AN ALGORITHM FOR LIFTING POINTS IN A TROPICAL VARIETY

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Abstract. The aim of this paper is to give a constructive proof of one of the basic theorems of tropical geometry: given a point on a tropical variety (defined using initial ideals), there exists a Puiseux-valued "lift" of this point in the algebraic variety. This theorem is so fundamental because it justifies why a tropical variety (defined combinatorially using initial ideals) carries information about algebraic varieties: it is the image of an algebraic variety over the Puiseux series under the valuation map. We have implemented the "lifting algorithm" using Singular and `Gfan` if the base field is $\mathbb{Q}$. As a byproduct we get an algorithm to compute the Puiseux expansion of a space curve singularity in $(K^{n+1}, 0)$.

1. Introduction

In tropical geometry, algebraic varieties are replaced by certain piecewise linear objects called tropical varieties. Many algebraic geometry theorems have been "translated" to the tropical world (see for example [Mik05], [Vig04], [SS04a], [GM07] and many more). Because new methods can be used in the tropical world — for example, combinatorial methods — and because the objects seem easier to deal with due to their piecewise linearity, tropical geometry is a promising tool for deriving new results in algebraic geometry. (For example, the Welschinger invariant can be computed tropically, see [Mik05]).

There are two ways to define the tropical variety $\operatorname{Trop}(J)$ for an ideal J in the polynomial ring $K\{\{t\}\}[x_1, \ldots, x_n]$ over the field of Puiseux series (see Definition 2.1). One way is to define the tropical variety combinatorially using t-initial ideals (see Definition 2.4 and Definition 2.10, resp. [SS04a]) — this definition is more helpful when computing and it is the definition we use in this paper. The other way to define tropical varieties is as the closure of the image of the algebraic variety $V(J)$ of J in $K\{\{t\}\}^n$ under the negative of the valuation map (see Remark 2.2, resp. [RGST03], Definition 2.1) — this gives more insight why tropical varieties carry information about algebraic varieties.

It is our main aim in this paper to give a constructive proof that these two concepts coincide (see Theorem 2.13), and to derive that way an algorithm which allows to lift a given point $\omega \in \operatorname{Trop}(J)$ to a point in $V(J)$ up to given order (see Algorithms

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3.8 and 4.8). The algorithm has been implemented using the commutative algebra system SINGULAR (see [GPS05]) and the programme `Gfan` (see [Jen]), which computes Gröbner fans and tropical varieties.

Theorem 2.13 has been proved in the case of a principal ideal by [EKL04], Theorem 2.1.1. There is also a constructive proof for a principal ideal in [Tab06], Theorem 2.4. For the general case, there is a proof in [SS04b], Theorem 2.1, which has a gap however. Furthermore, there is a proof in [Dra06], Theorem 4.2, using affinoid algebras, and in [Kat06], Lemma 5.2.2, using flat schemes. A more general statement is proved in [Pay07], Theorem 4.2. Our proof has the advantage that it is constructive and works for an arbitrary ideal J.

We describe our algorithm first in the case where the ideal is 0-dimensional. This algorithm can be viewed as a variant of an algorithm presented by Joseph Maurer in [Mau80], a paper from 1980. In fact, he uses the term "critical tropism" for a point in the tropical variety, even though tropical varieties were not defined by that time. Apparently, the notion goes back to Monique Lejeune-Jalabert and Bernard Teissier[1] (see [LJT73]).

This paper is organised as follows: In Section 2 we recall basic definitions and state the main result. In Section 3 we give a constructive proof of the main result in the 0-dimensional case and deduce an algorithm. In Section 4 we reduce the arbitrary case algorithmically to the 0-dimensional case, and in Section 5 we gather some simple results from commutative algebra for the lack of a better reference. The proofs of both cases heavily rely on a good understanding of the relation of the dimension of an ideal J over the Puiseux series with its t-initial ideal, respectively with its restriction to the rings $R_N[\underline{x}]$ introduced below (see Definition 2.1). This will be studied in Section 6. Some of the theoretical as well as the computational results use Theorem 2.8 which was proved in [Mar07] using standard bases in the mixed power series polynomial ring $K[[t]][\underline{x}]$. We give an alternative proof in Section 7.

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Our programme can be downloaded from the web page

www.mathematik.uni-kl.de/~keilen/en/tropical.html.

[1]Asked about this coincidence in the two notions Bernard Teissier sent us the following kind and interesting explanation: *As far as I know the term did not exist before. We tried to convey the idea that giving different weights to some variables made the space "anisotropic", and we were intrigued by the structure, for example, of anisotropic projective spaces (which are nowadays called weighted projective spaces). From there to "tropismes critiques" was a quite natural linguistic movement. Of course there was no "tropical" idea around, but as you say, it is an amusing coincidence. The Greek "Tropos" usually designates change, so that "tropisme critique" is well adapted to denote the values where the change of weights becomes critical for the computation of the initial ideal. The term "Isotropic", apparently due to Cauchy, refers to the property of presenting the same (physical) characters in all directions. Anisotropic is, of course, its negation. The name of Tropical geometry originates, as you probably know, from tropical algebra which honours the Brazilian computer scientist Imre Simon living close to the tropics, where the course of the sun changes back to the equator. In a way the tropics of Capricorn and Cancer represent, for the sun, critical tropisms.*

2. Basic Notations and the Main Theorem

In this section we will introduce the basic notations used throughout the paper.

Definition 2.1
Let K be an arbitrary field. We consider for $N \in \mathbb{N}_{>0}$ the discrete valuation ring

$$R_N = K\big[\big[t^{\frac{1}{N}}\big]\big] = \left\{ \sum_{\alpha=0}^{\infty} a_\alpha \cdot t^{\frac{\alpha}{N}} \;\middle|\; a_\alpha \in K \right\}$$

of formal power series in the unknown $t^{\frac{1}{N}}$ with *discrete valuation*

$$\mathrm{val}\left(\sum_{\alpha=0}^{\infty} a_\alpha \cdot t^{\frac{\alpha}{N}} \right) = \mathrm{ord}_t \left(\sum_{\alpha=0}^{\infty} a_\alpha \cdot t^{\frac{\alpha}{N}} \right) = \min\left\{ \frac{\alpha}{N} \;\middle|\; a_\alpha \neq 0 \right\} \in \frac{1}{N} \cdot \mathbb{Z} \cup \{\infty\},$$

and we denote by $L_N = \mathrm{Quot}(R_N)$ its quotient field. If $N \mid M$ then in an obvious way we can think of R_N as a subring of R_M, and thus of L_N as a subfield of L_M. We call the direct limit of the corresponding direct system

$$L = K\{\{t\}\} = \varinjlim L_N = \bigcup_{N>0} L_N$$

the *field of (formal) Puiseux series* over K.

Recall that if K is algebraically closed of characteristic 0, then L is algebraically closed.

Remark 2.2
If $0 \neq N \in \mathbb{N}$ then $S_N = \left\{1, t^{\frac{1}{N}}, t^{\frac{2}{N}}, t^{\frac{3}{N}}, \ldots \right\}$ is a multiplicatively closed subset of R_N, and obviously

$$L_N = S_N^{-1} R_N = \left\{ t^{\frac{-\alpha}{N}} \cdot f \;\middle|\; f \in R_N, \alpha \in \mathbb{N} \right\}.$$

The valuation of R_N extends to L_N, and thus L, by $\mathrm{val}\left(\frac{f}{g}\right) = \mathrm{val}(f) - \mathrm{val}(g)$ for $f, g \in R_N$ with $g \neq 0$. In particular, $\mathrm{val}(0) = \infty$.

Notation 2.3
Since an ideal $J \trianglelefteq L[\underline{x}]$ is generated by finitely many elements, the set

$$\mathcal{N}(J) = \left\{ N \in \mathbb{N}_{>0} \;\middle|\; \langle J \cap R_N[\underline{x}] \rangle_{L[\underline{x}]} = J \right\}$$

is non-empty, and if $N \in \mathcal{N}(J)$ then $N \cdot \mathbb{N}_{>0} \subseteq \mathcal{N}(J)$. We also introduce the notation $J_{R_N} = J \cap R_N[\underline{x}]$.

Remark and Definition 2.4
Let $N \in \mathbb{N}_{>0}$, $w = (w_0, \ldots, w_n) \in \mathbb{R}_{<0} \times \mathbb{R}^n$, and $q \in \mathbb{R}$.
We may consider the direct product

$$V_{q,w,N} = \prod_{\substack{(\alpha,\beta) \in \mathbb{N}^{n+1} \\ w \cdot (\frac{\alpha}{N}, \beta) = q}} K \cdot t^{\frac{\alpha}{N}} \cdot \underline{x}^\beta$$

of K-vector spaces and its subspace

$$W_{q,w,N} = \bigoplus_{\substack{(\alpha,\beta) \in \mathbb{N}^{n+1} \\ w \cdot (\frac{\alpha}{N}, \beta) = q}} K \cdot t^{\frac{\alpha}{N}} \cdot \underline{x}^\beta.$$

As a K-vector space the formal power series ring $K\big[\big[t^{\frac{1}{N}},\underline{x}\big]\big]$ is just

$$K\big[\big[t^{\frac{1}{N}},\underline{x}\big]\big]=\prod_{q\in\mathbb{R}}V_{q,w,N},$$

and we can thus write any power series $f\in K\big[\big[t^{\frac{1}{N}},\underline{x}\big]\big]$ in a unique way as

$$f=\sum_{q\in\mathbb{R}}f_{q,w}\quad\text{ with }\quad f_{q,w}\in V_{q,w,N}.$$

Note that this representation is independent of N in the sense that if $f\in K\big[\big[t^{\frac{1}{N'}},\underline{x}\big]\big]$ for some other $N'\in\mathbb{N}_{>0}$ then we get the same non-vanishing $f_{q,w}$ if we decompose f with respect to N'.
Moreover, if $0\neq f\in R_N[\underline{x}]\subset K\big[\big[t^{\frac{1}{N}},\underline{x}\big]\big]$, then there is a *maximal* $\hat{q}\in\mathbb{R}$ such that $f_{\hat{q},w}\neq 0$ and $f_{q,w}\in W_{q,w,N}$ for all $q\in\mathbb{R}$, since the $\underline{x}$-degree of the monomials involved in f is bounded. We call the elements $f_{q,w}$ *w-quasihomogeneous* of w-degree $\deg_w(f_{q,w})=q\in\mathbb{R}$,

$$\operatorname{in}_w(f):=f_{\hat{q},w}\in K\big[t^{\frac{1}{N}},\underline{x}\big]$$

the *w-initial form* of f, and

$$\operatorname{ord}_w(f):=\hat{q}=\max\{\deg_w(f_{q,w})\mid f_{q,w}\neq 0\}$$

the *w-order* of f. Set $\in_\omega(0)=0$. If $t^\beta x^\alpha\neq t^{\beta'}x^{\alpha'}$ are both monomials of $\operatorname{in}_w(f)$, then $\alpha\neq\alpha'$.
For $I\subseteq R_N[\underline{x}]$ we call

$$\operatorname{in}_w(I)=\big\langle \operatorname{in}_w(f)\;\big|\; f\in I\big\rangle\trianglelefteq K\big[t^{\frac{1}{N}},\underline{x}\big]$$

the *w-initial ideal* of I. Note that its definition depends on N.
Moreover, we call for $f\in R_N[\underline{x}]$

$$\text{t-in}_w(f)=\operatorname{in}_w(f)(1,\underline{x})=\operatorname{in}_w(f)_{|t=1}\in K[\underline{x}]$$

the *t-initial form of f w.r.t. w*, and if $f=t^{\frac{-\alpha}{N}}\cdot g\in L[\underline{x}]$ with $g\in R_N[\underline{x}]$ we set

$$\text{t-in}_w(f):=\text{t-in}_w(g).$$

This definition does not depend on the particular representation of f.
If $J\subseteq L[\underline{x}]$ is a subset of $L[\underline{x}]$, then

$$\text{t-in}_w(J)=\langle\text{t-in}_w(f)\mid f\in J\rangle\trianglelefteq K[\underline{x}]$$

is the *t-initial ideal* of J, which does not depend on any N.
For two w-quasihomogeneous elements $f_{q,w}\in W_{q,w,N}$ and $f_{q',w}\in W_{q',w,N}$ we have $f_{q,w}\cdot f_{q',w}\in W_{q+q',w,N}$. In particular, $\operatorname{in}_w(f\cdot g)=\operatorname{in}_w(f)\cdot\operatorname{in}_w(g)$ for $f,g\in R_N[\underline{x}]$, and $\text{t-in}_w(f\cdot g)=\text{t-in}_w(f)\cdot\text{t-in}_w(g)$ for $f,g\in L[\underline{x}]$.

Example 2.5
Let $w=(-1,-2,-1)$ and

$$f=\big(2t+t^{\frac{3}{2}}+t^2\big)\cdot x^2+(-3t^3+2t^4)\cdot y^2+t^5xy^2+\big(t+3t^2\big)\cdot x^7y^2.$$

Then $\operatorname{ord}_w(f)=-5$, $\operatorname{in}_w(f)=2tx^2-3t^3y^2$, and $\text{t-in}_w(f)=2x^2-3y^2$.

Notation 2.6
Throughout this paper we will mostly use the weight -1 for the variable t, and in order to simplify the notation we will then usually write for $\omega \in \mathbb{R}^n$

$$\mathrm{in}_\omega \quad \text{instead of} \quad \mathrm{in}_{(-1,\omega)}$$

and

$$\text{t-in}_\omega \quad \text{instead of} \quad \text{t-in}_{(-1,\omega)}\,.$$

The case that $\omega = (0,\ldots,0)$ is of particular interest, and we will simply write

$$\mathrm{in}_0 \quad \text{respectively} \quad \text{t-in}_0\,.$$

This should not lead to any ambiguity.

In general, the t-initial ideal of an ideal J is not generated by the t-initial forms of the given generators of J.

Example 2.7
Let $J = \langle tx+y, x+t\rangle \lhd L[x,y]$ and $\omega = (1,-1)$. Then $y - t^2 \in J$, but

$$y = \text{t-in}_\omega(y-t^2) \notin \langle \text{t-in}_\omega(tx+y), \text{t-in}_\omega(x+t)\rangle = \langle x\rangle.$$

We can compute the t-initial ideal using standard bases by [Mar07], Corollary 6.11.

Theorem 2.8
Let $J = \langle I\rangle_{L[\underline{x}]}$ with $I \unlhd K\big[t^{\frac{1}{N}}, \underline{x}\big]$, $\omega \in \mathbb{Q}^n$ and G be a standard basis of I with respect to $>_\omega$ (see Remark 3.7 for the definition of $>_\omega$).
Then $\text{t-in}_\omega(J) = \text{t-in}_\omega(I) = \big\langle\, \text{t-in}_\omega(G)\big\rangle \unlhd K[\underline{x}]$.

The proof of this theorem uses standard basis techniques in the ring $K[[t]][\underline{x}]$. We give an alternative proof in Section 7.

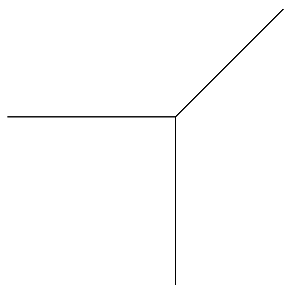
Example 2.9
In Example 2.7, $G = (tx+y, x+t, y-t^2)$ is a suitable standard basis and thus $\text{t-in}_\omega(J) = \langle x, y\rangle$.

Definition 2.10
Let $J \unlhd L[\underline{x}]$ be an ideal then the *tropical variety* of J is defined as

$$\mathrm{Trop}(J) = \{\omega \in \mathbb{R}^n \mid \text{t-in}_\omega(J) \text{ is monomial free}\}.$$

It is possible that $\mathrm{Trop}(J) = \emptyset$.

Example 2.11
Let $J = \langle x+y+1\rangle \subset L[x,y]$. As J is generated by one polynomial f which then automatically is a standard basis, the t-initial ideal $\text{t-in}_\omega(J)$ will be generated by $\text{t-in}_\omega(f)$ for any ω. Hence $\text{t-in}_\omega(J)$ contains no monomial if and only if $\text{t-in}_\omega(f)$ is not a monomial. This is the case for all ω such that $\omega_1 = \omega_2 \geq 0$, or $\omega_1 = 0 \geq \omega_2$, or $\omega_2 = 0 \geq \omega_1$. Hence the tropical variety $\mathrm{Trop}(J)$ looks as follows:



We need the following basic results about tropical varieties.

Lemma 2.12
Let $J, J_1, \ldots, J_k \trianglelefteq L[\underline{x}]$ be ideals. Then:

(a) $J_1 \subseteq J_2 \implies \mathrm{Trop}(J_1) \supseteq \mathrm{Trop}(J_2)$,
(b) $\mathrm{Trop}(J_1 \cap \ldots \cap J_k) = \mathrm{Trop}(J_1) \cup \ldots \cup \mathrm{Trop}(J_k)$,
(c) $\mathrm{Trop}(J) = \mathrm{Trop}\left(\sqrt{J}\right) = \bigcup_{P \in \mathrm{minAss}(J)} \mathrm{Trop}(P)$, *and*
(d) $\mathrm{Trop}(J_1 + J_2) \subseteq \mathrm{Trop}(J_1) \cap \mathrm{Trop}(J_2)$.

Proof: Suppose that $J_1 \subseteq J_2$ and $\omega \in \mathrm{Trop}(J_2) \setminus \mathrm{Trop}(J_1)$. Then $\text{t-in}_\omega(J_1)$ contains a monomial, but since $\text{t-in}_\omega(J_1) \subseteq \text{t-in}_\omega(J_2)$ this contradicts $\omega \in \mathrm{Trop}(J_2)$. Thus $\mathrm{Trop}(J_2) \subseteq \mathrm{Trop}(J_1)$. This shows (a).
Since $J_1 \cap \ldots \cap J_k \subseteq J_i$ for each $i = 1, \ldots, k$ the first assertion implies that

$$\mathrm{Trop}(J_1) \cup \ldots \cup \mathrm{Trop}(J_k) \subseteq \mathrm{Trop}(J_1 \cap \ldots \cap J_k).$$

Conversely, if $\omega \notin \mathrm{Trop}(J_i)$ for $i = 1, \ldots, k$ then there exist polynomials $f_i \in J_i$ such that $\text{t-in}_\omega(f_i)$ is a monomial. But then $\text{t-in}_\omega(f_1 \cdots f_k) = \text{t-in}_\omega(f_1) \cdots \text{t-in}_\omega(f_k)$ is a monomial and $f_1 \cdots f_k \in J_1 \cdots J_k \subseteq J_1 \cap \ldots \cap J_k$. Thus $\omega \notin \mathrm{Trop}(J_1 \cap \ldots \cap J_k)$, which shows (b).
For (c) it suffices to show that $\mathrm{Trop}(J) \subseteq \mathrm{Trop}\left(\sqrt{J}\right)$, since $J \subseteq \sqrt{J} = \bigcap_{P \in \mathrm{minAss}(J)} P$. If $\omega \notin \mathrm{Trop}\left(\sqrt{J}\right)$ then there is an $f \in \sqrt{J}$ such that $\text{t-in}_\omega(f)$ is a monomial and such that $f^m \in J$ for some m. But then $\text{t-in}_\omega(f^m) = \text{t-in}_\omega(f)^m$ is a monomial and thus $\omega \notin \mathrm{Trop}(J)$.
Finally (d) is obvious from the definition. □

We are now able to state our main theorem.

Theorem 2.13
If K is algebraically closed of characteristic zero and $J \trianglelefteq K\{\{t\}\}[\underline{x}]$ is an ideal then

$$\omega \in \mathrm{Trop}(J) \cap \mathbb{Q}^n \iff \exists\, p \in V(J) : -\mathrm{val}(p) = \omega \in \mathbb{Q}^n,$$

where val *is the coordinate-wise valuation.*

The proof of one direction is straight forward and it does not require that K is algebraically closed.

Proposition 2.14
If $J \trianglelefteq L[\underline{x}]$ is an ideal and $p \in V(J) \cap (L^*)^n$, then $-\mathrm{val}(p) \in \mathrm{Trop}(J)$.

Proof: Let $p = (p_1, \ldots, p_n)$, and let $\omega = -\mathrm{val}(p) \in \mathbb{Q}^n$. If $f \in J$, we have to show that $\text{t-in}_\omega(f)$ is not a monomial, but since this property is preserved when multiplying with some $t^{\frac{\alpha}{N}}$ we may as well assume that $f \in J_{R_N}$. As $p \in V(J)$, we know that $f(p) = 0$. In particular the terms of lowest t-order in $f(p)$ have to cancel. But the terms of lowest order in $f(p)$ are $\mathrm{in}_\omega(f)(a_1 \cdot t^{-\omega_1}, \ldots, a_n \cdot t^{-\omega_n})$, where $p_i = a_i \cdot t^{-\omega_i} + h.o.t.$. Hence $\mathrm{in}_\omega(f)(a_1 t^{-\omega_1}, \ldots, a_n t^{-\omega_n}) = 0$, which is only possible if $\mathrm{in}_\omega(f)$, and thus $\text{t-in}_\omega(f)$, is not a monomial. □

Essentially, this was shown by Newton in [New70].

Remark 2.15
If the base field K in Theorem 2.13 is not algebraically closed or not of characteristic zero, then the Puiseux series field is not algebraically closed (see e.g. [Ked01]). We therefore cannot expect to be able to lift each point in the tropical variety of an ideal $J \lhd K\{\{t\}\}[\underline{x}]$ to a point in $V(J) \subseteq K\{\{t\}\}^n$. However, if we replace $V(J)$

by the vanishing set, say W, of J over the algebraic closure $\overline{L}$ of $K\{\{t\}\}$ then it is still true that each point ω in the tropical variety of J can be lifted to a point $p \in W$ such that $\operatorname{val}(p) = -\omega$. For this we note first that if $\dim(J) = 0$ then the non-constructive proof of Theorem 3.1 works by passing from J to $\langle J \rangle_{\overline{L}[\underline{x}]}$, taking into account that the non-archimedian valuation of a field in a natural way extends to its algebraic closure. And if $\dim(J) > 0$ then we can add generators to J by Proposition 4.6 and Remark 4.5 so as to reduce to the zero dimensional case before passing to the algebraic closure of $K\{\{t\}\}$.
Note, it is even possible to apply Algorithm 3.8 in the case of positive characteristic. However, due to the weird nature of the algebraic closure of the Puiseux series field in that case we cannot guarantee that the result will coincide with a solution of J up to the order up to which it is computed. It may very well be the case that some intermediate terms are missing (see [Ked01] Section 5).

3. Zero-Dimensional Lifting Lemma

In this section we want to give a constructive proof of the Lifting Lemma 3.1.

Theorem 3.1 (Lifting Lemma)
Let K be an algebraically closed field of characteristic zero and $L = K\{\{t\}\}$. If $J \lhd L[\underline{x}]$ is a zero dimensional ideal and $\omega \in \operatorname{Trop}(J) \cap \mathbb{Q}^n$, then there is a point $p \in V(J)$ such that $-\operatorname{val}(p) = \omega$.

Non-Constructive Proof: If $\omega \in \operatorname{Trop}(J)$ then by Lemma 2.12 there is an associated prime $P \in \operatorname{minAss}(J)$ such that $\omega \in \operatorname{Trop}(P)$. But since $\dim(J) = 0$ the ideal P is necessarily a maximal ideal, and since L is algebraically closed it is of the form

$$P = \langle x_1 - p_1, \ldots, x_n - p_n \rangle$$

with $p_1, \ldots, p_n \in L$. Since $\omega \in \operatorname{Trop}(P)$ the ideal $\text{t-in}_\omega(P)$ does not contain any monomial, and therefore necessarily $\operatorname{ord}_t(p_i) = -\omega_i$ for all $i = 1, \ldots, n$. This shows that $p = (p_1, \ldots, p_n) \in V(P) \subseteq V(J)$ and $\operatorname{val}(p) = -\omega$. $\square$

The drawback of this proof is that in order to find p one would have to be able to find the associated primes of J which would amount to something close to primary decomposition over L. This is of course not feasible. We will instead adapt the constructive proof that L is algebraically closed, i.e. the Newton-Puiseux Algorithm for plane curves, which has already been generalised to space curves (see [Mau80], [AMNR92]) to our situation in order to compute the point p up to any given order. The idea behind this is very simple and the first recursion step was basically already explained in the proof of Proposition 2.14. Suppose we have a polynomial $f \in R_N[\underline{x}]$ and a point

$$p = \big(u_1 \cdot t^{\alpha_1} + v_1 \cdot t^{\beta_1} + \ldots, \ldots, u_n \cdot t^{\alpha_n} + v_n \cdot t^{\beta_n} + \ldots\big) \in (L^*)^n.$$

Then, a priori, the term of lowest t-order in $f(p)$ will be $\operatorname{in}_{-\alpha}(f)(u_1 \cdot t^{\alpha_1}, \ldots, u_n \cdot t^{\alpha_n})$. Thus, in order for $f(p)$ to be zero it is necessary that $\text{t-in}_{-\alpha}(f)(u_1, \ldots, u_n) = 0$. Let p' denote the tail of p, that is $p_i = u_i \cdot t^{\alpha_i} + t^{\alpha_i} \cdot p'_i$. Then p' is a zero of

$$f' = f\big(t^{\alpha_1} \cdot (u_1 + x_1), \ldots, t^{\alpha_n} \cdot (u_n + x_n)\big).$$

The same arguments then show that $\text{t-in}_{\alpha-\beta}(f')(v_1, \ldots, v_n) = 0$, and assuming now that none of the v_i is zero we find $\text{t-in}_{\alpha-\beta}(f')$ must be monomial free, that is $\alpha - \beta$ is a point in the tropical variety and all its components are strictly negative.

The basic idea for the algorithm which computes a suitable p is thus straight forward. Given $\omega = -\alpha$ in the tropical variety of an ideal J, compute a point $u \in V(\text{t-in}_\omega(J))$ apply the above transformation to J and compute a negative-valued point in the tropical variety of the transformed ideal. Then go on recursively. It may happen that the solution that we are about to construct this way has some component with only finitely many terms. Then after a finite number of steps there might be no suitable ω in the tropical variety. However, in that situation we can simply eliminate the corresponding variable for the further computations.

Example 3.2
Consider the ideal $J = \langle f_1, \ldots, f_4\rangle \lhd L[x, y]$ with

$$\begin{aligned} f_1 = &\quad y^2 + 4t^2y + (-t^3 + 2t^4 - t^5),\\ f_2 = &\quad (1 + t) \cdot x - y + (-t - 3t^2),\\ f_3 = &\quad xy + (-t + t^2) \cdot x + (t^2 - t^4),\\ f_4 = &\quad x^2 - 2tx + (t^2 - t^3).\end{aligned}$$

The t-initial ideal of J with respect to $\omega = \big(-1, -\frac{3}{2}\big)$ is

$$\text{t-in}_\omega(J) = \langle y^2 - 1, x - 1\rangle,$$

so that $\omega \in \text{Trop}(J)$ and $u = (1, 1)$ is a suitable choice. Applying the transformation $\gamma_{\omega,u} : (x, y) \mapsto \big(t \cdot (1 + x), t^{\frac{3}{2}} \cdot (1 + y)\big)$ to J we get $J' = \langle f_1', \ldots, f_4'\rangle$ with

$$\begin{aligned} f_1' &= t^3y^2 + \big(2t^3 + 4t^{\frac{7}{2}}\big) \cdot y + \big(4t^{\frac{7}{2}} + 2t^4 - t^5\big),\\ f_2' &= (t + t^2) \cdot x - t^{\frac{3}{2}} \cdot y + \big(-t^{\frac{3}{2}} - 2t^2\big),\\ f_3' &= t^{\frac{5}{2}} \cdot xy + \big(-t^2 + t^3 + t^{\frac{5}{2}}\big) \cdot x + t^{\frac{5}{2}} \cdot y + \big(t^{\frac{5}{2}} + t^3 - t^4\big),\\ f_4' &= t^2x^2 - t^3.\end{aligned}$$

This shows that the x-coordinate of a solution of J' necessarily is $x = \pm t^{\frac{1}{2}}$, and we could substitute this for x in the other equations in order to reduce by one variable. We will instead see what happens when we go on with our algorithm.
The t-initial ideal of J' with respect to $\omega' = \big(-\frac{1}{2}, -\frac{1}{2}\big)$ is

$$\text{t-in}_{\omega'}(J') = \langle y + 2, x - 1\rangle,$$

so that $\omega' \in \text{Trop}(J')$ and $u' = (1, -2)$ is our only choice. Applying the transformation $\gamma_{\omega',u'} : (x, y) \mapsto \big(t^{\frac{1}{2}} \cdot (1 + x), t^{\frac{1}{2}} \cdot (-2 + y)\big)$ to J' we get the ideal $J'' = \langle f_1'', \ldots, f_4''\rangle$ with

$$\begin{aligned} f_1'' &= t^4y^2 + 2t^{\frac{7}{2}}y + \big(-2t^4 - t^5\big),\\ f_2'' &= \big(t^{\frac{3}{2}} + t^{\frac{5}{2}}\big) \cdot x - t^2 \cdot y + t^{\frac{5}{2}},\\ f_3'' &= t^{\frac{7}{2}} \cdot xy + \big(-t^{\frac{5}{2}} + t^3 - t^{\frac{7}{2}}\big) \cdot x + \big(t^3 + t^{\frac{7}{2}}\big) \cdot y + \big(-t^{\frac{7}{2}} - t^4\big),\\ f_4'' &= t^3x^2 + 2t^3x.\end{aligned}$$

If we are to find an $\omega'' \in \text{Trop}(J'')$, then f_4'' implies that necessarily $\omega_1'' = 0$. But we are looking for an ω'' all of whose entries are strictly negative. The reason why this does not exist is that there is a solution of J'' with $x = 0$. We thus have to eliminate the variable x, and replace J'' by the ideal $J''' = \langle f'''\rangle$ with

$$f''' = y - t^{\frac{1}{2}}.$$

Then $\omega''' = -\frac{1}{2} \in \text{Trop}(J''')$ and $\text{t-in}_{\omega'''}(f''') = y - 1$. Thus $u''' = 1$ is our only choice, and since $f'''(u''' \cdot t^{-\omega'''}) = f'''(t^{\frac{1}{2}}) = 0$ we are done.

Backwards substitution gives

$$\begin{aligned} p &= \left(t^{\omega_1} \cdot \left(u_1 + t^{\omega_1'} \cdot (u_1' + 0)\right), t^{\omega_2} \cdot \left(u_2 + t^{\omega_2'} \cdot \left(u_2' + t^{\omega_2''} \cdot u'''\right)\right)\right) \\ &= \left(t \cdot \left(1 + t^{\frac{1}{2}}\right), t^{\frac{3}{2}} \cdot \left(1 + t^{\frac{1}{2}} \cdot \left(-2 + t^{\frac{1}{2}}\right)\right)\right) \\ &= \left(t + t^{\frac{3}{2}}, t^{\frac{3}{2}} - 2t^2 + t^{\frac{5}{2}}\right) \end{aligned}$$

as a point in $V(J)$ with $\mathrm{val}(p) = \left(1, \frac{3}{2}\right) = -\omega$. Note that in general the procedure will not terminate.

For the proof that this algorithm works we need two types of transformations which we are now going to introduce and study.

Definition and Remark 3.3
For $\omega' \in \mathbb{Q}^n$ let us consider the L-algebra isomorphism

$$\Phi_{\omega'} : L[\underline{x}] \longrightarrow L[\underline{x}] : x_i \mapsto t^{-\omega_i'} \cdot x_i,$$

and the isomorphism which it induces on L^n

$$\phi_{\omega'} : L^n \to L^n : (p_1', \ldots, p_n') \mapsto \left(t^{-\omega_1'} \cdot p_1', \ldots, t^{-\omega_n'} \cdot p_n'\right).$$

Suppose we have found a $p' \in V\big(\Phi_{\omega'}(J)\big)$, then $p = \phi_{\omega'}(p') \in V(J)$ and $\mathrm{val}(p) = \mathrm{val}(p') - \omega'$.
Thus choosing ω' appropriately we may in Theorem 3.1 assume that $\omega \in \mathbb{Q}^n_{<0}$, which due to Corollary 6.15 implies that the dimension of J behaves well when contracting to the power series ring $R_N[\underline{x}]$ for a suitable N.
Note also the following properties of $\Phi_{\omega'}$, which we will refer to quite frequently. If $J \trianglelefteq L[\underline{x}]$ is an ideal, then

$$\dim(J) = \dim\left(\Phi_{\omega'}(J)\right) \quad \text{and} \quad \text{t-in}_{\omega'}(J) = \text{t-in}_0\left(\Phi_{\omega'}(J)\right),$$

where the latter is due to the fact that

$$\deg_w\left(t^\alpha \cdot \underline{x}^\beta\right) = -\alpha + \omega' \cdot \beta = \deg_v\left(t^{\alpha - \omega' \cdot \beta} \cdot \underline{x}^\beta\right) = \deg_v\left(\Phi_{\omega'}(t^\alpha \cdot \underline{x}^\beta)\right)$$

with $w = (-1, \omega')$ and $v = (-1, 0, \ldots, 0)$.

Definition and Remark 3.4
For $u = (u_1, \ldots, u_n) \in K^n$, $\omega \in \mathbb{Q}^n$ and $w = (-1, \omega)$ we consider the L-algebra isomorphism

$$\gamma_{\omega,u} : L[\underline{x}] \longrightarrow L[\underline{x}] : x_i \mapsto t^{-\omega_i} \cdot (u_i + x_i),$$

and its effect on a w-quasihomogeneous element

$$f_{q,w} = \sum_{\substack{(\alpha,\beta) \in \mathbb{N}^{n+1} \\ -\frac{\alpha}{N} + \omega \cdot \beta = q}} a_{\alpha,\beta} \cdot t^{\frac{\alpha}{N}} \cdot \underline{x}^\beta.$$

If we set

$$p_\beta := \prod_{i=1}^n (u_i + x_i)^{\beta_i} - u^\beta \in \langle x_1, \ldots, x_n \rangle \lhd K[\underline{x}]$$

then

$$\begin{aligned}
\gamma_{\omega,u}(f_{q,w}) &= \sum_{-\frac{\alpha}{N}+\omega\cdot\beta=q} a_{\alpha,\beta}\cdot t^{\frac{\alpha}{N}}\cdot\prod_{i=1}^{n} t^{-\omega_i\cdot\beta_i}\cdot(u_i+x_i)^{\beta_i}\\
&= t^{-q}\cdot\sum_{-\frac{\alpha}{N}+\omega\cdot\beta=q} a_{\alpha,\beta}\cdot\left(u^\beta+p_\beta\right)\\
&= t^{-q}\cdot\left(f_{q,w}(1,u)+\sum_{-\frac{\alpha}{N}+\omega\cdot\beta=q} a_{\alpha,\beta}\cdot p_\beta\right)\\
&= t^{-q}\cdot f_{q,w}(1,u)+t^{-q}\cdot p_{f_{q,w},u},
\end{aligned}\tag{1}$$

with

$$p_{f_{q,w},u} := \sum_{-\frac{\alpha}{N}+w\cdot\beta=q} a_{\alpha,\beta}\cdot p_\beta \in \langle x_1,\ldots,x_n\rangle \lhd K[\underline{x}].$$

In particular, if $\omega\in\frac{1}{N}\cdot\mathbb{Z}^n$ and $f=\sum_{q\leq\hat{q}} f_{q,w}\in R_N[\underline{x}]$ with $\hat{q}=\operatorname{ord}_\omega(f)$ then

$$\gamma_{\omega,u}(f)=t^{-\hat{q}}\cdot g$$

where

$$g=\sum_{q\leq\hat{q}}\left(t^{\hat{q}-q}\cdot f_{q,w}(1,u)+t^{\hat{q}-q}\cdot p_{f_{q,w},u}\right)\in R_N[\underline{x}].$$

□

The following lemma shows that if we consider the transformed ideal $\gamma_{\omega,u}(J)\cap R_N[\underline{x}]$ in the power series ring $K\left[\left[t^{\frac{1}{N}},\underline{x}\right]\right]$ then it defines the germ of a space curve through the origin. This allows us then in Corollary 3.6 to apply normalisation to find a negative-valued point in the tropical variety of $\gamma_{\omega,u}(J)$.

Lemma 3.5
Let $J\lhd L[\underline{x}]$, let $\omega\in\operatorname{Trop}(J)\cap\frac{1}{N}\cdot\mathbb{Z}^n$, and $u\in V\big(\text{t-in}_\omega(J)\big)\subset K^n$. Then

$$\gamma_{\omega,u}(J)\cap R_N[\underline{x}]\subseteq\left\langle t^{\frac{1}{N}},x_1,\ldots,x_n\right\rangle\lhd R_N[\underline{x}].$$

Proof: Let $w=(-1,\omega)$ and $0\neq f=\gamma_{\omega,u}(h)\in\gamma_{\omega,u}(J)\cap R_N[\underline{x}]$ with $h\in J$. Since f is a polynomial in $\underline{x}$ we have

$$h=\gamma_{\omega,u}^{-1}(f)=f(t^{\omega_1}\cdot x_1-u_1,\ldots,t^{\omega_n}\cdot x_n-u_n)\in t^m\cdot R_N[\underline{x}]$$

for some $m\in\frac{1}{N}\cdot\mathbb{Z}$. We can thus decompose $g:=t^{-m}\cdot h\in J_{R_N}$ into its w-quasihomogeneous parts, say

$$t^{-m}\cdot h=g=\sum_{q\leq\hat{q}} g_{q,w},$$

where $\hat{q}=\operatorname{ord}_\omega(g)$ and thus $g_{\hat{q},w}=\operatorname{in}_\omega(g)$ is the w-initial form of g. As we have seen in Remark 3.4 there are polynomials $p_{g_{q,w},u}\in\langle x_1,\ldots,x_n\rangle\lhd K[\underline{x}]$ such that

$$\gamma_{\omega,u}(g_{q,w})=t^{-q}\cdot g_{q,w}(1,u)+t^{-q}\cdot p_{g_{q,w},u}.$$

But then

$$\begin{aligned}
f=\gamma_{\omega,u}(h)=\gamma_{\omega,u}(t^m\cdot g)=t^m\cdot\gamma_{\omega,u}(g)=t^m\cdot\gamma_{\omega,u}\left(\sum_{q\leq\hat{q}} g_{q,\omega}\right)\\
=t^m\cdot\sum_{q\leq\hat{q}}\left(t^{-q}\cdot g_{q,w}(1,u)+t^{-q}\cdot p_{g_{q,w},u}\right)\\
=t^{m-\hat{q}}\cdot g_{\hat{q},w}(1,u)+t^{m-\hat{q}}\cdot p_{g_{\hat{q},w},u}+\sum_{q<\hat{q}} t^{m-q}\cdot\left(g_{q,w}(1,u)+p_{g_{q,w},u}\right).
\end{aligned}$$

However, since $g \in J$ and $u \in V\big(\text{t-in}_\omega(J)\big)$ we have

$$g_{\hat{q},w}(1,u) = \text{t-in}_\omega(g)(u) = 0$$

and thus using (1) we get

$$p_{g_{\hat{q},w},u} = t^{\hat{q}} \cdot \big(\gamma_{\omega,u}(g_{\hat{q},w}) - t^{-\hat{q}} \cdot g_{\hat{q},w}(1,u)\big) = t^{\hat{q}} \cdot \gamma_{\omega,u}(g_{\hat{q},w}) \neq 0,$$

since $g_{\hat{q},w} = \text{in}_\omega(g) \neq 0$ and $\gamma_{\omega,u}$ is an isomorphism. We see in particular, that $m - \hat{q} \geq 0$ since $f \in R_N[\underline{x}]$ and $p_{g_{\hat{q},w},u} \in \langle x_1, \ldots, x_n\rangle \lhd K[\underline{x}]$, and hence

$$f = t^{m-\hat{q}} \cdot p_{g_{\hat{q},w},u} + \sum_{q<\hat{q}} t^{m-q} \cdot \big(g_{q,w}(1,u) + p_{g_{q,w},u}\big) \in \big\langle t^{\frac{1}{N}}, x_1, \ldots, x_n\big\rangle.$$

□

The following corollary assures the existence of a negative-valued point in the tropical variety of the transformed ideal – after possibly eliminating those variables for which the components of the solution will be zero.

Corollary 3.6
Suppose that K is an algebraically closed field of characteristic zero. Let $J \lhd L[\underline{x}]$ be a zero-dimensional ideal, let $\omega \in \text{Trop}(J) \cap \mathbb{Q}^n$, and $u \in V\big(\text{t-in}_\omega(J)\big) \subset K^n$. Then

$$\exists\, p = (p_1, \ldots, p_n) \in V\big(\gamma_{\omega,u}(J)\big) \;:\; \forall i: \; \text{val}(p_i) \in \mathbb{Q}_{>0} \cup \{\infty\}.$$

In particular, if $n_p = \#\{p_i \mid p_i \neq 0\} > 0$ and $\underline{x}_p = (x_i \mid p_i \neq 0)$, then

$$\text{Trop}\left(\gamma_{\omega,u}(J) \cap L[\underline{x}_p]\right) \cap \mathbb{Q}^{n_p}_{<0} \neq \emptyset.$$

Proof: We may choose an $N \in \mathcal{N}(\gamma_{\omega,u}(J))$ and such that $\omega \in \frac{1}{N} \cdot \mathbb{Z}^n_{\leq 0}$. Let $I = \gamma_{\omega,u}(J) \cap R_N[\underline{x}]$.
Since $\gamma_{\omega,u}$ is an isomorphism we know that

$$0 = \dim(J) = \dim\left(\gamma_{\omega,u}(J)\right),$$

and by Proposition 5.3 we know that

$$\text{Ass}(I) = \big\{P_{R_N} \;\big|\; P \in \text{Ass}\left(\gamma_{\omega,u}(J)\right)\big\}.$$

Since the maximal ideal

$$\mathfrak{m} = \big\langle t^{\frac{1}{N}}, x_1, \ldots, x_n\big\rangle_{R_N[\underline{x}]} \lhd R_N[\underline{x}]$$

contains the element $t^{\frac{1}{N}}$, which is a unit in $L[\underline{x}]$, it cannot be the contraction of a prime ideal in $L[\underline{x}]$. In particular, $\mathfrak{m} \notin \text{Ass}(I)$. Thus there must be a $P \in \text{Ass}(I)$ such that $P \subsetneqq \mathfrak{m}$, since by Lemma 3.5 $I \subset \mathfrak{m}$ and since otherwise $\mathfrak{m}$ would be minimal over I and hence associated to I.
The strict inclusion implies that $\dim(P) \geq 1$, while Theorem 6.10 shows that

$$\dim(P) \leq \dim(I) \leq \dim\left(\gamma_{\omega,u}(J)\right) + 1 = 1.$$

Hence the ideal P is a 1-dimensional prime ideal in $R_N[\underline{x}] \subset K\big[\big[t^{\frac{1}{N}}, \underline{x}\big]\big]$, where the latter is the completion of the former with respect to $\mathfrak{m}$. Since $P \subset \mathfrak{m}$, the completion $\hat{P}$ of P with respect to $\mathfrak{m}$ is also 1-dimensional and the normalisation

$$\psi : K\big[\big[t^{\frac{1}{N}}, \underline{x}\big]\big]/\hat{P} \hookrightarrow \widetilde{R} \simeq K[[s]]$$

gives a parametrisation where we may assume that $\psi\big(t^{\frac{1}{N}}\big) = s^M$ for some $M \in \mathbb{N}_{>0}$ since K is algebraically closed and of characteristic zero (see e.g. [DP00] Cor. 4.4.10 for $K = \mathbb{C}$). Let now $s_i = \psi(x_i) \in K[[s]]$ then necessarily $a_i = \text{ord}_s(s_i) > 0$, since

ψ is a local K-algebra homomorphism, and $f(s^M, s_1, \ldots, s_n) = \psi(f) = 0$ for all $f \in \hat{P}$. Taking $I \subseteq P \subset \hat{P}$ and $\gamma_{\omega,u}(J) = \langle I \rangle$ into account and replacing s by $t^{\frac{1}{N \cdot M}}$ we get

$$f\big(t^{\frac{1}{N}}, p\big) = 0 \quad \text{for all} \;\; f \in \gamma_{\omega,u}(J)$$

where

$$p = \Big(s_1\big(t^{\frac{1}{N\cdot M}}\big), \ldots, s_n\big(t^{\frac{1}{N\cdot M}}\big)\Big) \in R^n_{N\cdot M} \subseteq L^n.$$

Moreover,

$$\mathrm{val}(p_i) = \frac{a_i}{N \cdot M} \in \mathbb{Q}_{>0} \cup \{\infty\},$$

and every $f \in \gamma_{\omega,u}(J) \cap L[\underline{x}_p]$ vanishes at the point $p' = (p_i \mid p_i \neq 0)$. By Proposition 2.14

$$-\,\mathrm{val}(p') \in \mathrm{Trop}\left(\gamma_{\omega,u}(J) \cap L[\underline{x}_p]\right) \cap \mathbb{Q}^{n_p}_{<0}.$$

□

Constructive Proof of Theorem 3.1: Recall that by Remark 3.3 we may assume that $\omega \in \mathbb{Q}^n_{<0}$. It is our first aim to construct recursively sequences of the following objects for $\nu \in \mathbb{N}$:

- natural numbers $1 \leq n_\nu \leq n$,
- natural numbers $1 \leq i_{\nu,1} < \ldots < i_{\nu,n_\nu} \leq n$,
- subsets of variables $\underline{x}_\nu = (x_{i_{\nu,1}}, \ldots, x_{i_{\nu,n_\nu}})$,
- ideals $J'_\nu \lhd L[\underline{x}_{\nu-1}]$,
- ideals $J_\nu \lhd L[\underline{x}_\nu]$,
- vectors $\omega_\nu = (\omega_{\nu,i_{\nu,1}}, \ldots, \omega_{\nu,i_{\nu,n_\nu}}) \in \mathrm{Trop}(J_\nu) \cap (\mathbb{Q}_{<0})^{n_\nu}$, and
- vectors $u_\nu = (u_{\nu,i_{\nu,1}}, \ldots, u_{\nu,i_{\nu,n_\nu}}) \in V\big(\text{t-in}_{\omega_\nu}(J_\nu)\big) \cap (K^*)^{n_\nu}$.

We set $n_0 = n$, $\underline{x}_{-1} = \underline{x}_0 = \underline{x}$, $J_0 = J'_0 = J$, and $\omega_0 = \omega$, and since $\text{t-in}_\omega(J)$ is monomial free by assumption and K is algebraically closed we may choose a $u_0 \in V\big(\text{t-in}_{\omega_0}(J_0)\big) \cap (K^*)^{n_0}$. We then define recursively for $\nu \geq 1$

$$J'_\nu = \gamma_{\omega_{\nu-1},u_{\nu-1}}(J_{\nu-1}).$$

By Corollary 3.6 we may choose a point $q \in V(J'_\nu) \subset L^{n_{\nu-1}}$ such that $\mathrm{val}(q_i) = \mathrm{ord}_t(q_i) > 0$ for all $i = 1, \ldots, n_{\nu-1}$. As in Corollary 3.6 we set

$$n_\nu = \#\{q_i \mid q_i \neq 0\} \in \{0, \ldots, n_{\nu-1}\},$$

and we denote by

$$1 \leq i_{\nu,1} < \ldots < i_{\nu,n_\nu} \leq n$$

the indexes i such that $q_i \neq 0$.

If $n_\nu = 0$ we simply stop the process, while if $n_\nu \neq 0$ we set

$$\underline{x}_\nu = (x_{i_{\nu,1}}, \ldots, x_{i_{\nu,n_\nu}}) \subseteq \underline{x}_{\nu-1}.$$

We then set

$$J_\nu = \big(J'_\nu + \langle \underline{x}_{\nu-1} \setminus \underline{x}_\nu \rangle\big) \cap L[\underline{x}_\nu],$$

and by Corollary 3.6 we can choose

$$\omega_\nu = (\omega_{\nu,i_{\nu,1}}, \ldots, \omega_{\nu,i_{\nu,n_\nu}}) \in \mathrm{Trop}(J_\nu) \cap \mathbb{Q}^{n_\nu}_{<0}.$$

Then $\text{t-in}_{\omega_\nu}(J_\nu)$ is monomial free, so that we can choose a

$$u_\nu = (u_{\nu,i_{\nu,1}}, \ldots, u_{\nu,i_{\nu,n_\nu}}) \in V\big(\text{t-in}_{\omega_\nu}(J_\nu)\big) \cap (K^*)^{n_\nu}.$$

Next we define

$$\varepsilon_i = \sup\left\{\nu \;\middle|\; i \in \{i_{\nu,1},\ldots,i_{\nu,n_\nu}\}\right\} \in \mathbb{N}\cup\{\infty\} \text{ and}$$

$$p_{\mu,i} = \sum_{\nu=0}^{\min\{\varepsilon_i,\mu\}} u_{\nu,i}\cdot t^{-\sum_{j=0}^{\nu}\omega_{j,i}}$$

for $i = 1,\ldots,n$. All $\omega_{\nu,i}$ are strictly negative, which is necessary to see that the $p_{\mu,i}$ converge to a Puiseux series. Note that in the case $n = 1$ the described procedure is just the classical Puiseux expansion (see e.g. [DP00] Thm. 5.1.1 for the case $K = \mathbb{C}$). To see that the $p_{\mu,i}$ converge to a Puiseux series (i.e. that there exists a common denominator N for the exponents as μ goes to infinity), the general case can easily be reduced to the case $n = 1$ by projecting the variety to all coordinate lines, analogously to the proof in section 3 of [Mau80]. The ideal of the projection to one coordinate line is principal. Transformation and intersection commute.
It is also easy to see that at $p = (p_1,\ldots,p_n) \in L^n$ all polynomials in J vanish, where

$$p_i = \lim_{\mu\to\infty} p_{\mu,i} = \sum_{\nu=0}^{\infty} u_{\nu,i}\cdot t^{-\sum_{j=0}^{\nu}\omega_{j,i}} \in R_N \subset L.$$

□

Remark 3.7
The proof is basically an algorithm which allows to compute a point $p \in V(J)$ such that $\mathrm{val}(p) = -\omega$. However, if we want to use a computer algebra system like Singular for the computations, then we have to restrict to generators of J which are polynomials in $t^{\frac{1}{N}}$ as well as in $\underline{x}$. Moreover, we should pass from $t^{\frac{1}{N}}$ to t, which can be easily done by the K-algebra isomorphism

$$\Psi_N : L[\underline{x}] \longrightarrow L[\underline{x}] : t \mapsto t^N, x_i \mapsto x_i.$$

Whenever we do a transformation which involves rational exponents we will clear the denominators using this map with an appropriate N.
We will in the course of the algorithm have to compute the t-initial ideal of J with respect to some $\omega \in \mathbb{Q}^n$, and we will do so by a standard basis computation using the monomial ordering $>_\omega$, given by

$$t^\alpha\cdot\underline{x}^\beta \;>_\omega\; t^{\alpha'}\cdot\underline{x}^{\beta'} \iff$$
$$-\alpha+\omega\cdot\beta > -\alpha'+\omega\cdot\beta' \text{ or } (-\alpha+\omega\cdot\beta = -\alpha'+\omega\cdot\beta' \text{ and } \underline{x}^\beta > \underline{x}^{\beta'}),$$

where $>$ is some fixed global monomial ordering on the monomials in $\underline{x}$.

Algorithm 3.8 (ZDL – Zero Dimensional Lifting Algorithm)
Input: $(m, f_1,\ldots,f_k,\omega) \in \mathbb{N}_{>0}\times K[t,\underline{x}]^k\times\mathbb{Q}^n$ such that $\dim(J) = 0$ and $\omega \in \mathrm{Trop}(J)$ for $J = \langle f_1,\ldots,f_k\rangle_{L[\underline{x}]}$.

Output: $(N,p) \in \mathbb{N}\times K[t,t^{-1}]^n$ such that $p\big(t^{\frac{1}{N}}\big)$ coincides with the first m terms of a solution of $V(J)$ and such that $\mathrm{val}(p) = -\omega$.

Instructions:

- Choose $N \geq 1$ such that $N\cdot\omega \in \mathbb{Z}^n$.
- FOR $i = 1,\ldots,k$ DO $f_i := \Psi_N(f_i)$.
- $\omega := N\cdot\omega$
- IF some $\omega_i > 0$ THEN

- FOR $i=1,\ldots,k$ DO $f_i := \Phi_\omega(f_i)\cdot t^{-\operatorname{ord}_t\left(\Phi_\omega(f_i)\right)}$.
- $\tilde\omega := \omega$.
- $\omega := (0,\ldots,0)$.

- Compute a standard basis $(g_1,\ldots,g_l)$ of $\langle f_1,\ldots,f_k\rangle_{K[t,\underline{x}]}$ with respect to the ordering $>_\omega$.
- Compute a zero $u\in(K^*)^n$ of $\langle \text{t-in}_\omega(g_1),\ldots,\text{t-in}_\omega(g_l)\rangle_{K[\underline{x}]}$.
- IF $m=1$ THEN $(N,p) := \big(N, u_1\cdot t^{-\omega_1},\ldots,u_n\cdot t^{-\omega_n}\big)$.
- ELSE
  - Set $G=\big(\gamma_{\omega,u}(f_i)\;\big|\; i=1,\ldots,k\big)$.
  - FOR $i=1,\ldots,n$ DO
    - Compute a generating set G' of $\langle G,x_i\rangle_{K[t,\underline{x}]} : \langle t\rangle^\infty$.
    - IF $G'\subseteq\langle t,\underline{x}\rangle$ THEN
      - $\underline{x} := \underline{x}\setminus\{x_i\}$
      - Replace G by a generating set of $\langle G'\rangle\cap K[t,\underline{x}]$.
  - IF $\underline{x}=\emptyset$ THEN $(N,p) := \big(N, u_1\cdot t^{-\omega_1},\ldots,u_n\cdot t^{-\omega_n}\big)$.
  - ELSE
    - Compute a point ω' in the negative orthant of the tropical variety of $\langle G\rangle_{L[\underline{x}]}$.
    - $(N',p') = ZDL(m-1,G,\omega')$.
    - $N := N\cdot N'$.
    - FOR $j=1,\ldots,n$ DO
      - IF $x_i\in\underline{x}$ THEN $p_i := t^{-\omega_i\cdot N'}\cdot(u_i+p_i')$.
      - ELSE $p_i := t^{-\omega_i\cdot N'}\cdot u_i$.
- IF some $\tilde\omega_i>0$ THEN $p := \big(t^{-\tilde\omega_1}\cdot p_1,\ldots,t^{-\tilde\omega_n}\cdot p_n\big)$.

Proof: The algorithm which we describe here is basically one recursion step in the constructive proof of Theorem 3.1 given above, and thus the correctness follows once we have justified why our computations do what is required by the recursion step. Notice that step 4 and the last step make an adjusting change of variables to make all ω_i non-positive in the body of the algorithm. This together with step 3 guarantees that $t^{-\omega_i}$ is a polynomial.
If we compute a standard basis $(g_1,\ldots,g_l)$ of $\langle f_1,\ldots,f_k\rangle_{K[t,\underline{x}]}$ with respect to $>_\omega$, then by Theorem 2.8 the t-initial forms of the g_i generate the t-initial ideal of $J=\langle f_1,\ldots,f_k\rangle_{L[\underline{x}]}$. We thus compute a zero u of the t-initial ideal as required.
Next the recursion in the proof of Theorem 3.1 requires to find an $\omega\in\big(\mathbb{Q}_{>0}\cup\{\infty\}\big)^n$, which is $-\operatorname{val}(q)$ for some $q\in V(J)$, and we have to eliminate those components which are zero. Note that the solutions with first component zero are the solutions of $J+\langle x_1\rangle$. Checking if there is a solution with strictly positive valuation amounts by the proof of Corollary 3.6 to checking if $\big(J+\langle x_1\rangle\big)\cap K[[t]][\underline{x}]\subseteq\langle t,\underline{x}\rangle$, and the latter is equivalent to $G'\subseteq\langle t,\underline{x}\rangle$ by Lemma 3.9. If so, we eliminate the variable x_1 from $\langle G'\rangle_{K[t,\underline{x}]}$, which amounts to projecting all solutions with first component zero to L^{n-1}. We then continue with the remaining variables. That way we find a set of variables $\{x_{i_1},\ldots,x_{i_s}\}$ such that there is a solution of $V(J)$ with strictly positive valuation where precisely the other components are zero.
The rest follows from the constructive proof of Theorem 3.1. □

Lemma 3.9
Let $f_1,\ldots,f_k\in K[t,\underline{x}]$, $J=\langle f_1,\ldots,f_k\rangle_{L[\underline{x}]}$, $I=\langle f_1,\ldots,f_k\rangle_{K[t,\underline{x}]}:\langle t\rangle^\infty$, and let

G be a generating set of I. Then:

$$J \cap K[[t]][\underline{x}] \subseteq \langle t, \underline{x}\rangle \iff I \subseteq \langle t, \underline{x}\rangle \iff G \subseteq \langle t, \underline{x}\rangle.$$

Proof: The last equivalence is clear since I is generated by G, and for the first equivalence it suffices to show that $J \cap K[[t]][\underline{x}] = \langle I\rangle_{K[[t]][\underline{x}]}$.
For this let us consider the following two ideals $I' = \langle f_1, \ldots, f_k\rangle_{K[[t]][\underline{x}]} : \langle t\rangle^\infty$ and $I'' = \langle f_1, \ldots, f_k\rangle_{K[t]_{\langle t\rangle}[\underline{x}]} : \langle t\rangle^\infty$. By Lemma 6.6 we know that $J \cap K[[t]][\underline{x}] = I'$ and by [Mar07] Prop. 6.20 we know that $I' = \langle I''\rangle_{K[[t]][\underline{x}]}$. It thus suffice to show that $I'' = \langle I\rangle_{K[t]_{\langle t\rangle}[\underline{x}]}$. Obviously $I \subseteq I''$, which proves one inclusion. Conversely, if $f \in I''$ then f satisfies a relation of the form

$$t^m \cdot f \cdot u = \sum_{i=1}^{k} g_i \cdot f_i,$$

with $m \geq 0$, $u \in K[t]$, $u(0) = 1$ and $g_1, \ldots, g_k \in K[t, \underline{x}]$. Thus $f \cdot u \in I$ and $f = \frac{f \cdot u}{u} \in \langle I\rangle_{K[t]_{\langle t\rangle}[\underline{x}]}$. □

Remark 3.10
In order to compute the point ω' we may want to compute the tropical variety of $\langle G\rangle_{L[\underline{x}]}$. The tropical variety can be computed as a subcomplex of a Gröbner fan or more efficiently by applying Algorithm 5 in [BJS$^+$07] for computing tropical bases of tropical curves.

Remark 3.11
We have implemented the above algorithm in the computer algebra system Singular (see [GPS05]) since nearly all of the necessary computations are reduced to standard basis computations over $K[t, \underline{x}]$ with respect to certain monomial orderings. In Singular however we do not have an algebraically closed field K over which we can compute the zero u of an ideal. We get around this by first computing the absolute minimal associated primes of $\langle \text{t-in}_\omega(g_1), \ldots, \text{t-in}_\omega(g_k)\rangle_{K[t,\underline{x}]}$ all of which are maximal by Corollary 6.15, using the absolute primary decomposition in Singular. Choosing one of these maximal ideals we only have to adjoin one new variable, say a, to realise the field extension over which the zero lives, and the minimal polynomial, say m, for this field extension is provided by the absolute primary decomposition. In subsequent steps we might have to enlarge the minimal polynomial, but we can always get away with only one new variable.
The field extension should be the coefficient field of our polynomial ring in subsequent computations. Unfortunately, the program `gfan` which we use in order to compute tropical varieties does not handle field extensions. (It would not be a problem to actually implement field extensions — we would not have to come up with new algorithms.) But we will see in Lemma 3.12 that we can get away with computing tropical varieties of ideals in the polynomial ring over the extension field of K by computing just over K. More precisely, we want to compute a negative-valued point ω' in the tropical variety of a transformed ideal $\gamma_{\omega,u}(J)$. Instead, we compute a point $(\omega', 0)$ in the tropical variety of the ideal $\gamma_{\omega,u}(J) + \langle m\rangle$. So to justify this it is enough to show that ω is in the tropical variety of an ideal $J \trianglelefteq K[a]/\langle m\rangle\{\{t\}\}[\underline{x}]$ if and only if $(\omega, 0)$ is in the tropical variety of the ideal $J + \langle m\rangle \trianglelefteq K\{\{t\}\}[\underline{x}, a]$. Recall that $\omega \in \text{Trop}(J)$ if and only if $\text{t-in}_\omega(J)$ contains no monomial, and by Theorem 2.8, $\text{t-in}_\omega(J)$ is equal to $\text{t-in}_\omega(J_{R_N})$, where $N \in \mathcal{N}(J)$.

Lemma 3.12
Let $m \in K[a]$ be an irreducible polynomial, let $\varphi : K[t^{\frac{1}{N}}, \underline{x}, a] \to (K[a]/\langle m\rangle)[t^{\frac{1}{N}}, \underline{x}]$ take elements to their classes, and let $I \trianglelefteq (K[a]/\langle m\rangle)[t^{\frac{1}{N}}, \underline{x}]$. Then $\mathrm{in}_\omega(I)$ contains no monomial if and only if $\mathrm{in}_{(\omega,0)}(\varphi^{-1}(I))$ contains no monomial. In particular, the same holds for $\text{t-in}_\omega(I)$ and $\text{t-in}_{(\omega,0)}(\varphi^{-1}(I))$.

Proof: Suppose $\mathrm{in}_{(\omega,0)}\,\varphi^{-1}(I)$ contains a monomial. Then there exists an $f \in \varphi^{-1}(I)$ such that $\mathrm{in}_{(\omega,0)}(f)$ is a monomial. The polynomial $\varphi(f)$ is in I. When applying φ the monomial $\mathrm{in}_{(\omega,0)}(f)$ maps to a monomial whose coefficient in $K[a]/\langle m\rangle$ has a representative $h \in K[a]$ with just one term. The representative h cannot be 0 modulo $\langle m\rangle$ since $\langle m\rangle$ does not contain a monomial. Thus $\varphi\big(\mathrm{in}_{(\omega,0)(f)}\big) = \mathrm{in}_\omega(\varphi(f))$ is a monomial.
For the other direction, suppose $\mathrm{in}_\omega(I)$ contains a monomial. We must show that $\mathrm{in}_{(\omega,0)}(\varphi^{-1}(I))$ contains a monomial. This is equivalent to showing that $(\mathrm{in}_{(\omega,0)}(\varphi^{-1}(I)) : ((t^{\frac{1}{N}} \cdot x_1 \cdots x_n)^\infty)$ contains a monomial. By assumption there exists an $f \in I$ such that $\mathrm{in}_\omega(f)$ is a monomial. Let g be in $\varphi^{-1}(I)$ such that g maps to f under the surjection φ and with the further condition that the support of g projected to the $(t^{\frac{1}{N}}, \underline{x})$-coordinates equals the support of f. The initial form $\mathrm{in}_{(\omega,0)}(g)$ is a polynomial with all exponent vectors having the same $(t^{\frac{1}{N}}, \underline{x})$ parts as $\mathrm{in}_\omega(f)$ does. Let g' be $\mathrm{in}_{(\omega,0)}(g)$ with the common $(t^{\frac{1}{N}}, \underline{x})$-part removed from the monomials, that is $g' \in K[a]$. Notice that $\varphi(g') \neq 0$. We now have $g' \notin \langle m\rangle$ and hence $\langle g', m\rangle = k[a]$ since $\langle m\rangle$ is maximal. Now m and g' are contained in $(\mathrm{in}_{(\omega,0)}(\varphi^{-1}(I)) : (t^{\frac{1}{N}} \cdot x_1 \cdots x_n)^\infty)$, implying that $(\mathrm{in}_{(\omega,0)}(\varphi^{-1}(I)) : (t^{\frac{1}{N}} \cdot x_1 \cdots x_n)^\infty) \supseteq K[a]$. This shows that $\mathrm{in}_{(\omega,0)}(\varphi^{-1}(I))$ contains a monomial. $\Box$

Remark 3.13
In Algorithm 3.8 we choose zeros of the t-initial ideal and we choose points in the negative quadrant of the tropical variety. If we instead do the same computations for all zeros and points of the negative quadrant of the tropical variety, then we get Puiseux expansions of all branches of the space curve germ defined by the ideal $\langle f_1, \ldots, f_k\rangle_{K[[t,\underline{x}]]}$ in $(K^{n+1}, 0)$.

4. Reduction to the Zero Dimensional Case

In this section, we want to give a proof of the Lifting Lemma (Theorem 3.1) for any ideal J of dimension $\dim J = d > 0$, using our algorithm for the zero-dimensional case.
Given $\omega \in \mathrm{Trop}(J)$ we would like to intersect $\mathrm{Trop}(J)$ with another tropical variety $\mathrm{Trop}(J')$ containing ω, such that $\dim(J + J') = 0$ and apply the zero-dimensional algorithm to $J + J'$. However, we cannot conclude that $\omega \in \mathrm{Trop}(J + J')$ — we have $\mathrm{Trop}(J + J') \subseteq \mathrm{Trop}(J) \cap \mathrm{Trop}(J')$ but equality does not need to hold. For example, two plane tropical lines (given by two linear forms) which are not equal can intersect in a ray, even though the ideal generated by the two linear forms defines just a point.
So we have to find an ideal J' such that $J + J'$ is zero-dimensional and still $\omega \in \mathrm{Trop}(J + J')$ (see Proposition 4.6). We will use some ideas of [Kat06] Lemma 4.4.3 — the ideal J' will be generated by $\dim(J)$ sufficiently general linear forms. The proof of the proposition needs some technical preparations.

Notation 4.1
We denote by

$$V_\omega = \{a_0 + a_1 \cdot t^{\omega_1} \cdot x_1 + \ldots + a_n \cdot t^{\omega_n} \cdot x_n \mid a_i \in K\}$$

the $n+1$-dimensional K-vector space of *linear* polynomials over K, which in a sense are *scaled* by $\omega \in \mathbb{Q}^n$. Of most interest will be the case where $\omega = 0$.

The following lemma geometrically says that an affine variety of dimension at least one will intersect a generic hyperplane.

Lemma 4.2
Let K be an infinite field and $J \lhd L[\underline{x}]$ an equidimensional ideal of dimension $\dim(J) \geq 1$. Then there is a Zariski open dense subset U of V_0 such that $\langle f\rangle + Q \neq L[\underline{x}]$ for all $f \in U$ and $Q \in \operatorname{minAss}(J)$.

If V is an affine variety which meets $(K^*)^n$ in dimension at least 1, then a generic hyperplane section of V meets $(K^*)^n$ as well. The algebraic formulation of this geometric fact is the following lemma:

Lemma 4.3
Let K be an infinite field and $I \lhd K[\underline{x}]$ be an equidimensional ideal with $\dim(I) \geq 1$ and such that $x_1 \cdots x_n \notin \sqrt{I}$, then there is a Zariski open subset U of V_0 such that $x_1 \cdots x_n \notin \sqrt{I + \langle f\rangle}$ for $f \in U$.

The following lemma is an algebraic formulation of the geometric fact that given any affine variety none of its components will be contained in a generic hyperplane.

Lemma 4.4
Let K be an infinite field, let R be a ring containing K, and let $J \unlhd R[\underline{x}]$ be an ideal. Then there is a Zariski open dense subset U of V_0 such that $f \in U$ satisfies $f \notin P$ for $P \in \operatorname{minAss}(J)$.

Remark 4.5
If $\#K < \infty$ we can still find a suitable $f \in K[\underline{x}]$ which satisfies the conditions in Lemma 4.2, Lemma 4.3 and Lemma 4.4 due to Prime Avoidance. However, it may not be possible to choose a linear one.

With these preparations we can show that we can reduce to the zero dimensional case by cutting with generic hyperplanes.

Proposition 4.6
Suppose that K is an infinite field, and let $J \lhd L[\underline{x}]$ be an equidimensional ideal of dimension d and $\omega \in \operatorname{Trop}(J) \cap \mathbb{Q}^n$.
Then there exist Zariski open dense subsets $U_1, \ldots, U_d$ of V_ω such that $(f_1, \ldots, f_d) \in U_1 \times \ldots \times U_d$ and $J' = \langle f_1, \ldots, f_d\rangle_{L[\underline{x}]}$ satisfy:

- $\dim(J + J') = \dim\big(\text{t-in}_\omega(J) + \text{t-in}_\omega(J')\big) = 0$,
- $\dim\big(\text{t-in}_\omega(J')\big) = \dim(J') = n - d$,
- $x_1 \cdots x_n \notin \sqrt{\text{t-in}_\omega(J) + \text{t-in}_\omega(J')}$, *and*
- $\sqrt{\text{t-in}_\omega(J) + \text{t-in}_\omega(J')} = \sqrt{\text{t-in}_\omega(J + J')}$.

In particular, $\omega \in \operatorname{Trop}(J + J')$.

Proof: Applying Φ_ω to J first and then applying $\Phi_{-\omega}$ to J' later we may assume that $\omega = 0$. Moreover, we may choose an N such that $N \in \mathcal{N}(J)$ and $N \in \mathcal{N}(P)$ for

all $P\in\operatorname{minAss}(J)$. By Lemma 6.7 then also $\text{t-in}_0(J)=\text{t-in}_0(J_{R_N})$ and $\text{t-in}_0(P)=\text{t-in}_0(P_{R_N})$ for $P\in\operatorname{minAss}(J)$.
By Lemma 6.16

$$\operatorname{minAss}(J_{R_N})=\{P_{R_N}\mid P\in\operatorname{minAss}(J)\}. \tag{2}$$

In particular, all minimal associated primes P_{R_N} of J_{R_N} have codimension $n-d$ by Corollary 6.9.
Since $0\in\operatorname{Trop}(J)$ there exists a $P\in\operatorname{minAss}(J)$ with $0\in\operatorname{Trop}(P)$ by Lemma 2.12. Hence $1\not\in\text{t-in}_0(P)$ and we conclude by Corollary 6.17 that

$$\dim(J)=\dim\big(\text{t-in}_0(J)\big)=\dim(Q) \tag{3}$$

for all $Q\in\operatorname{minAss}\big(\text{t-in}_0(J)\big)$. In particular, all minimal associated prime ideals of $\text{t-in}_0(J)$ have codimension $n-d$.
Moreover, since $0\in\operatorname{Trop}(J)$ we know that $\text{t-in}_0(J)$ is monomial free, and in particular

$$x_1\cdots x_n\not\in\sqrt{\text{t-in}_0(J)}. \tag{4}$$

If $d=0$ then $J'=\langle\emptyset\rangle=\{0\}$ works due to (3) and (4). We may thus assume that $d>0$.
Since K is infinite we can apply Lemma 4.2 to J, Lemma 4.4 to $J\lhd L[\underline{x}]$, to $J_{R_N}\lhd R_N[\underline{x}]$ and to $\text{t-in}_0(J)\lhd K[\underline{x}]$ and Lemma 4.3 to $\text{t-in}_0(J)\lhd K[\underline{x}]$ (take (4) into account), and thus there exist Zariski open dense subsets U, U', U'', U''' and U'''' in V_0 such that no $f_1\in U_1=U\cap U'\cap U''\cap U'''\cap U''''$ is contained in any minimal associated prime of either J, J_{R_N} or $\text{t-in}_0(J)$, such that $1\not\in J+\langle f_1\rangle_{L[\underline{x}]}$ and such that $x_1\cdots x_n\not\in\sqrt{\text{t-in}_0(J)+\langle f_1\rangle}$. Since the intersection of four Zariski open and dense subsets is non-empty, there is such an f_1 and by Lemma 5.6 the minimal associated primes of the ideals $J+\langle f_1\rangle_{L[\underline{x}]}$, $J_{R_N}+\langle f_1\rangle_{R_N[\underline{x}]}$, and $\text{t-in}_0(J)+\langle f_1\rangle_{K[\underline{x}]}$ all have the same codimension $n-d+1$.
We claim that $t^{\frac{1}{N}}\not\in Q$ for any $Q\in\operatorname{minAss}(J_{R_N}+\langle f_1\rangle_{R_N[\underline{x}]})$. Suppose the contrary, then by Lemma 6.8 (b), (f) and (g)

$$\dim(Q)=n+1-\operatorname{codim}(Q)=d.$$

Consider now the residue class map

$$\pi:R_N[\underline{x}]\longrightarrow R_N[\underline{x}]/\langle t^{\frac{1}{N}}\rangle=K[\underline{x}].$$

Then $\text{t-in}_0(J)=\pi\big(J_{R_N}+\langle t^{\frac{1}{N}}\rangle\big)$, and we have

$$\text{t-in}_0(J)+\langle f_1\rangle_{K[\underline{x}]}\subseteq\pi\big(J_{R_N}+\langle t^{\frac{1}{N}},f_1\rangle_{R_N[\underline{x}]}\big)\subseteq\pi(Q).$$

Since $t^{\frac{1}{N}}\in Q$ the latter is again a prime ideal of dimension d. However, due to the choice of f_1 we know that every minimal associated prime of $\text{t-in}_0(J)+\langle f_1\rangle_{K[\underline{x}]}$ has codimension $n-d+1$ and hence the ideal itself has dimension $d-1$. But then it cannot be contained in an ideal of dimension d.
Applying the same arguments another $d-1$ times we find Zariski open dense subsets $U_2,\ldots,U_d$ of V_0 such that for all $(f_1,\ldots,f_d)\in U_1\times\cdots\times U_d$ the minimal associated primes of the ideals

$$J+\langle f_1,\ldots,f_k\rangle_{L[\underline{x}]}$$

respectively

$$J_{R_N}+\langle f_1,\ldots,f_k\rangle_{R_N[\underline{x}]}$$

respectively

$$\text{t-in}_0(J)+\langle f_1,\ldots,f_k\rangle_{K[\underline{x}]}$$

all have codimension $n-d+k$ for each $k = 1, \ldots, d$, such that $1 \notin J+\langle f_1, \ldots, f_k\rangle_{L[\underline{x}]}$, and such that

$$x_1 \cdots x_n \notin \sqrt{\text{t-in}_0(J) + \langle f_1, \ldots, f_k\rangle_{K[\underline{x}]}}.$$

Moreover, none of the minimal associated primes of $J_{R_N} + \langle f_1, \ldots, f_k\rangle_{R_N[\underline{x}]}$ contains $t^{\frac{1}{N}}$.

In particular, since $f_i \in K[\underline{x}]$ we have (see Theorem 2.8)

$$\text{t-in}_0(J') = \text{t-in}_0\left(\langle f_1, \ldots, f_d\rangle_{K[t,\underline{x}]}\right) = \langle f_1, \ldots, f_d\rangle_{K[\underline{x}]},$$

and J' obviously satisfies the first three requirements of the proposition.

For the fourth requirement it suffices to show

$$\text{minAss}\left(\text{t-in}_0(J) + \text{t-in}_0(J')\right) = \text{minAss}\left(\text{t-in}_0(J + J')\right).$$

For this consider the ring extension

$$R_N[\underline{x}] \subseteq S_N^{-1} R_N[\underline{x}] = L_N[\underline{x}]$$

given by localisation and denote by $I^c = I \cap R_N[\underline{x}]$ the contraction of an ideal I in $L_N[\underline{x}]$ and by $I^e = \langle I\rangle_{L_N[\underline{x}]}$ the extension of an ideal I in $R_N[\underline{x}]$. Moreover, we set $J_0 = J \cap L_N[\underline{x}]$ and $J_0' = J' \cap L_N[\underline{x}]$, so that $J_0^c = J_{R_N}$ and ${J_0'}^c = \langle f_1, \ldots, f_d\rangle_{R_N[\underline{x}]}$. Note then first that

$$(J_0^c + {J_0'}^c)^e = J_0^{ce} + {J_0'}^{ce} = J_0 + J_0',$$

and therefore by the correspondence of primary decomposition under localisation (see [AM69] Prop. 4.9)

$$\text{minAss}\left((J_0 + J_0')^c\right) = \left\{Q \in \text{minAss}(J_0^c + {J_0'}^c) \;\middle|\; t^{\frac{1}{N}} \notin Q\right\} = \text{minAss}\left(J_0^c + {J_0'}^c\right).$$

This then shows that

$$\sqrt{J_0^c + {J_0'}^c} = \sqrt{(J_0 + J_0')^c},$$

and since $\pi(J_0^c) = \text{t-in}_0(J_{R_N}) = \text{t-in}_0(J)$, $\pi({J_0'}^c) = \text{t-in}_0(J')$ and $\pi\big((J_0 + J_0')^c\big) = \text{t-in}_0(J + J')$ we get

$$\begin{aligned}\sqrt{\text{t-in}_0(J) + \text{t-in}_0(J')} &= \sqrt{\pi(J_0^c) + \pi({J_0'}^c)} = \pi\left(\sqrt{J_0^c + {J_0'}^c}\right)\\ &= \pi\left(\sqrt{(J_0 + J_0')^c}\right) = \sqrt{\pi\big((J_0 + J_0')^c\big)} = \sqrt{\text{t-in}_0(J + J')}.\end{aligned}$$

It remains to show the "in particular" part. However, since

$$x_1 \cdots x_n \notin \sqrt{\text{t-in}_\omega(J) + \text{t-in}_\omega(J')} = \sqrt{\text{t-in}_\omega(J + J')},$$

the ideal $\text{t-in}_\omega(J + J')$ is monomial free, or equivalently $\omega \in \text{Trop}(J + J')$. □

Remark 4.7

Proposition 4.6 shows that the ideal J' can be found by choosing d linear forms $f_j = \sum_{i=1}^n a_{ji} \cdot t^{\omega_i} \cdot x_i + a_{j0}$ with random $a_{ji} \in K$, and we only need that K is infinite.

We are now in the position to finish the proof of Theorem 2.13.

Proof of Theorem 2.13: If $\omega \in \text{Trop}(J) \cap \mathbb{Q}^n$ then there is a minimal associated prime ideal $P \in \text{minAss}(J)$ such that $\omega \in \text{Trop}(P)$ by Lemma 2.12. By assumption the field K is algebraically closed and therefore infinite, so that Proposition 4.6 applied to P shows that we can choose an ideal P' such that $\omega \in \text{Trop}(P+P')$ and $\dim(P+P') = 0$. By Theorem 3.1 there exists a point $p \in V(P+P') \subseteq V(J)$ such that $\text{val}(p) = -\omega$. This finishes the proof in view of Proposition 2.14. □

Algorithm 4.8 (RDZ - Reduction to Dimension Zero)
INPUT: a prime ideal $P \in K(t)[\underline{x}]$ and $\omega \in \text{Trop}(P)$.

OUTPUT: an ideal J such that $\dim(J) = 0$, $P \subset J$ and $\omega \in \text{Trop}(J)$.

INSTRUCTIONS:

- $d := \dim(P)$
- $J := P$
- WHILE $\dim(J) \neq 0$ OR $\text{t-in}_\omega(J)$ not monomial-free DO
  - FOR $j = 0$ TO d pick random values $a_{0,j}, \ldots, a_{n,j} \in K$, and define $f_j := a_{0,j} + \sum a_{i,j} \cdot t^{\omega_i} x_i$.
  - $J := P + \langle f_1, \ldots, f_d \rangle$

Proof: We only have to show that the random choices will lead to a suitable ideal J with probability 1. To see this, we want to apply Proposition 4.6. For this we only have to see that $P^e = \langle P \rangle_{L[\underline{x}]}$ is equidimensional of dimension $d = \dim(P)$. By [Mar07] Corollary 6.13 the intersection of P^e with $K(t)[\underline{x}]$, P^{ec}, is equal to P. Using Proposition 5.3 we see that

$$\{P\} = \text{minAss}(P^{ec}) \subseteq \{Q^c \mid Q \in \text{minAss}(P^e)\} \subseteq \text{Ass}(P^{ec}) = \{P\}.$$

By Lemma 5.4 we have $\dim Q = \dim(P) = d$ for every $Q \in \text{minAss}(P^e)$, hence P^e is equidimensional of dimension d. □

Remark 4.9
Note that we cannot perform primary decomposition over $L[\underline{x}]$ computationally. Given a d-dimensional ideal J and $\omega \in \text{Trop}(J)$ in our implementation of the lifting algorithm, we perform primary decomposition over $K(t)[\underline{x}]$. By Lemma 2.12, there must be a minimal associated prime P of J such that $\omega \in \text{Trop}(P)$. Its restriction to $K(t)[\underline{x}]$ is one of the minimal associated primes that we computed, and this prime is our input for algorithm 4.8.

Example 4.10
Assume $P = \langle x + y + t \rangle \trianglelefteq L[x, y]$, and $\omega = (-1, -2)$. Choose coefficients randomly and add for example the linear form $f = -2xt^{-1} + 2t^{-2}y - 1$. Then $J = \langle x+y+t, f \rangle$ has dimension 0 and ω is contained in $\text{Trop}(J)$. Note that the intersection of $\text{Trop}(P)$ with $\text{Trop}(f)$ is not transversal, as the vertex of the tropical line $\text{Trop}(f)$ is at ω.

5. Some Commutative Algebra

In this section we gather some simple results from commutative algebra for the lack of a better reference. They are primarily concerned with the dimension of an ideal under contraction respectively extension for certain ring extensions. The results in this section are independent of the previous sections

Notation 5.1
In this section we denote by $I^e = \langle I\rangle_{R'}$ the extension of $I \trianglelefteq R$ and by $J^c = \varphi^{-1}(J)$ the contraction of $J \trianglelefteq R'$, where $\varphi : R \to R'$ is a ring extension. If no ambiguity can arise we will not explicitly state the ring extension.

We first want to understand how primary decomposition behaves under restriction. The following lemma is an easy consequence of the definitions.

Lemma 5.2
If $\varphi : R \to R'$ is any ring extension and $Q \lhd R'$ a P-primary ideal, then Q^c is P^c-primary.

Proposition 5.3
Let $\varphi : R \to R'$ be any ring extension, let $J \trianglelefteq R'$ be an ideal such that $(J^c)^e = J$, and let $J = Q_1 \cap \ldots \cap Q_k$ be a minimal primary decomposition. Then

$$\mathrm{Ass}(J^c) = \left\{P^c \;\middle|\; P \in \mathrm{Ass}(J)\right\} = \left\{\sqrt{Q_i}^{\,c} \;\middle|\; i = 1,\ldots,k\right\},$$

and $J^c = \bigcap_{P\in\mathrm{Ass}(J^c)} Q_P$ is a minimal primary decomposition, where

$$Q_P = \bigcap_{\sqrt{Q_i}^{\,c}=P} Q_i^c.$$

Moreover, we have $\mathrm{minAss}(J^c) \subseteq \left\{P^c \;\middle|\; P \in \mathrm{minAss}(J)\right\}$.
Note that the $\sqrt{Q_i}^{\,c}$ are not necessarily pairwise different, and thus the cardinality of $\mathrm{Ass}(J^c)$ *may be strictly smaller than k.*

Proof: Let $\mathcal{P} = \left\{\sqrt{Q_i}^{\,c} \;\middle|\; i = 1,\ldots,k\right\}$ and let Q_P be defined as above for $P \in \mathcal{P}$. Since contraction commutes with intersection we have

$$J^c = \bigcap_{P\in\mathcal{P}} Q_P. \tag{5}$$

By Lemma 5.2 the Q_i^c with $P = \sqrt{Q_i}^{\,c}$ are P-primary, and thus so is their intersection, so that (5) is a primary decomposition. Moreover, by construction the radicals of the Q_P are pairwise different. It thus remains to show that none of the Q_P is superfluous. Suppose that there is a $P = \sqrt{Q_i}^{\,c} \in \mathcal{P}$ such that

$$J^c = \bigcap_{P'\in\mathcal{P}\setminus\{P\}} Q_{P'} \subseteq \bigcap_{j\neq i} Q_j^c,$$

then

$$J = (J^c)^e \subseteq \bigcap_{j\neq i}(Q_j^c)^e \subseteq \bigcap_{j\neq i} Q_j$$

in contradiction to the minimality of the given primary decomposition of J. This shows that (5) is a minimal primary decomposition and that $\mathrm{Ass}(J^c) = \mathcal{P}$.
Finally, if $P \in \mathrm{Ass}(J)$ such that P^c is minimal over J^c then necessarily there is a $\tilde{P} \in \mathrm{minAss}(J)$ such that $P^c = \tilde{P}^c$.

□

We will use this result to show that dimension behaves well under extension for polynomial rings over a field extension.

Lemma 5.4
If $F \subseteq F'$ is a field extension, $I \trianglelefteq F[\underline{x}]$ is an ideal and $I^e = \langle I \rangle_{F'[\underline{x}]}$ then

$$\dim(I^e) = \dim(I).$$

Moreover, if I is prime then $\dim(P) = \dim(I)$ for all $P \in \operatorname{minAss}(I^e)$.

Proof: Choose any global degree ordering $>$ on the monomials in $\underline{x}$ and compute a standard basis G' of I with respect to $>$. Then G' is also a standard basis of I^e by Buchberger's Criterion. If M is the set of leading monomials of elements of G' with respect to $>$, then the dimension of the ideal generated by M does not depend on the base field but only on M (see e.g. [GP02] Prop. 3.5.8). Thus we have (see e.g. [GP02] Cor. 5.3.14)

$$\dim(I) = \dim\big(\langle M \rangle_{F[\underline{x}]}\big) = \dim\big(\langle M \rangle_{F'[\underline{x}]}\big) = \dim(I^e). \tag{6}$$

Let now I be prime. It remains to show that I^e is equidimensional.
If we choose a maximal independent set $\underline{x}' \subseteq \underline{x}$ of $L_>(I^e) = \langle M \rangle_{F'[\underline{x}]}$ then by definition (see [GP02] Def. 3.5.3) $\langle M \rangle \cap F'[\underline{x}'] = \{0\}$, so that necessarily $\langle M \rangle_{F[\underline{x}]} \cap F[\underline{x}'] = \{0\}$. This shows that $\underline{x}'$ is an independent set of $L_>(I) = \langle M \rangle_{F[\underline{x}]}$, and it is maximal since its size is $\dim(I^e) = \dim(I)$ by (6). Moreover, by [GP02] Ex. 3.5.1 $\underline{x}'$ is a maximal independent set of both I and I^e. Choose now a global monomial ordering $>'$ on the monomials in $\underline{x}'' = \underline{x} \setminus \underline{x}'$.
We claim that if $G = \{g_1, \ldots, g_k\} \subset F[\underline{x}]$ is a standard basis of $\langle I \rangle_{F(\underline{x}')[\underline{x}'']}$ with respect to $>'$ and if $0 \neq h = \operatorname{lcm}\big(\operatorname{lc}_{>'}(g_1), \ldots, \operatorname{lc}_{>'}(g_k)\big) \in F[\underline{x}']$, then $I^e : \langle h \rangle^\infty = I^e$. For this we consider a minimal primary decomposition $I^e = Q_1 \cap \ldots \cap Q_l$ of I^e. Since $I^{ece} = I^e$ we may apply Proposition 5.3 to get

$$\Big\{ \sqrt{Q_i}^{\,c} \;\Big|\; i = 1, \ldots, l \Big\} = \operatorname{Ass}(I^{ec}) = \{I\}, \tag{7}$$

where the latter equality is due to $I^{ec} = I$ (see e.g. [Mar07] Cor. 6.13) and to I being prime. Since $\underline{x}'$ is an independent set of I we know that $h \notin I$ and thus (7) shows that $h^m \notin \sqrt{Q_i}$ for any $i = 1, \ldots, l$ and any $m \in \mathbb{N}$. Let now $f \in I^e : \langle h \rangle^\infty$, then there is an $m \in \mathbb{N}$ such that $h^m \cdot f \in I^e \subseteq Q_i$ and since Q_i is primary and $h^m \notin \sqrt{Q_i}$ this forces $f \in Q_i$. But then $f \in Q_1 \cap \ldots \cap Q_l = I^e$, which proves the claim.
With the same argument as at the beginning of the proof we see that G is a standard basis of $\langle I^e \rangle_{F'(\underline{x}')[\underline{x}'']}$, and we may thus apply [GP02] Prop. 4.3.1 to the ideal I^e which shows that $I^e : \langle h \rangle^\infty$ is equidimensional. We are thus done by the claim. $\square$

If the field extension is algebraic then dimension also behaves well under restriction.

Lemma 5.5
Let $F \subseteq F'$ be an algebraic field extension and let $J \lhd F'[\underline{x}]$ be an ideal, then $\dim(J) = \dim(J \cap F[\underline{x}])$.

Proof: Since the field extension is algebraic the ring extension $F[\underline{x}] \subseteq F'[\underline{x}]$ is integral again. But then the ring extension $F[\underline{x}]/J \cap F[\underline{x}] \hookrightarrow F'[\underline{x}]/J$ is integral again (see [AM69] Prop. 5.6), and in particular they have the same dimension (see [Eis96] Prop. 9.2). $\square$

For Section 4 — where we want to intersect an ideal of arbitrary dimension to get a zero-dimensional ideal — we need to understand how dimension behaves when we intersect. The following result is concerned with that question. Geometrically it just means that intersecting an equidimensional variety with a hypersurface which does not contain any irreducible component leads again to an equidimensional variety of dimension one less. We need this result over R_N instead of a field K.

Lemma 5.6
Let R be a catenary integral domain, let $I \lhd R$ with $\operatorname{codim}(Q) = d$ for all $Q \in \operatorname{minAss}(I)$, and let $f \in R$ such that $f \notin Q$ for all $Q \in \operatorname{minAss}(I)$. Then

$$\operatorname{minAss}(I + \langle f\rangle) = \bigcup_{Q\in\operatorname{minAss}(I)} \operatorname{minAss}(Q + \langle f\rangle).$$

In particular, $\operatorname{codim}(Q') = d + 1$ *for all* $Q' \in \operatorname{minAss}(I + \langle f\rangle)$.

Proof: If $Q' \in \operatorname{minAss}(I + \langle f\rangle)$ then Q' is minimal among the prime ideals containing $I + \langle f\rangle$. Moreover, since $I \subseteq Q'$ there is a minimal associated prime $Q \in \operatorname{minAss}(I)$ of I which is contained in Q'. And, since $f \in Q'$ we have $Q + \langle f\rangle \subseteq Q'$ and Q' must be minimal with this property since it is minimal over $I + \langle f\rangle$. Hence $Q' \in \operatorname{minAss}(Q + \langle f\rangle)$.
Conversely, if $Q' \in \operatorname{minAss}(Q + \langle f\rangle)$ where $Q \in \operatorname{minAss}(I)$, then $I + \langle f\rangle \subseteq Q'$. Thus there exists a $Q'' \in \operatorname{minAss}(I + \langle f\rangle)$ such that $Q'' \subseteq Q'$. Then $I \subseteq Q''$ and therefore there exists a $\tilde{Q} \in \operatorname{minAss}(I)$ such that $\tilde{Q} \subseteq Q''$. Moreover, since $f \notin \tilde{Q}$ but $f \in Q''$ this inclusion is strict which implies

$$\operatorname{codim}(Q') \geq \operatorname{codim}(Q'') \geq \operatorname{codim}\left(\tilde{Q}\right) + 1 = \operatorname{codim}(Q) + 1,$$

where the first inequality comes from $Q'' \subseteq Q'$ and the last equality is due to our assumption on I. But by Krull's Principal Ideal Theorem (see [AM69] Cor. 11.17) we have

$$\operatorname{codim}(Q'/Q) = 1,$$

since Q'/Q by assumption is minimal over f in R/Q where f is neither a unit (otherwise $Q + \langle f\rangle = R$ and no Q' exists) nor a zero divisor. Finally, since R is catenary and thus all maximal chains of prime ideals from $\langle 0\rangle$ to Q' have the same length this implies

$$\operatorname{codim}(Q') = \operatorname{codim}(Q) + 1. \tag{8}$$

This forces that $\operatorname{codim}(Q') = \operatorname{codim}(Q'')$ and thus $Q' = Q'' \in \operatorname{minAss}(I + \langle f\rangle)$.
The "in particular" part follows from (8). □

6. Good Behaviour of the Dimension

In this section we want to show (see Theorem 6.14) that for an ideal $J \trianglelefteq L[\underline{x}]$, $N \in \mathcal{N}(J)$ and a point $\omega \in \operatorname{Trop}(P) \cap \mathbb{Q}^n_{\leq 0}$ in the non-positive quadrant of the tropical variety of an associated prime P of maximal dimension we have

$$\dim(J_{R_N}) = \dim\left(\text{t-in}_\omega(J)\right) + 1 = \dim(J) + 1.$$

The results in this section are independent of Sections 2, 3 and 4.
Let us first give examples which show that the hypotheses on ω are necessary.

Example 6.1
Let $J = \langle 1 + tx\rangle \lhd L[x]$ and consider $\omega = 1 \in \operatorname{Trop}(J)$. Then $\text{t-in}_\omega(J) = \langle 1 + x\rangle$ has dimension zero in $K[x]$, and

$$I = J \cap R_1[x] = \langle 1 + tx\rangle_{R_1[x]}$$

has dimension zero as well by Lemma 6.8 (d).

Example 6.2
Let $J = \langle x-1\rangle \lhd L[x]$ and $\omega = -1 \notin \mathrm{Trop}(J)$, then $\text{t-in}_\omega(J) = \langle 1\rangle$ has dimension -1, while $J \cap R_1[x] = \langle x-1\rangle$ has dimension 1.

Example 6.3
Let $J = P\cdot Q = P\cap Q \lhd L[x,y,z]$ with $P = \langle tx-1\rangle$ and $Q = \langle x-1, y-1, z-1\rangle$, and let $\omega = (0,0,0) \in \mathrm{Trop}(Q)\cap \mathbb{Q}^3_{\leq 0}$. Then $\text{t-in}_\omega(J) = \langle x-1, y-1, z-1\rangle \lhd K[x,y,z]$ has dimension zero, while

$$J \cap R_1[x,y,z] = (P\cap R_1[x,y,z]) \cap (Q \cap R_1[x,y,z])$$

has dimension two by Lemma 6.8 (d).

Before now starting with studying the behaviour of dimension we have to collect some technical results used throughout the proofs.

Lemma 6.4
Let $J \unlhd L[\underline{x}]$ be an ideal and $\mathrm{Trop}(J)\cap \mathbb{Q}^n_{\leq 0} \neq \emptyset$, then $1 \notin \mathrm{in}_0(J_{R_N})$.

Proof: Let $\omega \in \mathrm{Trop}(J)\cap\mathbb{Q}^n_{\leq 0}$ and suppose that $f \in J_{R_N}$ with $\mathrm{in}_0(f) = 1$. If $t^\alpha \cdot \underline{x}^\beta$ is a monomial of f with $t^\alpha\cdot\underline{x}^\beta \neq 1$, then $\mathrm{in}_0(f) = 1$ implies $\alpha > 0$, and hence $-\alpha + \beta_1\cdot\omega_1 + \ldots + \beta_n\cdot\omega_n < 0$, since $\omega_1,\ldots,\omega_n \leq 0$ and $\beta_1,\ldots,\beta_n \geq 0$. But this shows that $\mathrm{in}_\omega(f) = 1$, and therefore $1 \in \text{t-in}_\omega(J)$, in contradiction to our assumption that $\text{t-in}_\omega(J)$ is monomial free. $\square$

Lemma 6.5
Let $I \unlhd R_N[\underline{x}]$ be an ideal such that $I = I : \left\langle t^{\frac{1}{N}}\right\rangle^\infty$ and let $P \in \mathrm{Ass}(I)$, then $P = P : \left\langle t^{\frac{1}{N}}\right\rangle^\infty$ and $t^{\frac{1}{N}} \notin P$.

Proof: Since $R_N[\underline{x}]$ is noetherian and P is an associated prime there is an $f \in R_N[\underline{x}]$ such that $P = I : \langle f\rangle$ (see [AM69] Prop. 7.17).
Suppose that $t^{\frac{\alpha}{N}}\cdot g \in P$ for some $g \in R_N[\underline{x}]$ and $\alpha > 0$. Then $t^{\frac{\alpha}{N}}\cdot g\cdot f \in I$, and since I is saturated with respect to $t^{\frac{1}{N}}$ it follows that $g\cdot f \in I$. This, however, implies that $g \in P$. Thus P is saturated with respect to $t^{\frac{1}{N}}$. If $t^{\frac{1}{N}} \in P$ then $1 \in P$, which contradicts the fact that P is a prime ideal. $\square$

Contractions of ideals in $L[\underline{x}]$ to $R_N[\underline{x}]$ are always $t^{\frac{1}{N}}$-saturated.

Lemma 6.6
Let $I \unlhd R_N[\underline{x}]$ be an ideal in $R_N[\underline{x}]$ and $J = \langle I\rangle_{L[\underline{x}]}$, then $J_{R_N} = I : \left\langle t^{\frac{1}{N}}\right\rangle^\infty$.

Proof: Since $L_N \subset L$ is a field extension [Mar07] Corollary 6.13 implies $J\cap L_N[\underline{x}] = \langle I\rangle_{L_N[\underline{x}]}$, and it suffices to see that $\langle I\rangle_{L_N[\underline{x}]} \cap R_N[\underline{x}] = I : \left\langle t^{\frac{1}{N}}\right\rangle^\infty$. If $I \cap S_N \neq \emptyset$ then both sides of the equation coincide with $R_N[\underline{x}]$, so that we may assume that $I\cap S_N$ is empty. Recall that $L_N = S_N^{-1}R_N$, so that if $f \in R_N[\underline{x}]$ with $t^{\frac{\alpha}{N}}\cdot f \in I$ for some α, then

$$f = \frac{t^{\frac{\alpha}{N}}\cdot f}{t^{\frac{\alpha}{N}}} \in \langle I\rangle_{L_N[\underline{x}]} \cap R_N[\underline{x}].$$

Conversely, if

$$f = \frac{g}{t^{\frac{\alpha}{N}}} \in \langle I\rangle_{L_N[\underline{x}]} \cap R_N[\underline{x}]$$

with $g \in I$, then $g = t^{\frac{\alpha}{N}}\cdot f \in I$ and thus f is in the right hand side. $\square$

Lemma 6.7
Let $J \trianglelefteq L[\underline{x}]$ and $N \in \mathcal{N}(J)$. Then $\text{t-in}_0(J) = \text{t-in}_0(J_{R_N})$, *and*

$$1 \notin \text{t-in}_0(J) \quad \Longleftrightarrow \quad 1 \notin \text{in}_0(J_{R_N}).$$

Proof: Suppose that $f \in J_{R_N} \subset J$ then $\text{t-in}_0(f) \in \text{t-in}_0(J)$, and if in addition $\text{in}_0(f) = 1$, then by definition $1 = \text{t-in}_0(f) \in \text{t-in}_0(J)$.
Let now $f \in J$, then by assumption there are $f_1, \ldots, f_k \in R_{N\cdot M}[\underline{x}]$ for some $M \geq 1$, $g_1, \ldots, g_k \in J_{R_N}$ and some $\alpha \geq 0$ such that

$$t^{\frac{\alpha}{M\cdot N}} \cdot f = f_1 \cdot g_1 + \ldots + f_k \cdot g_k \in R_{N\cdot M}[\underline{x}].$$

By [Mar07] Corollary 6.17 we thus get

$$\text{t-in}_0(f) = \text{t-in}_0\left(t^{\frac{\alpha}{N\cdot M}} \cdot f\right) \in \text{t-in}_0(J_{R_{N\cdot M}}) = \text{t-in}_0(J_{R_N}).$$

Moreover, if we assume that $1 = \text{t-in}_0(f) = \text{t-in}_0\left(t^{\frac{\alpha}{N\cdot M}} \cdot f\right)$ then there is an $\alpha' \geq 0$ such that

$$t^{\frac{\alpha'}{M\cdot N}} \cdot \text{t-in}_0(f) = \text{in}_0\left(t^{\frac{\alpha}{N\cdot M}} \cdot f\right) \in \text{in}_0(J_{R_{N\cdot M}}).$$

This necessarily implies that each monomial in $t^{\frac{\alpha}{N\cdot M}} \cdot f$ is divisible by $t^{\frac{\alpha'}{N\cdot M}}$, or by Lemma 6.5 equivalently that $t^{\frac{\alpha-\alpha'}{N\cdot M}} \cdot f \in J_{R_{N\cdot M}}$. But then

$$1 = \text{in}_0\left(t^{\frac{\alpha-\alpha'}{N\cdot M}} \cdot f\right) \in \text{in}_0(J_{R_{N\cdot M}}),$$

and thus by [Mar07] Corollary 6.19 also $1 \in \text{in}_0(J_{R_N})$. □

In the following lemma we gather the basic information on the ring $R_N[\underline{x}]$ which is necessary to understand how the dimension of an ideal in $L[\underline{x}]$ behaves when restricting to $R_N[\underline{x}]$.

Lemma 6.8
Consider the ring extension $R_N[\underline{x}] \subset L_N[\underline{x}]$. Then:

(a) *R_N is universally catenary, and thus $R_N[\underline{x}]$ is catenary.*
(b) *If $I \trianglelefteq R_N[\underline{x}]$, then the following are equivalent:*
  (1) $1 \notin \text{in}_0(I)$.
  (2) $\forall\, p \in R_N[\underline{x}] \;:\; 1 + t^{\frac{1}{N}} \cdot p \notin I$.
  (3) $I + \left\langle t^{\frac{1}{N}} \right\rangle \subsetneqq R_N[\underline{x}]$.
  (4) $\exists\, P \lhd R_N[\underline{x}]$ *maximal such that* $I \subseteq P$ *and* $t^{\frac{1}{N}} \in P$.
  (5) $\exists\, P \lhd R_N[\underline{x}]$ *maximal such that* $I \subseteq P$ *and* $1 \notin \text{in}_0(P)$.
  In particular, if $P \lhd R_N[\underline{x}]$ is a maximal ideal, then

$$1 \notin \text{in}_0(P) \quad \Longleftrightarrow \quad t^{\frac{1}{N}} \in P.$$

(c) *If $P \lhd R_N[\underline{x}]$ is a maximal ideal such that $1 \notin \text{in}_0(P)$, then every maximal chain of prime ideals contained in P has length $n+2$.*
(d) *If $I \trianglelefteq R_N[\underline{x}]$ is any ideal with $1 \in \text{in}_0(I)$, then $R_N[\underline{x}]/I \cong L_N[\underline{x}]/\langle I\rangle$, and $I \cap S_N = \emptyset$ unless $I = R_N[\underline{x}]$. In particular,* $\dim(I) = \dim\left(\langle I\rangle_{L_N[\underline{x}]}\right)$.
(e) *If $P \lhd R_N[\underline{x}]$ is a maximal ideal such that $1 \in \text{in}_0(P)$, then every maximal chain of prime ideals contained in P has length $n+1$.*
(f) $\dim(R_N[\underline{x}]) = n + 1$.
(g) *If $P \lhd R_N[\underline{x}]$ is a prime ideal such that $1 \notin \text{in}_0(P)$, then*

$$\dim(P) + \text{codim}(P) = \dim(R_N[\underline{x}]) = n + 1.$$

(h) *If $P \lhd R_N[\underline{x}]$ is a prime ideal such that $1 \in \mathrm{in}_0(P)$, then*

$$\dim(P) + \mathrm{codim}(P) = n.$$

Proof: For (a), see [Mat86] Thm. 29.4.
In (b), the equivalence of (1) and (2) is obvious from the definitions. Let us now use this to show that for a maximal ideal $P \lhd R_N[\underline{x}]$

$$1 \notin \mathrm{in}_0(P) \iff t^{\frac{1}{N}} \in P.$$

If $t^{\frac{1}{N}} \notin P$ then $t^{\frac{1}{N}}$ is a unit in the field $R_N[\underline{x}]/P$ and thus there is a $p \in R_N[\underline{x}]$ such that $1 \equiv t^{\frac{1}{N}} \cdot p \pmod P$, or equivalently that $1 - t^{\frac{1}{N}} \cdot p \in P$. If on the other hand $t^{\frac{1}{N}} \in P$ then $1 + t^{\frac{1}{N}} \cdot p \in P$ would imply that $1 = (1 + t^{\frac{1}{N}} \cdot p) - t^{\frac{1}{N}} \cdot p \in P$. This proves the claim and shows at the same time the equivalence of (4) and (5).
If there is a maximal ideal P containing I and such that $1 \notin \mathrm{in}_0(P)$, then of course also $1 \notin \mathrm{in}_0(I)$. Therefore (5) implies (1).
Let now I be an ideal such that $1 \notin \mathrm{in}_0(I)$. Suppose that $I + \langle t^{\frac{1}{N}} \rangle = R_N[\underline{x}]$. Then $1 = q + t^{\frac{1}{N}} \cdot p$ with $q \in I$ and $p \in R_N[\underline{x}]$, and thus $q = 1 - t^{\frac{1}{N}} \cdot p \in I$, which contradicts our assumption. Thus $I + \langle t^{\frac{1}{N}} \rangle \neq R_N[\underline{x}]$, and (1) implies (3).
Finally, if $I + \langle t^{\frac{1}{N}} \rangle \neq R_N[\underline{x}]$, then there exists a maximal ideal P such that $I + \langle t^{\frac{1}{N}} \rangle \subseteq P$. This shows that (3) implies (4), and we are done.
To see (c), note that if $1 \notin \mathrm{in}_0(P)$, then $t^{\frac{1}{N}} \in P$ by (b), and we may consider the surjection $\psi : R_N[\underline{x}] \longrightarrow R_N[\underline{x}]/\langle t^{\frac{1}{N}} \rangle = K[\underline{x}]$. The prime ideals of $K[\underline{x}]$ are in $1:1$-correspondence with those prime ideals of $R_N[\underline{x}]$ which contain $t^{\frac{1}{N}}$. In particular, $P/\langle t^{\frac{1}{N}} \rangle = \psi(P)$ is a maximal ideal of $K[\underline{x}]$ and thus any maximal chain of prime ideals in P which starts with $\langle t^{\frac{1}{N}} \rangle$, say $\langle t^{\frac{1}{N}} \rangle = P_0 \subset \ldots \subset P_n = P$ has precisely $n+1$ terms since every maximal chain of prime ideals in $K[\underline{x}]$ has that many terms. Moreover, by Krull's Principal Ideal Theorem (see e.g. [AM69] Cor. 11.17) the prime ideal $\langle t^{\frac{1}{N}} \rangle$ has codimension 1, so that the chain of prime ideals

$$\langle 0 \rangle \subset \langle t^{\frac{1}{N}} \rangle = P_0 \subset \ldots \subset P_n = P$$

is maximal. Since by (a) the ring $R_N[\underline{x}]$ is catenary every maximal chain of prime ideals in between $\langle 0 \rangle$ and P has the same length $n+2$.
For (d), we assume that there exists an element $1 + t^{\frac{1}{N}} \cdot p \in I$ due to (b). But then $t^{\frac{1}{N}} \cdot (-p) \equiv 1 \pmod I$. Thus the elements of $S_N = \left\{1, t^{\frac{1}{N}}, t^{\frac{2}{N}}, \ldots\right\}$ are invertible modulo I. Therefore

$$R_N[\underline{x}]/I \cong S_N^{-1}(R_N[\underline{x}]/I) \cong S_N^{-1} R_N[\underline{x}]/S_N^{-1} I = L_N[\underline{x}]/\langle I \rangle.$$

In particular, if $I \neq R_N[\underline{x}]$ then $\langle I \rangle \neq L_N[\underline{x}]$ and thus $I \cap S_N = \emptyset$.
To show (e), note that by assumption there is an element $1 + t^{\frac{1}{N}} \cdot p \in P$ due to (b), and since P is maximal $p \notin R_N$. Choose a prime ideal Q contained in P which is minimal w.r.t. the property that it contains $1 + t^{\frac{1}{N}} \cdot p$. Since $1 + t^{\frac{1}{N}} \cdot p$ is neither a unit nor a zero divisor Krull's Principal Ideal Theorem implies that $\mathrm{codim}(Q) = 1$. Moreover, since $Q \cap S_N = \emptyset$ by Part (d) the ideal $\langle Q \rangle_{L_N[\underline{x}]}$ is a prime ideal which is minimal over $1 + t^{\frac{1}{N}} \cdot p$ by the one-to-one correspondence of prime ideals under localisation. Since every maximal chain of primes in $L_N[\underline{x}]$ has length n , and by Part (d) we have $\dim(Q) = \dim\left(\langle Q \rangle_{L_N[\underline{x}]}\right) = n-1$. Hence there is a maximal chain of prime ideals of length n from $\langle Q \rangle_{L_N[\underline{x}]}$ to $\langle P \rangle_{L_N[\underline{x}]}$. Since $\mathrm{codim}(Q) = 1$ it follows that there is a chain of prime ideals of length $n+1$ starting at $\langle 0 \rangle$ and

ending at P which cannot be prolonged. But by (a) the ring $R_N[\underline{x}]$ is catenary, and thus every maximal chain of prime ideals in P has length $n+1$.
Claim (f) follows from (c) and (e).
To see (g), note that by (b) there exists a maximal ideal Q containing P and $t^{\frac{1}{N}}$. If $k = \operatorname{codim}(P)$ then we may choose a maximal chain of prime ideals of length $k+1$ in P, and we may prolong it by at most $\dim(P)$ prime ideal to a maximal chain of prime ideals in Q, which by (b) and (c) has length $n+2$. Taking (f) into account this shows that

$$\dim(P) \geq (n+2) - (k+1) = \dim(R_N[\underline{x}]) - \operatorname{codim}(P).$$

However, the converse inequality always holds, which finishes the proof.
For (h) note that by (b) there is no maximal ideal which contains $t^{\frac{1}{N}}$ so that every maximal ideal containing P has codimension n. The result then follows as in (g). $\square$

Corollary 6.9
Let $P \lhd L[\underline{x}]$ be a prime ideal and $N \geq 1$, then

$$\dim(P_{R_N}) = \dim(P) + 1 \iff 1 \notin \operatorname{in}_0(P_{R_N}), \quad \textit{and}$$
$$\dim(P_{R_N}) = \dim(P) \iff 1 \in \operatorname{in}_0(P_{R_N}).$$

In any case

$$\operatorname{codim}(P_{R_N}) = \operatorname{codim}(P).$$

Proof: Since the field extension $L_N \subset L$ is algebraic by Lemma 5.5 we have

$$\dim(P) = \dim\big(P \cap L_N[\underline{x}]\big) \tag{9}$$

in any case. If $1 \in \operatorname{in}_0\big(P_{R_N}\big)$, then Lemma 6.8(d) implies

$$\dim\big(P_{R_N}\big) = \dim\big(\langle P_{R_N}\rangle_{L_N[\underline{x}]}\big) = \dim\big(P \cap L_N[\underline{x}]\big),$$

since $L_N[\underline{x}]$ is a localisation of $R_N[\underline{x}]$.
It thus suffices to show that $\dim\big(P_{R_N}\big) = \dim(P) + 1$ if $1 \notin \operatorname{in}_0\big(P_{R_N}\big)$.
Since $P \neq L[\underline{x}]$ we know that $S_N \cap P = \emptyset$. The $1:1$-correspondence of prime ideals under localisation thus shows that

$$l := \operatorname{codim}\big(P \cap L_N[\underline{x}]\big) = \operatorname{codim}\big(P_{R_N}\big).$$

Hence there exists a maximal chain of prime ideals

$$\langle 0 \rangle = Q_0 \subsetneqq \ldots \subsetneqq Q_l = P_{R_N}$$

of length $l+1$ in $R_N[\underline{x}]$. Note also that by (9)

$$l = \operatorname{codim}\big(P \cap L_N[\underline{x}]\big) = n - \dim\big(P \cap L_N[\underline{x}]\big) = n - \dim(P), \tag{10}$$

since $L_N[\underline{x}]$ is a polynomial ring over a field.
Moreover, since $1 \notin \operatorname{in}_0\big(P_{R_N}\big)$ by Lemma 6.8(b), there exists a maximal ideal $Q \lhd R_N[\underline{x}]$ containing P_{R_N} such that $1 \notin \operatorname{in}_0(Q)$. Choose a maximal chain of prime ideals

$$P_{R_N} = Q_l \subsetneqq Q_{l+1} \subsetneqq \ldots \subsetneqq Q_k = Q$$

in $R_N[\underline{x}]$ from P_{R_N} to Q, so that taking (10) into account

$$\dim(P_{R_N}) \geq k - l = k - n + \dim(P). \tag{11}$$

Finally, since the sequence

$$\langle 0 \rangle = Q_0 \subsetneqq Q_1 \subsetneqq \ldots \subsetneqq Q_l \subsetneqq \ldots \subsetneqq Q_k = Q$$

cannot be prolonged and since $1 \notin \mathrm{in}_0(Q)$, Lemma 6.8(c) implies that $k = n+1$. But since we always have

$$\dim\left(P_{R_N}\right) \leq \dim\left(R_N[\underline{x}]\right) - \mathrm{codim}\left(P_{R_N}\right) = n+1-l,$$

it follows from (10) and (11)

$$\dim(P)+1 \leq \dim\left(P_{R_N}\right) \leq n+1-l = \dim(P)+1.$$

The claim for the codimensions then follows from Lemma 6.8 (g) and (h). □

As an immediate corollary we get one of the main results of this section.

Theorem 6.10
Let $J \trianglelefteq L[\underline{x}]$ and $N \in \mathcal{N}(J)$. Then $\dim\left(J_{R_N}\right) = \dim(J)+1$ if and only if $\exists\, P \in \mathrm{Ass}(J)$ s.t. $\dim(P) = \dim(J)$ and $1 \notin \mathrm{in}_0\left(P_{R_N}\right)$. Otherwise $\dim\left(J_{R_N}\right) = \dim(J)$.

Proof: If there is such a $P \in \mathrm{Ass}(J)$ then Corollary 6.9 implies

$$\begin{aligned}\dim\left(P_{R_N}\right) &= \dim(P)+1 = \dim(J)+1 \text{ and}\\ \dim\left(P'_{R_N}\right) &\leq \dim(P')+1 \leq \dim(J)+1\end{aligned}$$

for any other $P' \in \mathrm{Ass}(J)$. This shows that

$$\dim\left(J_{R_N}\right) = \max\{\dim\left(P'_{R_N}\right) \bigm| P' \in \mathrm{Ass}(J)\} = \dim(J)+1,$$

due to Proposition 5.3.
If on the other hand $1 \in \mathrm{in}_0\left(P_{R_N}\right)$ for all $P \in \mathrm{Ass}(J)$ with $\dim(P) = \dim(J)$, then again by Corollary 6.9 $\dim(P_{R_N}) \leq \dim(J)$ for all associated primes with equality for some, and we are done with Proposition 5.3. □

It remains to show that also the dimension of the t-initial ideal behaves well.

Proposition 6.11
Let $I \trianglelefteq R_N[\underline{x}]$ be an ideal such that $I = I : \langle t^{\frac{1}{N}}\rangle^\infty$ and such that $1 \notin \mathrm{in}_0(P)$ for some $P \in \mathrm{Ass}(I)$ with $\dim(P) = \dim(I)$. Then

$$\dim(I) = \dim\left(\text{t-in}_0(I)\right)+1.$$

More precisely, $\dim(Q') = \dim(P)-1$ *for all* $Q' \in \mathrm{minAss}\left(\text{t-in}_0(P)\right)$.

Proof: We first want to show that

$$\text{t-in}_0(I) = \left(I + \left\langle t^{\frac{1}{N}}\right\rangle\right) \cap K[\underline{x}].$$

Any element $f \in \langle t^{\frac{1}{N}}\rangle + I$ can be written as $f = t^{\frac{1}{N}} \cdot g + h$ with $g \in R_N[\underline{x}]$ and $h \in I$ such that $\mathrm{in}_0(h) \in K[\underline{x}]$, and if in addition $f \in K[\underline{x}]$ then obviously $f = \mathrm{in}_0(h) = \text{t-in}_0(h) \in \text{t-in}_0(I)$. If, on the other hand, $g = \text{t-in}_0(f) \in \text{t-in}_0(I)$ for some $f \in I$, then $t^{\frac{\alpha}{N}} \cdot g = \mathrm{in}_0(f) \in \mathrm{in}_0(I)$ for some $\alpha \geq 0$, and every monomial in f is necessarily divisible by $t^{\frac{\alpha}{N}}$. Thus $f = t^{\frac{\alpha}{N}} \cdot h$ for some $h \in R_N[\underline{x}]$ and $g = \mathrm{in}_0(h) \equiv h \pmod{\langle t^{\frac{1}{N}}\rangle}$. But since I is saturated with respect to $t^{\frac{1}{N}}$ it follows that $h \in I$, and thus g is in the right hand side. This proves the claim.
Therefore, the inclusion $K[\underline{x}] \hookrightarrow R_N[\underline{x}]$ induces an isomorphism

$$K[\underline{x}]/\,\text{t-in}_0(I) \cong R_N[\underline{x}]/\left(\langle t^{\frac{1}{N}}\rangle + I\right) \tag{12}$$

which shows that

$$\dim\left(K[\underline{x}]/\,\text{t-in}_0(I)\right) = \dim\left(R_N[\underline{x}]/\left(I + \left\langle t^{\frac{1}{N}}\right\rangle\right)\right). \tag{13}$$

Next, we want to show that

$$\dim\left(P+\langle t^{\frac{1}{N}}\rangle\right)=\dim(P)-1=\dim(I)-1. \tag{14}$$

For this we consider an arbitrary $P'\in\operatorname{minAss}\left(P+\langle t^{\frac{1}{N}}\rangle\right)$. By Lemma 6.8 (b), $1\not\in\operatorname{in}_0(P')$. Applying Lemma 6.8 (g) to P and P' we get

$$\dim(R_N[\underline{x}])=\dim(P)+\operatorname{codim}(P)\quad\text{and}\quad\dim(R_N[\underline{x}])=\dim(P')+\operatorname{codim}(P').$$

Moreover, since I is saturated with respect to $t^{\frac{1}{N}}$ by Lemma 6.5 P does not contain $t^{\frac{1}{N}}$. Thus $t^{\frac{1}{N}}$ is neither a zero divisor nor a unit in $R_N[\underline{x}]/P$, and by Krull's Principal Ideal Theorem (see [AM69] Cor. 11.17) we thus get $\operatorname{codim}(P')=\operatorname{codim}(P)+1$, since by assumption P' is minimal over $t^{\frac{1}{N}}$ in $R_N[\underline{x}]/P$. Plugging the two previous equations in we get

$$\dim(P')=\dim(P)-1. \tag{15}$$

This proves (14), since P' was an arbitrary minimal associated prime of $P+\langle t^{\frac{1}{N}}\rangle$. We now claim that

$$\dim\left(P+\langle t^{\frac{1}{N}}\rangle\right)=\dim\left(I+\langle t^{\frac{1}{N}}\rangle\right). \tag{16}$$

Suppose this is not the case, then there is a $P'\in\operatorname{Ass}\left(I+\langle t^{\frac{1}{N}}\rangle\right)$ such that

$$\dim(P')>\dim\left(P+\langle t^{\frac{1}{N}}\rangle\right)=\dim(I)-1,$$

and since $I\subset P'$ it follows that

$$\dim(P')=\dim(I).$$

But then P' is necessarily a minimal associated prime of I in contradiction to Lemma 6.5, since P' contains $t^{\frac{1}{N}}$. This proves (16).

Equations (13), (14) and (16) finish the proof of the first claim. For the "more precisely" part notice that replacing I by P in (12) we see that there is a dimension preserving $1:1$-correspondence between $\operatorname{minAss}\big(P+\langle t^{\frac{1}{N}}\rangle\big)$ and $\operatorname{minAss}\big(\text{t-in}_0(P)\big)$. The result then follows from (15). □

Remark 6.12

The condition that I is saturated with respect to $t^{\frac{1}{N}}$ in Proposition 6.11 is equivalent to the fact that I is the contraction of the ideal $\langle I\rangle_{L_N[\underline{x}]}$. Moreover, it implies that $R_N[\underline{x}]/I$ is a flat R_N-module, or alternatively that the family

$$\iota^*:\operatorname{Spec}\big(R_N[\underline{x}]/I\big)\longrightarrow\operatorname{Spec}(R_N)$$

is flat, where the generic fibre is just $\operatorname{Spec}\big(L_N[\underline{x}]/\langle I\rangle\big)$ and the special fibre is $\operatorname{Spec}\big(K[\underline{x}]/\text{t-in}_0(I)\big)$. The condition $1\not\in\operatorname{in}_0(P)$ implies that the component of $\operatorname{Spec}\big(R_N[\underline{x}]/I\big)$ defined by P surjects onto $\operatorname{Spec}(R_N)$. With this interpretation the proof of Proposition 6.11 is basically exploiting the dimension formula for local flat extensions.

Corollary 6.13

Let $J\lhd L[\underline{x}]$ and $\omega\in\mathbb{Q}^n$, then

$$\dim\big(\text{t-in}_\omega(J)\big)=\max\big\{\dim(P)\;\big|\;P\in\operatorname{Ass}(J)\;:\;1\not\in\text{t-in}_\omega(P)\big\}.$$

Moreover, if J is prime, $1\not\in\text{t-in}_\omega(J)$ and $Q'\in\operatorname{minAss}\big(\text{t-in}_\omega(J)\big)$ then

$$\dim(Q')=\dim(J).$$

Proof: Let $J = Q_1 \cap \ldots \cap Q_k$ be a minimal primary decomposition of J, and

$$\Phi_\omega(J) = \Phi_\omega(Q_1) \cap \ldots \cap \Phi_\omega(Q_k)$$

the corresponding minimal primary decomposition of $\Phi_\omega(J)$. If we define a new ideal

$$J' = \bigcap_{1 \not\in \text{t-in}_0\left(\sqrt{\Phi_\omega(Q_i)}\right)} \Phi_\omega(Q_i),$$

then this representation is already a minimal primary decomposition of J'. Choose an N such that $N \in \mathcal{N}(J)$, $N \in \mathcal{N}(J')$ and $N \in \mathcal{N}\big(\Phi_\omega(Q_i)\big)$ for all $i = 1, \ldots, k$. By Lemma 6.7 we have

$$1 \not\in \text{t-in}_0\left(\sqrt{\Phi_\omega(Q_i)}\right) \iff 1 \not\in \text{in}_0\left(\sqrt{\Phi_\omega(Q_i)} \cap R_N[\underline{x}]\right). \tag{17}$$

Proposition 5.3 implies

$$\text{Ass}(J_{R_N}) = \left\{ \sqrt{\Phi_\omega(Q_i)} \cap R_N[\underline{x}] \;\middle|\; i = 1, \ldots, k \right\}$$

where the $\sqrt{\Phi_\omega(Q_i)} \cap R_N[\underline{x}]$ are not necessarily pairwise different, and

$$\text{Ass}(J'_{R_N}) = \left\{ \sqrt{\Phi_\omega(Q_i)} \cap R_N[\underline{x}] \;\middle|\; 1 \not\in \text{in}_0\left(\sqrt{\Phi_\omega(Q_i)} \cap R_N[\underline{x}]\right) \right\},$$

for which we have to take (17) into account.
Moreover, by Lemma 6.6 J'_{R_N} is saturated with respect to $t^{\frac{1}{N}}$. Thus we can apply Proposition 6.11 to J'_{R_N} to deduce $\dim(J'_{R_N}) = \dim\left(\text{t-in}_0(J'_{R_N})\right) + 1$.
Taking (17) into account we can apply Theorem 6.10 to J' and deduce that then $\dim(J'_{R_N}) = \dim(J') + 1$, but

$$\begin{aligned} \dim(J') &= \max\left\{ \dim\left(\sqrt{\Phi_\omega(Q_i)}\right) \mid 1 \not\in \text{t-in}_0\left(\sqrt{\Phi_\omega(Q_i)}\right)\right\} \\ &= \max\left\{ \dim\left(\sqrt{Q_i}\right) \mid 1 \not\in \text{t-in}_\omega\left(\sqrt{Q_i}\right)\right\}. \end{aligned}$$

It remains to show that $\text{t-in}_0(J'_{R_N}) = \text{t-in}_\omega(J)$. By Lemma 6.7 and Definition 3.3 we have $\text{t-in}_0(J'_{R_N}) = \text{t-in}_0(J')$ and

$$\text{t-in}_\omega(J) = \text{t-in}_0\left(\Phi_\omega(J)\right) \subseteq \text{t-in}_0(J'),$$

since $J \subseteq J'$. By assumption for any $\sqrt{\Phi_\omega(Q_i)} \not\in \text{Ass}(J')$ there is an $f_i \in \sqrt{\Phi_\omega(Q_i)}$ such that $\text{t-in}_0(f_i) = 1$ and there is some m_i such that $f_i^{m_i} \in \Phi_\omega(Q_i)$. If $f \in J'$ is any element, then for

$$g := f \cdot \prod_{\sqrt{\Phi_\omega(Q_i)} \not\in \text{Ass}(J')} f_i^{m_i} \in \Big(J' \cdot \prod_{\sqrt{\Phi_\omega(Q_i)} \not\in \text{Ass}(J')} \Phi_\omega(Q_i)\Big) \subseteq J$$

we have

$$\text{t-in}_0(f) = \text{t-in}_0(f) \cdot \prod_{\sqrt{\Phi_\omega(Q_i)} \not\in \text{Ass}(J')} \text{t-in}_0(f_i)^{m_i} = \text{t-in}_0(g) \in \text{t-in}_0(J).$$

This finishes the proof of the first claim.
For the "moreover" part note that by Lemma 6.7

$$\text{t-in}_\omega(J) = \text{t-in}_0\left(\Phi_\omega(J)\right) = \text{t-in}_0\left(\Phi_\omega(J) \cap R_N[\underline{x}]\right)$$

and $\Phi_\omega(J) \cap R_N[\underline{x}]$ is saturated and prime. Applying Proposition 6.11 to

$$Q' \in \text{minAss}\Big(\text{t-in}_0\left(\Phi_\omega(J) \cap R_N[\underline{x}]\right)\Big) = \text{minAss}\left(\text{t-in}_\omega(J)\right)$$

we get

$$\dim(Q') = \dim\big(\Phi_\omega(J) \cap R_N[\underline{x}]\big) - 1 = \dim(J),$$

where the latter equality is due to Corollary 6.9. □

Theorem 6.14
Let $J \lhd L[\underline{x}]$, $N \in \mathcal{N}(J)$ and $\omega \in \mathbb{Q}^n_{\leq 0}$.
If there is a $P \in \mathrm{Ass}(J)$ with $\dim(P) = \dim(J)$ and $\omega \in \mathrm{Trop}(P)$, then

$$\dim(J_{R_N}) = \dim(J) + 1 = \dim\big(\text{t-in}_\omega(J)\big) + 1.$$

Proof: By Lemma 6.4 the condition $\omega \in \mathrm{Trop}(P) \cap \mathbb{Q}^n_{\leq 0}$ implies that $1 \notin \mathrm{in}_0(P_{R_N})$. The result then follows from Theorem 6.10 and Corollary 6.13. □

Corollary 6.15
If $J \unlhd L[\underline{x}]$ is zero dimensional and $\omega \in \mathrm{Trop}(J)$, then $\dim\big(\text{t-in}_\omega(J)\big) = \dim(J) = 0$. If in addition $\mathrm{Trop}(J) \cap \mathbb{Q}^n_{\leq 0} \neq \emptyset$ and $N \in \mathcal{N}(J)$ $\dim\big(J_{R_N}\big) = 1$.

Proof: Since $\dim(J) = 0$ also $\dim(P) = 0$ for every associated prime P. By 2.12 there exists a P with $\omega \in \mathrm{Trop}(P)$. The first assertion thus follows from Corollary 6.13. The second assertion follows from Theorem 6.14. □

When cutting down the dimension we need to understand how the minimal associated primes of J and J_{R_N} relate to each other.

Lemma 6.16
Let $J \lhd L[\underline{x}]$ be equidimensional and $N \in \mathcal{N}(J)$. Then

$$\mathrm{minAss}(J_{R_N}) = \{P_{R_N} \mid P \in \mathrm{minAss}(J)\}.$$

Proof: The left hand side is contained in the right hand side by default (see Proposition 5.3). Let therefore $P \in \mathrm{minAss}(J)$ be given. By Proposition 5.3 $P_{R_N} \in \mathrm{Ass}(J)$, and it suffices to show that it is minimal among the associated primes. Suppose therefore we have $Q \in \mathrm{Ass}(J)$ such that $Q_{R_N} \subseteq P_{R_N}$. By Corollary 6.9 and the assumption we have

$$\mathrm{codim}(P_{R_N}) = \mathrm{codim}(P) \leq \mathrm{codim}(Q) = \mathrm{codim}(Q_{R_N}),$$

so that indeed $P_{R_N} = Q_{R_N}$. □

Another consequence is that the t-initial ideal of an equidimensional ideal is again equidimensional.

Corollary 6.17
Let $J \lhd L[\underline{x}]$ be an equidimensional ideal and $\omega \in \mathbb{Q}^n$, then

$$\mathrm{minAss}\big(\text{t-in}_\omega(J)\big) = \bigcup_{P \in \mathrm{minAss}(J)} \mathrm{minAss}\big(\text{t-in}_\omega(P)\big).$$

In particular, if there is a $P \in \mathrm{minAss}(J)$ such that $1 \notin \text{t-in}_\omega(P)$ then $\text{t-in}_\omega(J)$ is equidimensional of dimension $\dim(J)$.

Proof: Applying Φ_ω we may assume that $\omega = 0$, and we then may choose an $N \in \mathcal{N}(J)$ and $N \in \mathcal{N}(P)$ for all $P \in \mathrm{minAss}(J)$.
Denoting by

$$\pi : R_N[\underline{x}] \longrightarrow R_N[\underline{x}]/\langle t^{\frac{1}{N}} \rangle = K[\underline{x}]$$

the residue class map we get

$$\begin{aligned}\text{t-in}_0(J) &= \text{t-in}_0(J_{R_N}) = \pi\big(J_{R_N} + \langle t^{\frac{1}{N}}\rangle\big) \text{ and}\\ \text{t-in}_0(P) &= \text{t-in}_0(P_{R_N}) = \pi\big(P_{R_N} + \langle t^{\frac{1}{N}}\rangle\big)\end{aligned}$$

for all $P \in \operatorname{minAss}(J)$, where the first equality in both cases is due to Lemma 6.7 and where the last equality uses Lemma 6.6. Since there is a one-to-one correspondence between prime ideals in $K[\underline{x}]$ and prime ideals in $R_N[\underline{x}]$ which contain $t^{\frac{1}{N}}$, it suffices to show that

$$\operatorname{minAss}\big(J_{R_N} + \langle t^{\frac{1}{N}}\rangle\big) = \bigcup_{P\in\operatorname{minAss}(J)} \operatorname{minAss}\big(P_{R_N} + \langle t^{\frac{1}{N}}\rangle\big).$$

However, since the P_{R_N} are saturated with respect to $t^{\frac{1}{N}}$ by Lemma 6.6 they do not contain $t^{\frac{1}{N}}$. By Corollary 6.9 all P_{R_N} have the same codimension, since the P do by assumption. By Lemma 6.16,

$$\operatorname{minAss}(J_{R_N}) = \{P_{R_N} \mid P \in \operatorname{minAss}(J)\}.$$

Hence the result follows by Lemma 5.6.
The "in particular" part follows from Corollary 6.13. □

7. Computing t-Initial Ideals

This section is devoted to an alternative proof of Theorem 2.8 which does not need standard basis in the mixed power series polynomial ring $K[[t]][\underline{x}]$.
The following lemma is easy to show.

Lemma 7.1
Let $w \in \mathbb{R}_{<0} \times \mathbb{R}^n$, $0 \neq f = \sum_{i=1}^k g_i \cdot h_i$ with $f, g_i, h_i \in R_N[\underline{x}]$ and $\operatorname{ord}_w(f) \geq \operatorname{ord}_w(g_i \cdot h_i)$ for all $i = 1,\ldots,k$. Then

$$\operatorname{in}_w(f) \in \big\langle \operatorname{in}_w(g_1),\ldots,\operatorname{in}_w(g_k)\big\rangle \lhd K\big[t^{\frac{1}{N}},\underline{x}\big].$$

Proposition 7.2
Let $I \unlhd K\big[t^{\frac{1}{N}},\underline{x}\big]$, $\omega \in \mathbb{Q}^n$ and G be a standard basis of I with respect to the monomial ordering $>_\omega$ introduced in Remark 3.7. Then

$$\operatorname{in}_\omega(I) = \big\langle \operatorname{in}_\omega(G)\big\rangle \unlhd K\big[t^{\frac{1}{N}},\underline{x}\big] \quad \textit{and} \quad \text{t-in}_\omega(I) = \big\langle \text{t-in}_\omega(G)\big\rangle \unlhd K[\underline{x}].$$

Proof: It suffices to show that $\operatorname{in}_\omega(f) \in \langle\operatorname{in}_\omega(G)\rangle$ for every $f \in I$. Since $f \in I$ and G is a standard basis of I there exists a weak standard representation

$$u \cdot f = \sum_{g\in G} q_g \cdot g \tag{18}$$

of f where the leading term of u with respect to $>_\omega$ is $\operatorname{lt}_{>_\omega}(u) = 1$. But then the definition of $>_\omega$ implies that automatically $\operatorname{in}_\omega(u) = 1$. Since (18) is a standard representation we have $\operatorname{lm}_{>_\omega}(u\cdot f) \geq \operatorname{lm}_{>_\omega}(q_g \cdot g)$ for all g. But this necessarily implies that $\operatorname{ord}_w(f) \geq \operatorname{ord}_w(q_g \cdot g)$ where $w = (-1,\omega)$. Since $K\big[t^{\frac{1}{N}},\underline{x}\big] \subset R_N[\underline{x}]$ we can use Lemma 7.1 to show

$$\operatorname{in}_w(f) = \operatorname{in}_w(u\cdot f) \in \langle \operatorname{in}_w(g) \mid g \in G\rangle \unlhd K\big[t^{\frac{1}{N}},\underline{x}\big].$$

□

Proposition 7.3
Let $I \subseteq K[t,x]$ be an ideal, $J = \langle I\rangle_{L[x]}$ and $\omega \in \mathbb{R}^n$. Then $\text{t-in}_\omega(I) = \text{t-in}_\omega(J)$.

Proof: We need to prove the inclusion $\text{t-in}_\omega(I) \supseteq \text{t-in}_\omega(J)$. The other inclusion is clear since $I \subseteq J$. The right hand side is generated by elements of the form $f = \text{t-in}_\omega(g)$ where $g \in J$. Consider such f and g. The polynomial g must be of the form $g = \sum_i c_i \cdot g_i$ where $g_i \in I$ and $c_i \in L$. Let d be the $(-1,\omega)$-degree of $\text{in}_\omega(g)$. The degrees of terms in g_i are bounded. Terms $a \cdot t^\beta$ in c_i of large enough t-degree will make the $(-1,\omega)$-degree of $a \cdot t^\beta \cdot g_i$ drop below d since the degree of t is negative. Consequently, these terms can simply be ignored since they cannot affect the initial form of $g = \sum_i c_i \cdot g_i$. Renaming and possibly repeating some g_i's we may write g as a finite sum $g = \sum_i c'_i \cdot g_i$ where $c'_i = a_i \cdot t^{\beta_i}$ and $g_i \in I$ with $a_i \in K$ and $\beta_i \in \mathbb{Q}$. We will split the sum into subsums grouping together the c'_i's that have the same t-exponent modulo $\mathbb{Z}$. For suitable index sets A_j we let $g = \sum_j G_j$ where $G_j = \sum_{i \in A_j} c'_i \cdot g_i$. Notice that all t-exponents in a G_j are congruent modulo $\mathbb{Z}$ while t-exponents from different G_j's are not. In particular there is no cancellation in the sum $g = \sum_j G_j$. As a consequence $\text{in}_\omega(g) = \sum_{j \in S} \text{in}_\omega(G_j)$ for a suitable subset S. We also have $\text{t-in}_\omega(g) = \sum_{j \in S} \text{t-in}_\omega(G_j)$. We wish to show that each $\text{t-in}_\omega(G_j)$ is in $\text{t-in}(I)$. We can write $t^{\gamma_j} \cdot G_j = \sum_{i \in A_j} t^{\gamma_j} \cdot c'_i \cdot g_i$ for suitable $\gamma_j \in \mathbb{Q}$ such that $t^{\gamma_j} \cdot c'_i \in K[t]$ for all $i \in A_j$. Observe that

$$\text{t-in}_\omega(G_j) = \text{t-in}_\omega(t^{\gamma_j} \cdot G_j) = \text{t-in}_\omega\Big(\sum_{i \in A_j} t^{\gamma_j} \cdot c'_i \cdot g_i\Big) \in \text{t-in}_\omega(I).$$

Applying $\text{t-in}_\omega(g) = \sum_{j \in S} \text{t-in}_\omega(G_j)$ we see that $f = \text{t-in}_\omega(g) \in \text{t-in}_\omega(I)$. □

By substituting $t := t^{\frac{1}{n}}$ and scaling ω we get Theorem 2.8 as a corollary.

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