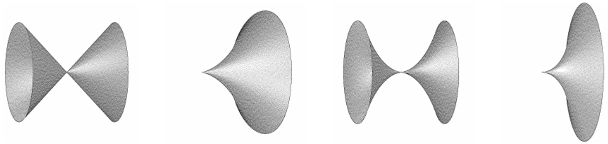
Determining whether an element is stable is connected to the study of its singular locus.

Theorem (Shah '81)

If the singular locus of a quartic surface contains at most rational double points, then the surface is stable.

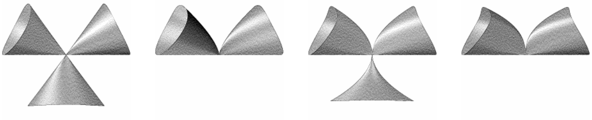
Arnold's classification '72

A_k: $x^2 + y^2 + z^{k+1}$



A_1 A_2 A_3 A_4

D_k: $x^2 + y^2 z + z^{k-1}$

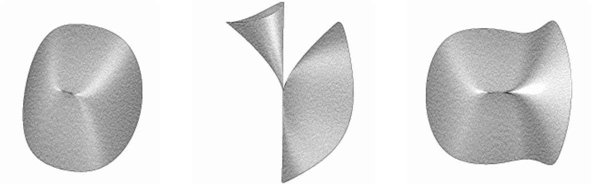


D_4 D_5 D_6 D_7

Pictures from Greuel-Lossen-Shustin "Introduction to Singularities and Deformation".

Arnold's classification '72

E_6: $x^2 + y^3 + z^4$, E_7: $x^2 + y^3 + yz^3$, and E_8: $x^2 + y^3 + z^5$



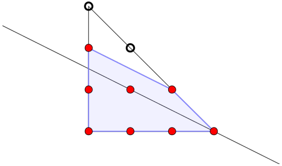
E_6 E_7 E_8

Pictures from Greuel-Lossen-Shustin "Introduction to Singularities and Deformation".

A combinatorial criterion

Theorem (Mumford '77)

A point f in $\mathbb{HS}_{4,3}$ is stable if and only if, for every choice of coordinates, and for all planes H through $\mathbf{p}$, each open halfspace of H contains a monomial of f.



A reflexive lattice polytope P contained in $4\Delta_3$ is called minimal if it does not properly contain any reflexive polytope.

There are precisely 115 minimal reflexive polytopes in $4\Delta_3$.

Stability

Theorem (Balletti-P-Sturmfels)

Let $f \in \mathbb{C}[x, y, z, w]$ *be a generic homogeneous quartic surface whose Newton polytope arises from a smooth tropical surface. Then the quartic surface* $V(f)$ *in* $\mathbb{P}^3$ *is stable.*

- We show the stability of surfaces having a minimal polytope as Newton polytope by studying their singular locus.
- We use Mumford's criterion to conclude that also generic surfaces with Newton polytope containing a minimal one are stable.

Thank you!