An Experimental Classification of Maximal Mediated Sets

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SFB-TRR 195, Tübingen

September 28, 2018

Nonnegative Polynomials

A polynomial $f \in \mathbb{R}[\boldsymbol{x}]$ is called nonnegative if $f(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$. Certifying the nonnegativity of a polynomial is crucial for polynomial optimization.

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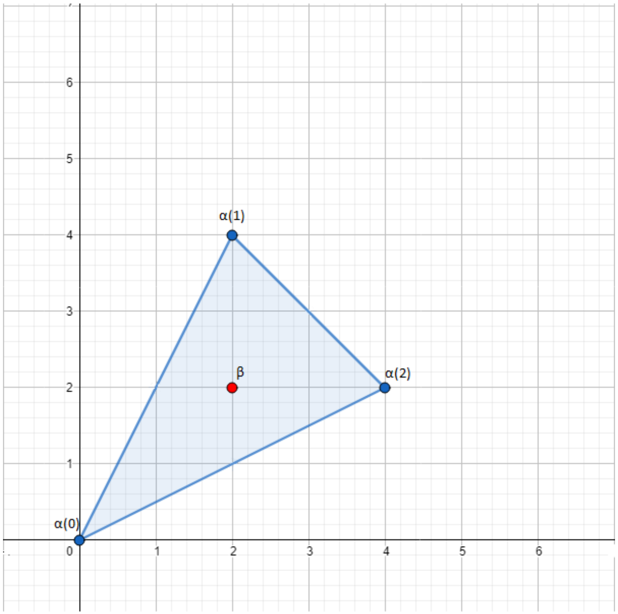
- Hilbert(1888) showed that there exists nonnegative polynomials that cannot be represented as sum of squares.
- The AM-GM inequality can be used to check the nonnegativity the circuit polynomials.

Polynomials Supported on a Circuit

Motzkin Polynomial (1967): Consider the Motzkin polynomial

$$f(x,y) = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$$



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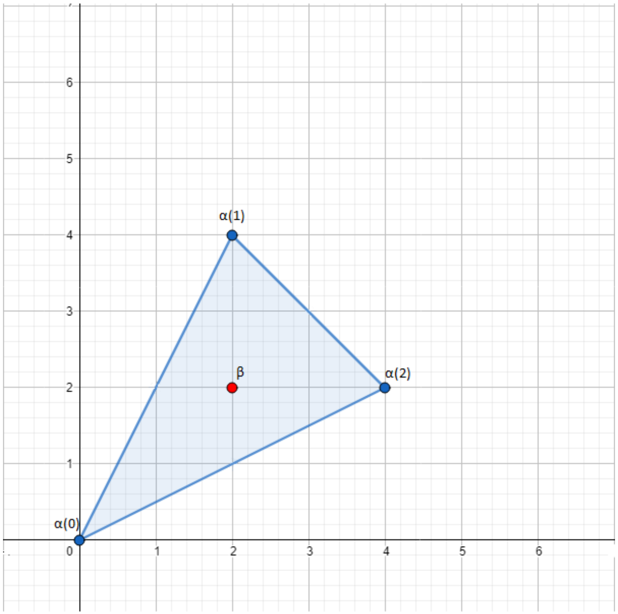
$f(x,y) \geq 0$ due to the classical AM-GM inequality.



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Theorem (Reznick (1989), de Wolff, Iliman (2014))

A nonnegative circuit polynomial f is SOS if and only if "inner term" is in MMS.

MMS Algorithm

Given a set of points $L \subset \mathbb{Z}^n$, we define a set of averages:

$$\overline{A}(L) = \left\{ \frac{\boldsymbol{s}+\boldsymbol{t}}{2} \,|\, \boldsymbol{s},\boldsymbol{t} \in L \cap (2\mathbb{Z})^n, \boldsymbol{s} \neq \boldsymbol{t} \right\}$$

Reznick's MMS algorithm(1989):

Input: Δ: finite set of points in $(2\mathbb{Z})^n$
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```
Δ^0 ← Conv(Δ) ∩ Z^n
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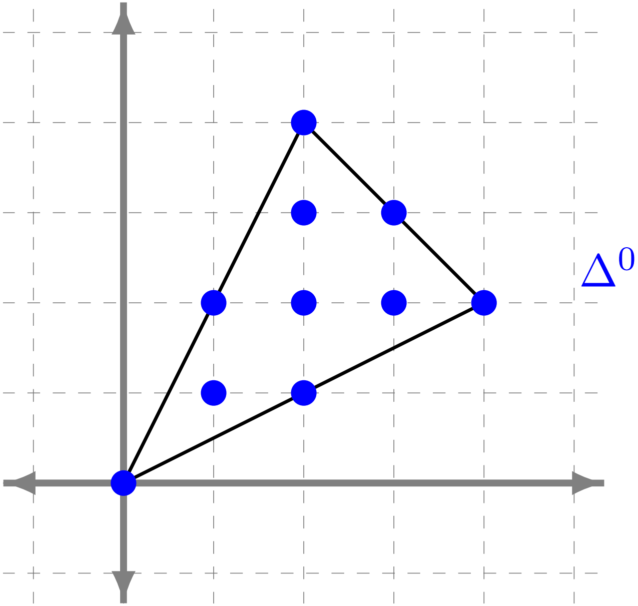

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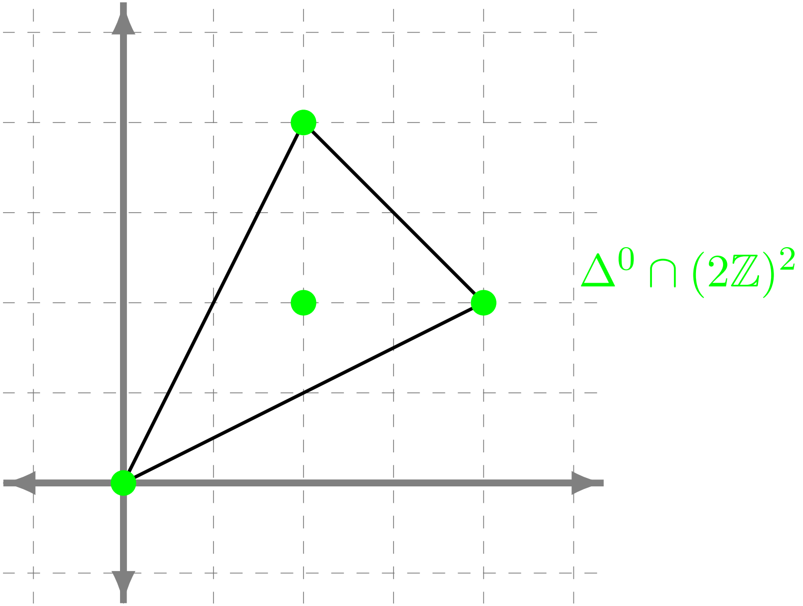

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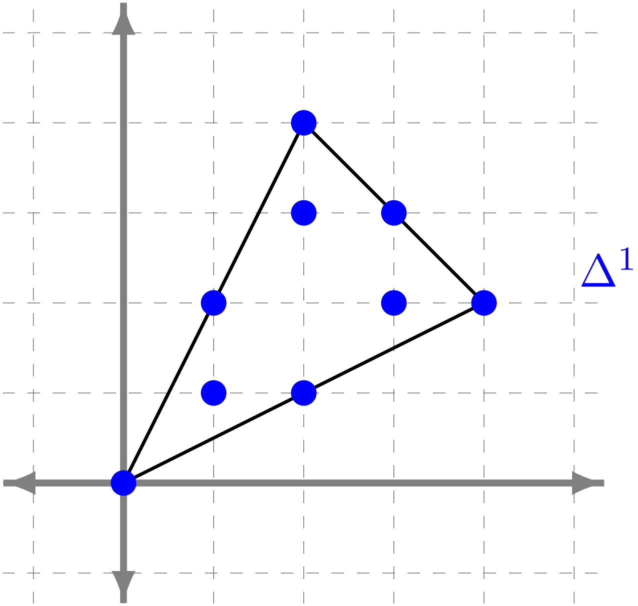

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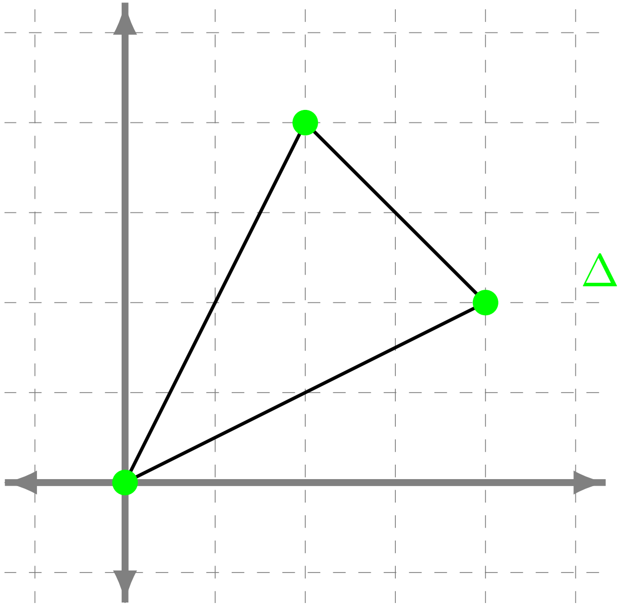

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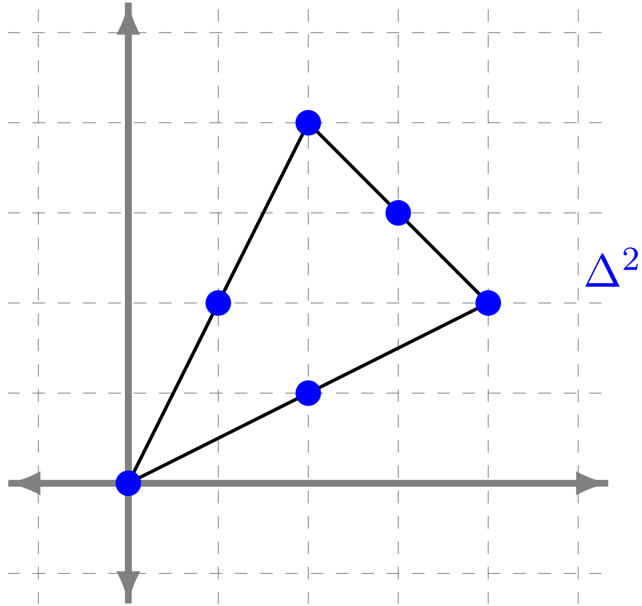

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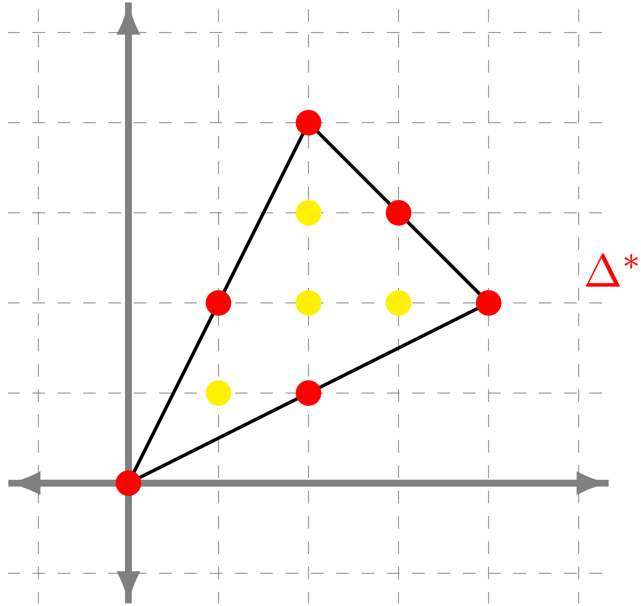

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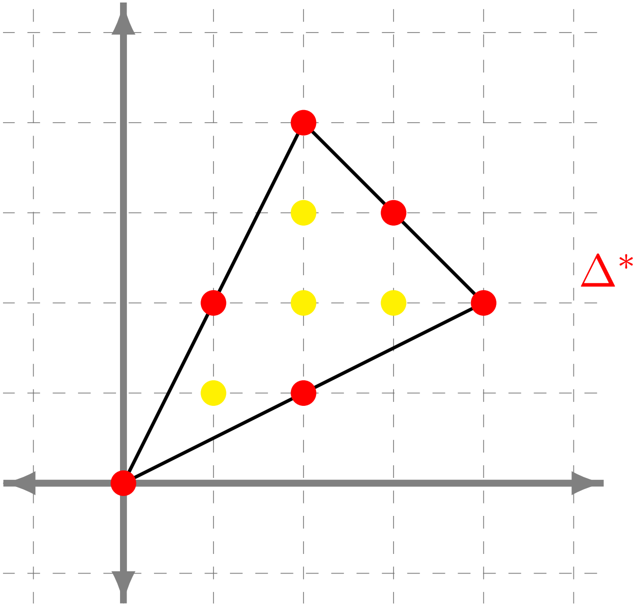

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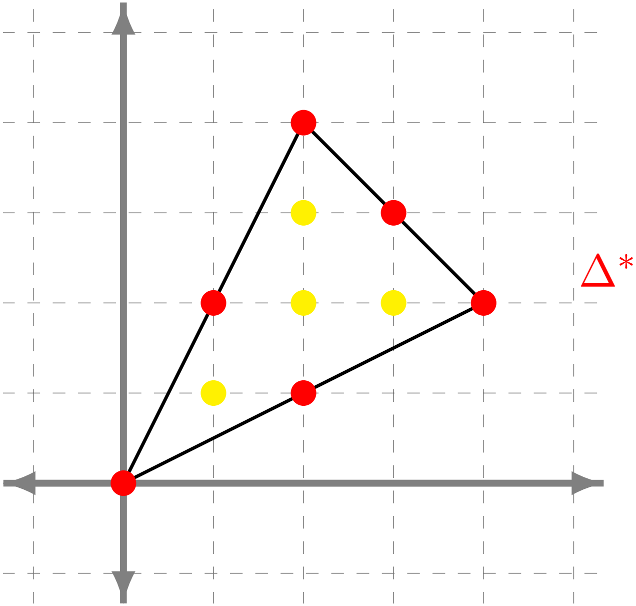

Fact:

- $\Delta \cup \overline{A}(\Delta) \subseteq \Delta^* \subseteq \text{conv}(\Delta) \cap \mathbb{Z}^n$

Task: Decide how dense Δ^* is in $\text{conv}(\Delta) \cap \mathbb{Z}^n$, so we define the h-ratio:

$$\mathcal{H}(\Delta) = \frac{|\Delta^* - (\Delta \cup \overline{A}(\Delta))|}{|(\text{conv}(\Delta) \cap \mathbb{Z}^n) - (\Delta \cup \overline{A}(\Delta))|}$$

Main Question: What is the distribution on $\mathcal{H}$ when $n > 2$?

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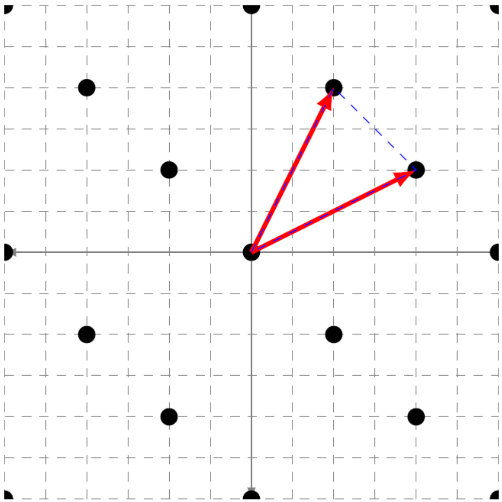
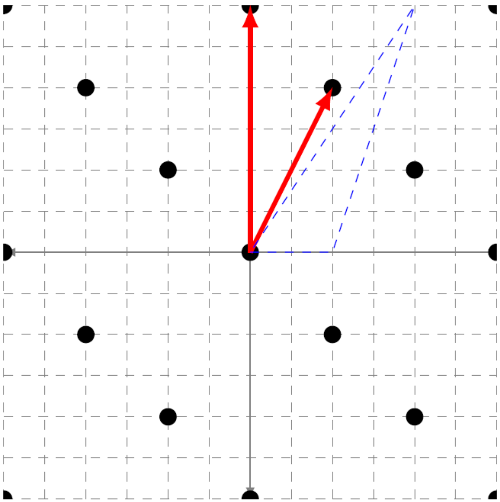
Observations:

- Even though the algoritm looks easy, there are too many simplices to consider. In dimension 4, maximal degree 8 there are more than 300000 simplices to check. This makes it hard to write to database.
- Thus, we need to get rid of the redundant data. In fact, instead of simplicies one can consider the underlying lattice.





$$\Delta_1 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\}$$

$$M_{\Delta_1} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$$

$$L_{\Delta_1} = \langle (2,4), (4,2) \rangle$$

$$\Delta_2 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix} \right\}$$

$$M_{\Delta_2} = \begin{bmatrix} 2 & 4 \\ 0 & 6 \end{bmatrix}$$

$$L_{\Delta_2} = \langle (2,4), (0,6) \rangle$$

$$T(\boldsymbol{x}) = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \boldsymbol{x}$$

Thank you for your attention!