SMOOTHNESS OF EQUISINGULAR FAMILIES OF CURVES

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Abstract. Francesco Severi (cf. [Sev21]) showed that equisingular families of plane nodal curves are T-smooth, i. e. smooth of the expected dimension, whenever they are non-empty. For families with more complicated singularities this is no longer true. Given a divisor D on a smooth projective surface Σ it thus makes sense to look for conditions which ensure that the family $V_{|D|}^{irr}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$ of irreducible curves in the linear system $|D|_l$ with precisely r singular points of types $\mathcal{S}_1,\ldots,\mathcal{S}_r$ is T-smooth. Considering different surfaces including the projective plane, general surfaces in $\mathbb{P}^3_{\mathbb{C}}$, products of curves and geometrically ruled surfaces, we produce a sufficient condition of the type

$$\sum_{i=1}^{r}\gamma_\alpha(\mathcal{S}_i) < \gamma\cdot(D-K_\Sigma)^2,$$

where γ_α is some invariant of the singularity type and γ is some constant. This generalises the results in [GLS01] for the plane case, combining their methods and the method of Bogomolov instability, used in [ChS97] and [GLS97]. For many singularity types the γ_α-invariant leads to essentially better conditions than the invariants used in [GLS97], and for most classes of geometrically ruled surfaces our results are the first known for T-smoothness at all.

1. Introduction

The varieties $V_{|D|}(rA_1)$ (respectively the open subvarieties $V_{|D|}^{irr}(rA_1)$) of reduced (respectively reduced and irreducible) nodal curves in a fixed linear system $|D|_l$ on a smooth projective surface Σ are also called *Severi varieties.* When $\Sigma=\mathbb{P}^2_{\mathbb{C}}$ Severi showed that these varieties are smooth of the expected dimension, whenever they are non-empty – that is, nodes always impose independent conditions. It seems natural to study this question on other surfaces, but it is not surprising that the situation becomes harder.

Tannenbaum showed in [Tan82] that also on K3-surfaces $V_{|D|}(rA_1)$ is always smooth, that, however, the dimension is larger than the expected one and thus $V_{|D|}(rA_1)$ is not T-smooth in this situation. If we restrict our attention to the subvariety $V_{|D|}^{irr}(rA_1)$ of *irreducible* curves with r nodes, then we gain T-smoothness again whenever the variety is non-empty. That is, while on a K3-surface the conditions which nodes impose on irreducible curves are always independent, they impose dependent conditions on reducible curves.

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On more complicated surfaces the situation becomes even worse. Chiantini and Sernesi study in [ChS97] Severi varieties on surfaces in $\mathbb{P}^3_{\mathbb{C}}$. They show that on a generic quintic Σ in $\mathbb{P}^3_{\mathbb{C}}$ with hyperplane section H the variety $V^{irr}_{|dH|}\big(\frac{5d(d-2)}{4}\cdot A_1\big)$ has a non-smooth reduced component of the expected dimension, if d is even. They construct their examples by intersecting a general cone over Σ in $\mathbb{P}^4_{\mathbb{C}}$ with a general complete intersection surface of type $\big(2,\frac{d}{2}\big)$ in $\mathbb{P}^4_{\mathbb{C}}$ and projecting the resulting curve to Σ in $\mathbb{P}^3_{\mathbb{C}}$. Moreover, Chiantini and Ciliberto give in [ChC99] examples showing that the Severi varieties $V^{irr}_{|dH|}(rA_1)$ on a surface in $\mathbb{P}^3_{\mathbb{C}}$ also may have components of dimension larger than the expected one.

Hence, one can only ask for numerical conditions ensuring that $V^{irr}_{|dH|}(rA_1)$ is T-smooth, and Chiantini and Sernesi answer this question by showing that on a surface of degree $n\geq 5$ the condition

$$r<\frac{d(d-2n+8)n}{4} \tag{1.1}$$

implies that $V^{irr}_{|dH|}(rA_1)$ is T-smooth for $d>2n-8$. Note that the above example shows that this bound is even sharp. Actually Chiantini and Sernesi prove a somewhat more general result for surfaces with ample canonical divisor K_Σ and curves which are in $|pK_\Sigma|_l$ for some $p\in\mathbb{Q}$. For their proof they suppose that for some curve $C\in V^{irr}_{|dH|}(rA_1)$ the cohomology group $H^1\big(\Sigma,\mathcal{J}_{X^*(C)/\Sigma}(D)\big)$ does not vanish and derive from this the existence of a Bogomolov unstable rank-two bundle E. This bundle in turn provides them with a curve Δ of small degree realising a large part of the zero-dimensional scheme $X^*(C)$, which leads to the desired contradiction.

This is basically the same approach used in [GLS97]. However, they allow arbitrary singularities rather than only nodes, and get in the case of a surface in $\mathbb{P}^3_{\mathbb{C}}$ of degree n

$$\sum_{i=1}^r\big(\tau^*_{ci}(\mathcal{S}_i)+1\big)^2<d\cdot\big(d-(n-4)\cdot\max\big\{\tau^*_{ci}(\mathcal{S}_i)+1\;\big|\;i=1,\ldots,r\big\}\big)\cdot n$$

as main condition for T-smoothness of $V^{irr}_{|dH|}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$, which for nodal curves coincides with (1.1). Moreover, for families of plane curves of degree d their result gives

$$\sum_{i=1}^r\big(\tau^*_{ci}(\mathcal{S}_i)+1\big)^2<d^2+6d$$

as sufficient condition for T-smoothness, which is weaker than the sufficient condition

$$\sum_{i=1}^r\gamma_1^*(\mathcal{S}_i)\leq(d+3)^2 \tag{1.2}$$

derived in [GLS00] and [GLS01] using the Castelnuovo function in order to provide a curve of small degree which realises a large part of $X^*(C)$. The advantage of the γ_1^*-invariant is that, while always bounded from above by $(\tau^*_{ci}+1)^2$, in many cases it is substantially smaller – e. g. for an ordinary m-fold point M_m, $m\geq 3$, we have $\gamma_1^{es}(M_m)=2m^2$, while

$$\big(\tau^{es}_{ci}(M_m)+1\big)^2\geq\frac{(m^2+2m+4)^2}{16}.$$

In this paper we combine the methods of [GLS00] and the method of Bogomolov instability to reproduce the result (1.2) in the plane case, and to derive a similar sufficient condition,

$$\sum_{i=1}^{r} \gamma_\alpha(\mathcal{S}_i) < \gamma \cdot (D - K_\Sigma)^2,$$

for T-smoothness on other surfaces – involving a generalisation γ_α^* of the γ_1^*-invariant which is always bounded from above by the latter one.

Note that a series of irreducible plane curves of degree d with r singularities of type A_k, k arbitrarily large, satisfying

$$r \cdot k^2 = \sum_{i=1}^{r} \tau^*(A_k)^2 = 9d^2 + \text{ terms of lower order}$$

constructed by Shustin (cf. [Shu97]) shows that asymptotically we cannot expect to do essentially better in general. For a survey on other known results on $\Sigma = \mathbb{P}^2_{\mathbb{C}}$ we refer to [GLS00] and [GLS01], and for results on Severi varieties on other surfaces see [Tan80, GrK89, GLS98, FlM01, Fla01].

In this section we introduce the basic concepts and notations used throughout the paper, and we state several important known facts. Section 2 contains the main results and Section 3 their proofs.

1.1. **General Assumptions and Notations.** Throughout this article Σ will denote a smooth projective surface over $\mathbb{C}$.

We will denote by $\mathrm{Div}(\Sigma)$ the group of divisors on Σ and by K_Σ its canonical divisor. If D is any divisor on Σ, $\mathcal{O}_\Sigma(D)$ shall be the corresponding invertible sheaf and we will sometimes write $H^\nu(X, D)$ instead of $H^\nu\big(X, \mathcal{O}_X(D)\big)$. A *curve* $C \subset \Sigma$ will be an effective (non-zero) divisor, that is a one-dimensional locally principal scheme, not necessarily reduced; however, an *irreducible curve* shall be reduced by definition. $|D|_l$ denotes the system of curves linearly equivalent to D. We will use the notation $\mathrm{Pic}(\Sigma)$ for the *Picard group* of Σ, that is $\mathrm{Div}(\Sigma)$ modulo linear equivalence (denoted by $\sim_l$), and $\mathrm{NS}(\Sigma)$ for the *Néron–Severi group*, that is $\mathrm{Div}(\Sigma)$ modulo algebraic equivalence (denoted by $\sim_a$). Given a reduced curve $C \subset \Sigma$ we will write $g(C)$ for its *geometric genus*.

Given any closed subscheme X of a scheme Y, we denote by $\mathcal{J}_X = \mathcal{J}_{X/Y}$ the *ideal sheaf* of X in $\mathcal{O}_Y$. If X is zero-dimensional we denote by $\deg(X) = \sum_{z\in Y} \dim_{\mathbb{C}}(\mathcal{O}_{Y,z}/\mathcal{J}_{X/Y,z})$ its *degree*. If $X \subset \Sigma$ is a zero-dimensional scheme on Σ and $D \in \mathrm{Div}(\Sigma)$, we denote by $\big|\mathcal{J}_{X/\Sigma}(D)\big|_l$ the linear system of curves C in $|D|_l$ with $X \subset C$.

Given two curves C and D in Σ and a point $z \in \Sigma$, and let $f, g \in \mathcal{O}_{\Sigma,z}$ be local equations at z of C and D respectively, then we will denote by $i(C, D; z) = i(f, g) = \dim_{\mathbb{C}}(\mathcal{O}_{\Sigma,z}/\langle f, g\rangle)$ the intersection multiplicity of C and D at z.

1.2. **Singularity Types.** The germ $(C, z) \subset (\Sigma, z)$ of a reduced curve $C \subset \Sigma$ at a point $z \in \Sigma$ is called a *plane curve singularity*, and two plane curve singularities (C, z) and $\big(C', z'\big)$ are said to be *topologically* (respectively *analytically equivalent*) if there is homeomorphism (respectively an analytical isomorphism) $\Phi : (\Sigma, z) \to$

(Σ, z') such that $\Phi(C) = C'$. We call an equivalence class with respect to these equivalence relations a *topological* (respectively *analytical*) *singularity type.*
When dealing with numerical conditions for T-smoothness some topological (respectively analytical) invariants of the singularities play an important role. We gather some results on them here for the convenience of the reader.
Let (C, z) be the germ at z of a reduced curve $C \subset \Sigma$ and let $f \in R = \mathcal{O}_{\Sigma,z}$ be a representative of (C, z) in local coordinates x and y. For the analytical type of the singularity the *Tjurina ideal*

$$I^{ea}(f) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f \right\rangle$$

plays a very important role, as does the *equisingularity ideal*

$$I^{es}(f) = \left\{ g \in R \;\middle|\; f + \varepsilon g \text{ is equisingular over } \mathbb{C}[\varepsilon]/(\varepsilon^2) \right\} \supseteq I^{ea}(f)$$

for the topological type. They give rise to the following invariants of the topological (respectively analytical) singularity type $\mathcal{S}$ of (C, z).

- (a) Analytical Invariants:
  - (1) $\tau(\mathcal{S}) = \dim_{\mathbb{C}} \big(R/I^{ea}(f)\big)$ is the *Tjurina number*, i. e. the dimension of the base space of the semiuniversal deformation of (C, z).
  - (2) $\tau_{ci}(\mathcal{S}) = \max\big\{ \dim_{\mathbb{C}}(R/I) \;\big|\; I^{ea}(f) \subseteq I \text{ a complete intersection}\big\}$.
  - (3) $\gamma_\alpha^{ea}(\mathcal{S}) = \max\big\{\gamma_\alpha(f; I) \;\big|\; I^{ea}(f) \subseteq I \text{ a complete intersection}\big\}$.
- (b) Topological Invariants:
  - (1) $\tau^{es}(\mathcal{S}) = \dim_{\mathbb{C}} \big(R/I^{es}(f)\big)$ is the codimension of the μ-constant stratum in the semiuniversal deformation of (C, z).
  - (2) $\tau_{ci}^{es}(\mathcal{S}) = \max\big\{ \dim_{\mathbb{C}}(R/I) \;\big|\; I^{es}(C, z) \subseteq I \text{ a complete intersection}\big\}$.
  - (3) $\gamma_\alpha^{es}(\mathcal{S}) = \max\big\{\gamma_\alpha(f; I) \;\big|\; I^{es}(C, z) \subseteq I \text{ a complete intersection}\big\}$.

Here, for an ideal I containing $I^{ea}(f)$ and a rational number $0 \leq \alpha \leq 1$ we define

$$\gamma_\alpha(f; I) = \max\big\{(1+\alpha)^2 \cdot \dim_{\mathbb{C}}(R/I),\ \lambda_\alpha(f; I, g) \;\big|\; g \in I, i(f, g) \leq 2 \cdot \dim_{\mathbb{C}}(R/I)\big\},$$

where for $g \in I$

$$\lambda_\alpha(f; I, g) = \frac{\big(\alpha \cdot i(f, g) - (1-\alpha) \cdot \dim_{\mathbb{C}}(R/I)\big)^2}{i(f, g) - \dim_{\mathbb{C}}(R/I)}.$$

Note that by Lemma 1.1 $i(f, g) > \dim_{\mathbb{C}}(R/I)$ for all $g \in I$ and $\gamma_\alpha(f, g)$ is thus a well-defined positive rational number.

> Throughout this article we will frequently treat topological and analytical singularities at the same time. Whenever we do so, we will write $\tau^*(\mathcal{S})$ for $\tau^{es}(\mathcal{S})$ respectively for $\tau(\mathcal{S})$, and analogously we use the notation $\tau_{ci}^*(\mathcal{S})$ and $\gamma_\alpha^*(\mathcal{S})$.

One easily sees the following relations:

$$(1+\alpha)^2 \cdot \tau_{ci}^*(\mathcal{S}) \leq \gamma_\alpha^*(\mathcal{S}) \leq \big(\tau_{ci}^*(\mathcal{S}) + \alpha\big)^2 \leq \big(\tau^*(\mathcal{S}) + \alpha\big)^2. \tag{1.3}$$

In [LoK03] the γ_α^*-invariant is calculated for the simple singularities,

$\mathcal{S}$		$\gamma_\alpha^{ea}(\mathcal{S}) = \gamma_\alpha^{es}(\mathcal{S})$
$A_k,$	$k \geq 1$	$(k+\alpha)^2$
$D_k,$	$4 \leq k \leq 4 + \sqrt{2}\cdot(2+\alpha)$	$\frac{(k+2\alpha)^2}{2}$
$D_k,$	$k \geq 4 + \sqrt{2}\cdot(2+\alpha)$	$(k-2+\alpha)^2$
$E_k,$	$k = 6, 7, 8$	$\frac{(k+2\alpha)^2}{2}$

and for the topological singularity type M_m of an ordinary m-fold point

$$\gamma_\alpha^{es}(M_m) = 2\cdot(m-1+\alpha)^2.$$

Moreover, upper and lower bounds for the γ_0^{es}-invariant and for the γ_1^{es}-invariant of a topological singularity type given by a convenient semi-quasihomogeneous power series can be found there. They also show that

$$\tau_{ci}^{es}(M_m) = \begin{cases} \frac{(m+1)^2}{4}, & \text{if } m \geq 3 \text{ odd},\\ \frac{m^2+2m}{4}, & \text{if } m \geq 4 \text{ even},\\ 1, & \text{if } m = 2. \end{cases}$$

These results show in particular that the upper bound for $\gamma_\alpha^*(\mathcal{S})$ in (1.3) may be attained, while it may as well be far from the actual value.

Lemma 1.1
Let (C,z) be a reduced plane curve singularity given by $f \in \mathcal{O}_{\Sigma,z}$ and let $I \subseteq \mathfrak{m}_{\Sigma,z} \subset \mathcal{O}_{\Sigma,z}$ be an ideal containing the Tjurina ideal $I^{ea}(C,z)$. Then for any $g \in I$ we have

$$\dim_{\mathbb{C}}(\mathcal{O}_{\Sigma,z}/I) < \dim_{\mathbb{C}}\big(\mathcal{O}_{\Sigma,z}/(f,g)\big) = i(f,g).$$

Proof: Cf. [Shu97] Lemma 4.1. □

1.3. **Singularity Schemes.** For a reduced curve $C \subset \Sigma$ we recall the definition of the zero-dimensional schemes $X^{es}(C)$ and $X^{ea}(C)$ from [GLS00]. They are defined by the ideal sheaves $\mathcal{J}_{X^{es}(C)/\Sigma}$ and $\mathcal{J}_{X^{ea}(C)/\Sigma}$ respectively, given by the stalks $\mathcal{J}_{X^{es}(C)/\Sigma,z} = I^{es}(f)$ and $\mathcal{J}_{X^{ea}(C)/\Sigma,z} = I^{ea}(f)$ respectively, where $f \in \mathcal{O}_{\Sigma,z}$ is a local equation of C at z. We call $X^{es}(C)$ the *equisingularity scheme* of C and $X^{ea}(C)$ the *equianalytical singularity scheme* of C.

Throughout this article we will frequently treat topological and analytical singularities at the same time. Whenever we do so, we will write $X^*(C)$ for $X^{es}(C)$ respectively for $X^{ea}(C)$.

1.4. **Equisingular Families.** Given a divisor $D \in \text{Div}(\Sigma)$ and topological or analytical singularity types $\mathcal{S}_1, \ldots, \mathcal{S}_r$, we denote by $V = V_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ the locally closed subspace of $|D|_l$ of reduced curves in the linear system $|D|_l$ having precisely r singular points of types $\mathcal{S}_1, \ldots, \mathcal{S}_r$. By $V^{irr} = V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ we denote the open subset of V of irreducible curves. If a type $\mathcal{S}$ occurs $k > 1$ times, we rather write $k\mathcal{S}$ than $\mathcal{S}, \overset{k}{\ldots}, \mathcal{S}$. We call these families of curves *equisingular families of curves*.
We say that V is *T-smooth* at $C \in V$ if the germ (V, C) is smooth of the (expected) dimension $\dim |D|_l - \deg\big(X^*(C)\big)$. By [Los98] Proposition 2.1 (see also [GrK89], [GrL96], [GLS00]) T-smoothness of V at C follows from the vanishing of $H^1\big(\Sigma, \mathcal{J}_{X^*(C)/\Sigma}(C)\big)$, since the tangent space of V at C may be identified with $H^0\big(\Sigma, \mathcal{J}_{X^*(C)/\Sigma}(C)\big)/H^0(\Sigma, \mathcal{O}_\Sigma)$.

2. The Main Results

In this section we give sufficient conditions for the T-smoothness of equisingular families of curves on certain surfaces with Picard number one, including the projective plane, general surfaces in $\mathbb{P}^3_{\mathbb{C}}$ and general K3-surfaces –, on general products of curves, and on geometrically ruled surfaces.

2.1. Surfaces with Picard Number One.

Theorem 2.1
Let Σ be a surface such that $\text{NS}(\Sigma) = L \cdot \mathbb{Z}$ with L ample, let $D = d \cdot L \in \text{Div}(\Sigma)$, let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types, and let $K_\Sigma = \kappa \cdot L$. Suppose that $d \geq \max\{\kappa + 1, -\kappa\}$ and

$$\sum_{i=1}^{r} \gamma^*_\alpha(\mathcal{S}_i) < \alpha \cdot (D - K_\Sigma)^2 = \alpha \cdot (d - \kappa)^2 \cdot L^2 \quad \text{with } \alpha = \tfrac{1}{\max\{1, 1+\kappa\}}. \tag{2.1}$$

Then either $V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is empty or it is T-smooth. □

Corollary 2.2
Let $d \geq 3$, $H \subset \mathbb{P}^2_{\mathbb{C}}$ be a line, and $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types. Suppose that

$$\sum_{i=1}^{r} \gamma^*_1(\mathcal{S}_i) < (d + 3)^2. \tag{2.2}$$

Then either $V^{irr}_{|dH|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is empty or T-smooth. □

As soon as for one of the singularities we have $\gamma^*_1(\mathcal{S}_i) > 4 \cdot \tau^*_{ci}(\mathcal{S}_i)$, e. g. simple singularities or ordinary multiple points which are not simple double points, then the strict inequality in (2.2) can be replaced by "$\leq$", which then is the same sufficient condition as in [GLS01] Theorem 1 (see also (1.2)).
In particular, $V^{irr}_{|dH|}(M_{m_1}, \ldots, M_{m_r})$, $m_i \geq 3$, is therefore T-smooth as soon as

$$\sum_{i=1}^{r} 2 \cdot m_i^2 \leq (d + 3)^2.$$

Moreover, this condition has the right assymptotics, as the examples in [GLS01] show. For further results in the plane case see [Wah74, GrK89, Lue87a, Lue87b, Shu87, Vas90, Shu91, Shu94, GrL96, Shu96, Shu97, GLS98, Los98, GLS00, GLS01]. A smooth complete intersection surface with Picard number one satisfies the assumptions of Theorem 2.1. Thus by the Theorem of Noether the result applies in particular to general surfaces in $\mathbb{P}^3_{\mathbb{C}}$. Moreover, if in Theorem 2.1 we have $\kappa > 0$, i. e. $\alpha < 1$, then the strict inequality in Condition (2.1) may be replaced by "$\leq$", since in (3.9) the second inequality is strict, as is the second inequality in (3.10).

Corollary 2.3
Let $\Sigma \subset \mathbb{P}^3_{\mathbb{C}}$ be a smooth hypersurface of degree $n \geq 4$, let $H \subset \Sigma$ be a hyperplane section, and suppose that the Picard number of Σ is one. Let $d \geq n-3$ and let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types. Suppose that

$$\sum_{i=1}^{r} \gamma^*_{\frac{1}{n-3}}(\mathcal{S}_i) \leq \frac{n}{n-3} \cdot (d-n+4)^2.$$

Then either $V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is empty or it is T-smooth. □

In particular, $V^{irr}_{|dH|}(M_{m_1}, \ldots, M_{m_r})$, $m_i \geq 3$, is therefore T-smooth as soon as

$$\sum_{i=1}^{r} 2 \cdot \left(m_i - \frac{n-2}{n-3}\right)^2 < \frac{n}{n-3} \cdot (d-n+4)^2,$$

which is better than the conditions derived from [GLS97]. The condition

$$r \leq \frac{n \cdot (n-3)}{(n-2)^2} \cdot (d-n+4)^2,$$

which gives the T-smoothness of $V_{|dH|}(rA_1)$ is weaker than the condition provided in [ChS97], but for $n = 5$ it reads $r \leq \frac{10}{9} \cdot (d-1)^2$ and comes still close to the sharp bound $\frac{5}{4} \cdot (d-1)^2$ provided there for odd d.
A general K3-surface has also Picard number one..

Corollary 2.4
Let Σ be a smooth K3-surface with $\mathrm{NS}(\Sigma) = L \cdot \mathbb{Z}$, L ample, and set $n = L^2$. Let $d \geq 1$, and let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types. Suppose that

$$\sum_{i=1}^{r} \gamma^*_1(\mathcal{S}_i) < d^2 n.$$

Then either $V^{irr}_{|dL|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is empty or it is T-smooth. □

The best previously known condition for T-smoothness on K3-surfaces

$$\sum_{i=1}^{r} \left(\tau^*_{ci}(\mathcal{S}_i) + 1\right)^2 < d^2 n$$

is thus completely replaced.

2.2. **Products of Curves.** If $\Sigma = C_1 \times C_2$ is the product of two smooth projective curves, then for a general choice of C_1 and C_2 the Néron–Severi group will be generated by two fibres of the canonical projections, by abuse of notation also denoted by C_1 and C_2. If both curves are elliptic, then "general" just means that the two curves are non-isogenous.

Theorem 2.5
Let C_1 and C_2 be two smooth projective curves of genera g_1 and g_2 with $g_1 \geq g_2$, such that for $\Sigma = C_1 \times C_2$ the Néron–Severi group is $\mathrm{NS}(\Sigma) = C_1\mathbb{Z} \oplus C_2\mathbb{Z}$.
Let $D \in \mathrm{Div}(\Sigma)$ such that $D \sim_a aC_1 + bC_2$ with $a \geq \max\{2-2g_2, 2g_2-1\}$ and $b \geq \max\{2-2g_1, 2g_1-1\}$, let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types. Suppose that

$$\sum_{i=1}^{r} \gamma_0^*(\mathcal{S}_i) \;<\; \gamma \cdot (D - K_\Sigma)^2, \tag{2.3}$$

where the constant γ may be read off the following table with $A = \frac{a-2g2+2}{b-2g1+2}$

g_1	g_2	γ
$0,1$	$0,1$	$\frac{1}{4}$
≥ 2	$0,1$	$\min\left\{\frac{1}{4g_1}, \frac{1}{4\cdot(g_1-1)\cdot A}\right\}$
≥ 2	≥ 2	$\min\left\{\frac{1}{4g_1+4g_2-4}, \frac{A}{4\cdot(g_2-1)}, \frac{1}{4\cdot(g_1-1)\cdot A}\right\}$

Then either $V_{|D|}^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is empty or it is T-smooth. □

In particular, on a product of non-isogenous elliptic curves for nodal curves we reproduce the previous sufficient condition

$$r < \frac{ab}{2},$$

for T-smoothness of $V_{|aC_1+bC_2|}^{irr}(rA_1)$ from [GLS97], while the previous general condition

$$\frac{\left(m_i^2 + 2m_i + 5\right)^2}{32} < ab$$

for T-smoothness of $V_{|aC_1+bC_2|}^{irr}(M_{m_1}, \ldots, M_{m_r})$, $m_i \geq 3$, has been replaced by

$$\sum_{i=1}^{r} 4 \cdot (m_i - 1)^2 < ab,$$

which is better from $m_i = 7$ on.
Note that the constant γ in Theorem 2.5 depends on the ratio of a and b unless both g_1 and g_2 are at most one. This means that in general an asymptotical behaviour can only be examined if the ratio remains unchanged.

2.3. **Geometrically Ruled Surfaces.** Let $\pi : \Sigma = \mathbb{P}_{\mathbb{C}}(\mathcal{E}) \to C$ be a geometrically ruled surface with normalised bundle $\mathcal{E}$ (in the sense of [Har77] V.2.8.1). The Néron–Severi group of Σ is $\mathrm{NS}(\Sigma) = C_0\mathbb{Z} \oplus F\mathbb{Z}$ with intersection matrix $\left(\begin{smallmatrix} -e & 1 \\ 1 & 0 \end{smallmatrix}\right)$ where $F \cong \mathbb{P}_{\mathbb{C}}^1$ is a fibre of π, C_0 a section of π with $\mathcal{O}_\Sigma(C_0) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, $g = g(C)$ the genus of C, $\mathfrak{e} = \Lambda^2\mathcal{E}$ and $e = -\deg(\mathfrak{e}) \geq -g$. For the canonical divisor we have $K_\Sigma \sim_a -2C_0 + (2g-2-e)\cdot F$.

Theorem 2.6
Let $\pi : \Sigma \to C$ be a geometrically ruled surface with $g = g(C)$. Let $D \in \mathrm{Div}(\Sigma)$ such that $D \sim_a aC_0 + bF$ with $b > \max\{2g-2, 2-2g\} + \frac{ae}{2}$ and $a > 2$, and let $\mathcal{S}_1, \ldots, \mathcal{S}_r$ be topological or analytical singularity types. Suppose that

$$\sum_{i=1}^{r} \gamma_0^*(\mathcal{S}_i) \; < \; \gamma \cdot (D - K_\Sigma)^2, \tag{2.4}$$

where with $A = \frac{a+2}{b+2-2g-\frac{ae}{2}}$ the constant γ satisfies

$$\gamma = \begin{cases} \frac{1}{4}, & \text{if } g \in \{0,1\}, \\ \min\left\{\frac{1}{4g}, \frac{1}{4\cdot(g-1)\cdot A}\right\}, & \text{if } g \geq 2. \end{cases}$$

Then either $V_{|D|}^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ is empty or it is T-smooth. □

The results of [GLS97] only applied to eight Hirzebruch surfaces and a few classes of fibrations over elliptic curves, while our results apply to all geometrically ruled surfaces. Moreover, the results are in general better, e. g. for the Hirzebruch surface $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ already the previous sufficient condition for T-smoothness of families of curves with r cusps and $b = 3a$ the condition

$$9r < 2a^2 + 8a$$

has been replaced by the slightly better condition

$$8r < 3a^2 + 8a + 4.$$

For ordinary multiple points the difference will become more significant. Even for families of nodal curves the new conditions would always be slightly better, but for those families T-smoothness is guaranteed anyway by [Tan80].
Note that, as for products of curves, the constant γ in Theorem 2.6 depends on the ratio of a and b unless g is at most one.

3. The Proofs

The following Lemma is the technical key to the above results. Using the method of Bogomolov unstable vector bundles, it gives us a "small" curve which passes through a "large" part of $X^*(C)$, provided that $h^1\big(\Sigma, \mathcal{J}_{X^*(C)/\Sigma}(D)\big) \neq 0$. We will then show that its existence contradicts (2.1), (2.3), or (2.4) respectively.

Lemma 3.1
Let Σ a smooth projective surface, and let $D \in \mathrm{Div}(\Sigma)$ and $X \subset \Sigma$ be a zero-dimensional scheme satisfying

(0) *$D - K_\Sigma$ is big and nef, and $D + K_\Sigma$ is nef,*
(1) *$\exists\, C \in |D|_l$ irreducible : $X \subseteq X^*(C)$,*
(2) *$h^1\big(\Sigma, \mathcal{J}_{X/\Sigma}(D)\big) > 0$, and*
(3) *$4\cdot\deg(X_0) < (D - K_\Sigma)^2$ for all local complete intersection schemes $X_0 \subseteq X$.*

Then there exists a curve $\Delta \subset \Sigma$ *and a zero-dimensional local complete intersection scheme* $X_0 \subseteq X \cap \Delta$ *such that with the notation* $\mathrm{supp}(X_0) = \{z_1, \ldots, z_s\}$, $X_i = X_{0,z_i}$ *and*[1] $\varepsilon_i = \min\{\deg(X_i), i(C,\Delta;z_i) - \deg(X_i)\} \geq 1$ *we have*

(a) $D.\Delta \geq \deg(X_0) + \sum_{i=1}^{s} \varepsilon_i$,
(b) $\deg\big(X_0\big) \geq \big(D - K_\Sigma - \Delta\big).\Delta$,
(c) $\big(D - K_\Sigma - 2\cdot\Delta\big)^2 > 0$, *and*
(d) $\big(D - K_\Sigma - 2\cdot\Delta\big).H > 0$ *for all* $H \in \mathrm{Div}(\Sigma)$ *ample.*

Moreover, it follows

$$0 \leq \tfrac{1}{4}\cdot(D - K_\Sigma)^2 - \deg\big(X_0\big) \leq \big(\tfrac{1}{2}\cdot(D-K_\Sigma) - \Delta\big)^2. \tag{3.1}$$

Proof: Choose $X_0 \subseteq X$ minimal such that still $h^1\big(\Sigma, \mathcal{J}_{X_0/\Sigma}(D)\big) > 0$. By Assumption (0) the divisor $D - K_\Sigma$ is big and nef, and thus $h^1\big(\Sigma, \mathcal{O}_\Sigma(D)\big) = 0$ by the Kawamata–Viehweg Vanishing Theorem. Hence X_0 cannot be empty.
Due to the Grothendieck-Serre duality we have

$$0 \neq H^1\big(\Sigma, \mathcal{J}_{X_0/\Sigma}(D)\big) \cong \mathrm{Ext}^1\big(\mathcal{J}_{X_0/\Sigma}(D-K_\Sigma), \mathcal{O}_\Sigma\big).$$

That is, there is an extension

$$0 \to \mathcal{O}_\Sigma \to E \to \mathcal{J}_{X_0/\Sigma}(D - K_\Sigma) \to 0. \tag{3.2}$$

The minimality of X_0 implies that E is locally free and X_0 is a local complete intersection scheme (cf. [Laz97] Proposition 3.9). Moreover, we have

$$c_1(E) = D - K_\Sigma \quad \text{and} \quad c_2(E) = \deg(X_0). \tag{3.3}$$

By Assumption (3) and (3.3) we have

$$c_1(E)^2 - 4\cdot c_2(E) = (D-K_\Sigma)^2 - 4\cdot\deg(X_0) > 0,$$

and thus E is Bogomolov unstable (cf. [Laz97] Theorem 4.2). This, however, implies that there exists a divisor $\Delta_0 \in \mathrm{Div}(\Sigma)$ and a zero-dimensional scheme $Z \subset \Sigma$ such that

$$0 \to \mathcal{O}_\Sigma(\Delta_0) \to E \to \mathcal{J}_{Z/\Sigma}(D - K_\Sigma - \Delta_0) \to 0 \tag{3.4}$$

is exact, and such that

$$(2\Delta_0 - D + K_\Sigma)^2 \geq c_1(E)^2 - 4\cdot c_2(E) > 0 \tag{3.5}$$

and

$$(2\Delta_0 - D + K_\Sigma).H > 0 \quad \text{for all ample} \quad H \in \mathrm{Div}(\Sigma). \tag{3.6}$$

Tensoring (3.4) with $\mathcal{O}_\Sigma(-\Delta_0)$ leads to the following exact sequence

$$0 \to \mathcal{O}_\Sigma \to E(-\Delta_0) \to \mathcal{J}_{Z/\Sigma}\left(D - K_\Sigma - 2\Delta_0\right) \to 0, \tag{3.7}$$

and we deduce $h^0\big(\Sigma, E(-\Delta_0)\big) \neq 0$.
Now tensoring (3.2) with $\mathcal{O}_\Sigma(-\Delta_0)$ leads to

$$0 \to \mathcal{O}_\Sigma(-\Delta_0) \to E(-\Delta_0) \to \mathcal{J}_{X_0/\Sigma}\left(D - K_\Sigma - \Delta_0\right) \to 0. \tag{3.8}$$

[1] Since $X_0 \subseteq X^*(C) \subseteq X^{ea}(C)$, Lemma 1.1 applies to the local ideals of X_0, that is for the points $z \in \mathrm{supp}(X_0)$ we have $i(C,\Delta;z) \geq \deg(X_0,z) + 1$.

Let H be some ample divisor. By (3.6) and since $D - K_\Sigma$ is nef by (0):

$$-\Delta_0.H < -\tfrac{1}{2} \cdot (D - K_\Sigma).H \leq 0.$$

Hence $-\Delta_0$ cannot be effective, that is $H^0\big(\Sigma, \mathcal{O}_\Sigma(-\Delta_0)\big) = 0$. But the long exact cohomology sequence of (3.8) then implies

$$0 \neq H^0\big(\Sigma, E(-\Delta_0)\big) \hookrightarrow H^0\left(\Sigma, \mathcal{J}_{X_0/\Sigma}(D - K_\Sigma - \Delta_0)\right).$$

In particular we may choose a curve

$$\Delta \in \big|\mathcal{J}_{X_0/\Sigma}(D - K_\Sigma - \Delta_0)\big|_l.$$

Thus (c) and (d) follow from (3.5) and (3.6). It remains to show (a) and (b).
We note that $C \in |D|_l$ is irreducible and that Δ cannot contain C as an irreducible component: otherwise applying (3.6) with some ample divisor H we would get the following contradiction, since $D + K_\Sigma$ is nef by (0),

$$0 \leq (\Delta - C).H < -\tfrac{1}{2} \cdot (D + K_\Sigma).H \leq 0.$$

Since $X_0 \subset C \cap \Delta$ the Theorem of Bézout implies (a):

$$D.\Delta = C.\Delta = \sum_{z \in C\cap\Delta} i(C, \Delta; z) \geq \sum_{i=1}^{s} \big(\deg(X_i) + \varepsilon_i\big) = \deg(X_0) + \sum_{i=1}^{s} \varepsilon_i.$$

Finally, by (3.3) and (3.4) we get (b):

$$\deg(X_0) = c_2(E) = \Delta_0.(D - K_\Sigma - \Delta_0) + \deg(Z) \geq (D - K_\Sigma - \Delta).\Delta.$$

Equation (3.1) is just a reformulation of (b). □

Using this result we can now prove the main theorems.

Proof of Theorem 2.1: Let $C \in V^{irr}_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$. It suffices to show that the cohomology group $h^1\big(\Sigma, \mathcal{J}_{X^*(C)/\Sigma}(D)\big)$ vanishes.
Suppose this is not the case. Since for $X_0 \subseteq X^*(C)$ any local complete intersection scheme and $z \in \operatorname{supp}(X_0)$ we have

$$4 \cdot \deg(X_z) \leq \frac{4}{(1+\alpha)^2} \cdot \gamma^*_\alpha(C, z) \leq \frac{1}{\alpha} \cdot \gamma^*_\alpha(C, z) \tag{3.9}$$

Lemma 3.1 applies and there is curve $\Delta \in |\delta \cdot L|_l$ and a local complete intersection scheme $X_0 \subseteq X^*(C)$ satisfying the assumptions (a)-(d) there and Equation (3.1). That is, fixing the notation $l = \sqrt{L^2}$, $\operatorname{supp}(X_0) = \{z_1, \ldots, z_s\}$, $X_i = X_{0,z_i}$ and $\varepsilon_i = \min\{\deg(X_i), i(C, \Delta; z_i) - \deg(X_i)\} \geq 1$, we have

(a) $d \cdot \delta \cdot l^2 \geq \deg(X_0) + \sum_{i=1}^s \varepsilon_i$,
(b) $\deg(X_0) \geq (d - \kappa - \delta) \cdot \delta \cdot l^2$,

and

$$\delta \cdot l \leq \tfrac{(d-\kappa)\cdot l}{2} - \sqrt{\tfrac{(d-\kappa)^2 \cdot l^2}{4} - \deg(X_0)} = \frac{2 \cdot \deg(X_0)}{(d-\kappa)\cdot l + \sqrt{(d-\kappa)^2 \cdot l^2 - 4 \cdot \deg(X_0)}}.$$

But then together with (a) and (b) we deduce

$$\sum_{i=1}^{s} \varepsilon_i \leq \delta \cdot (\delta + \kappa) \cdot l^2 \leq \frac{1}{\alpha} \cdot \left(\frac{2 \cdot \deg(X_0)}{(d-\kappa)\cdot l + \sqrt{(d-\kappa)^2 \cdot l^2 - 4 \cdot \deg(X_0)}}\right)^2. \tag{3.10}$$

Applying the Cauchy inequality this leads to

$$\sum_{i=1}^{s}\frac{\deg(X_i)^2}{\varepsilon_i}\geq\frac{\deg(X_0)^2}{\sum_{i=1}^s\varepsilon_i}\geq\frac{\alpha\cdot(d-\kappa)^2\cdot l^2}{4}\cdot\left(1+\sqrt{1-\tfrac{4\cdot\deg(X_0)}{(d-\kappa)^2\cdot l^2}}\right)^2.$$

Setting

$$\beta=\frac{\sum_{i=1}^s\frac{\deg(X_i)^2}{\varepsilon_i}}{\alpha\cdot(d-\kappa)^2\cdot l^2},\quad\gamma=\frac{\sum_{i=1}^s\frac{\deg(X_i)^2}{\varepsilon_i}}{\alpha\cdot\deg(X_0)},$$

we thus have

$$\beta\geq\frac{1}{4}\cdot\left(1+\sqrt{1-\tfrac{4\beta}{\gamma}}\right)^2,$$

and hence, $\beta\geq\left(\frac{\gamma}{\gamma+1}\right)^2$. But then, applying the Cauchy inequality once more, we find

$$\begin{aligned}\alpha\cdot(d-\kappa)^2\cdot l^2&=\frac{\alpha\cdot\gamma}{\beta}\cdot\deg(X_0)\leq\alpha\cdot\left(\gamma+2+\frac{1}{\gamma}\right)\cdot\deg(X_0)\\&\leq\sum_{i=1}^{s}\left(\frac{\deg(X_i)^2}{\varepsilon_i}+2\alpha\deg(X_i)+\alpha^2\varepsilon_i\right)\leq\sum_{i=1}^{r}\gamma_\alpha^*(\mathcal{S}_i),\end{aligned}$$

in contradiction to Equation (2.1). □

Proof of Theorem 2.5: Let $C\in V_{|D|}^{irr}(\mathcal{S}_1,\ldots,\mathcal{S}_r)$. It suffices to show that the cohomology group $h^1\big(\Sigma,\mathcal{J}_{X^*(C)/\Sigma}(D)\big)$ vanishes.

Suppose this is not the case. Since for $X_0\subseteq X^*(C)$ any local complete intersection scheme and $z\in\operatorname{supp}(X)$ we have

$$\deg(X_z)\leq\gamma_0^*(C,z),$$

and since $\gamma\leq\frac{1}{4}$, Lemma 3.1 applies and there is curve $\Delta\sim_a\alpha\cdot C_1+\beta\cdot C_2$ and a local complete intersection scheme $X_0\subseteq X^*(C)$ satisfying the assumptions (a)-(d) there and Equation (3.1). That is, fixing the notation $\operatorname{supp}(X_0)=\{z_1,\ldots,z_s\}$, $X_i=X_{0,z_i}$ and $\varepsilon_i=\min\{\deg(X_i),i(C,\Delta;z_i)-\deg(X_i)\}\geq 1$, we have

(a) $a\beta+b\alpha\geq\deg(X_0)+\sum_{i=1}^s\varepsilon_i$,
(b) $\deg(X_0)\geq(a-2g_2+2-\alpha)\cdot\beta+(b-2g_1+2-\beta)\cdot\alpha$, and
(c) $0\leq\alpha\leq\frac{a-2g_2+2}{2}$ and $0\leq\beta\leq\frac{b-2g_1+2}{2}$.

The last inequalities follow from (d) in Lemma 3.1 replacing the ample divisor H by the nef divisors C_2 respectively C_1.

From (b) and (c) we deduce

$$\deg(X_0)\geq\frac{a-2g_2+2}{2}\cdot\beta+\frac{b-2g_1+2}{2}\cdot\alpha,$$

and thus

$$\deg(X_0)^2\geq 4\cdot\frac{a-2g_2+2}{2}\cdot\frac{b-2g_1+2}{2}\cdot\alpha\cdot\beta=\frac{(D-K_\Sigma)^2}{2}\cdot\alpha\cdot\beta.\tag{3.11}$$

Considering now (a) and (b) we get

$$0<\sum_{i=1}^{s}\varepsilon_i\leq\Delta.(\Delta+K_\Sigma)=2\alpha\beta+(2g_1-2)\cdot\alpha+(2g_2-2)\cdot\beta\leq\frac{\alpha\beta}{2\gamma},$$

where the last inequality holds only if $\alpha \neq 0 \neq \beta$. In particular, we see $\alpha \neq 0$ if $g_2 \leq 1$ and $\beta \neq 0$ if $g_1 \leq 1$. But this together with (3.11) gives

$$\sum_{i=1}^{s} \varepsilon_i \leq \frac{\deg(X_0)^2}{\gamma \cdot (D - K_\Sigma)^2}.$$

If $\alpha = 0$, then from (a) and (b) we deduce again

$$0 < \sum_{i=1}^{s} \varepsilon_i \leq (2g_2 - 2) \cdot \beta \leq \frac{4 \cdot (g_1 - 1)}{A} \cdot \frac{\deg(X_0)^2}{(D - K_\Sigma)^2} \leq \frac{\deg(X_0)^2}{\gamma \cdot (D - K_\Sigma)^2},$$

and similarly, if $\beta = 0$,

$$0 < \sum_{i=1}^{s} \varepsilon_i \leq (2g_1 - 2) \cdot \alpha \leq 4 \cdot (g_1 - 1) \cdot A \cdot \frac{\deg(X_0)^2}{(D - K_\Sigma)^2} \leq \frac{\deg(X_0)^2}{\gamma \cdot (D - K_\Sigma)^2}.$$

Applying the Cauchy inequality, we finally get

$$\gamma \cdot (D - K_\Sigma)^2 \leq \frac{\deg(X_0)^2}{\sum_{i=1}^{s} \varepsilon_i} \leq \sum_{i=1}^{s} \frac{\deg(X_i)^2}{\varepsilon_i} \leq \sum_{i=1}^{r} \gamma_0^*(\mathcal{S}_i),$$

in contradiction to Assumption (2.3). □

Proof of Theorem 2.6: Let $C \in V_{|D|}^{irr}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$. It suffices to show that the cohomology group $h^1\big(\Sigma, \mathcal{J}_{X^*(C)/\Sigma}(D)\big)$ vanishes.
Suppose this is not the case. Since for $X_0 \subseteq X^*(C)$ any local complete intersection scheme and $z \in \operatorname{supp}(X)$ we have

$$\deg(X_z) \leq \gamma_0^*(C, z),$$

and since $\gamma \leq \frac{1}{4}$, Lemma 3.1 applies and there is curve $\Delta \sim_a \alpha \cdot C_0 + \beta \cdot F$ and a local complete intersection scheme $X_0 \subseteq X^*(C)$ satisfying the assumptions (a)-(d) there and Equation (3.1).
Remember that the Néron–Severi group of Σ is generated by a section C_0 of π and a fibre F with intersection pairing given by $\left(\begin{smallmatrix} -e & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $K_\Sigma \sim_a -2C_0 + (2g - 2 - e) \cdot F$. Note that

$$\alpha \geq 0 \quad \text{ and } \quad \beta' := \beta - \tfrac{e}{2}\alpha \geq 0.$$

If we set $b' = b - \frac{ae}{2}$, $\kappa_1 = a + 2$ and $\kappa_2 = b + 2 - 2g - \frac{ae}{2} = b' + 2 - 2g$, we get

$$(D - K_\Sigma)^2 = -e \cdot (a+2)^2 + 2 \cdot (a+2) \cdot (b + 2 + e - 2g) = 2 \cdot \kappa_1 \cdot \kappa_2. \qquad (3.12)$$

Fixing the notation $\operatorname{supp}(X_0) = \{z_1, \ldots, z_s\}$, $X_i = X_{0,z_i}$, and $\varepsilon_i = \min\{\deg(X_i), i(C, \Delta; z_i) - \deg(X_i)\} \geq 1$, the conditions on Δ and $\deg(X_0)$ take the form

(a) $a\beta' + b'\alpha \geq \deg(X_0) + \sum_{i=1}^{s} \varepsilon_i$,
(b) $\deg(X_0) \geq \kappa_1 \cdot \beta' + \kappa_2 \cdot \alpha - 2\alpha\beta'$, and
(c) $0 \leq \alpha \leq \frac{\kappa_1}{2}$ and $0 \leq \beta' \leq \frac{\kappa_2}{2}$.

The last inequalities follow from (d) in Lemma 3.1 replacing the ample divisor H by the nef divisors F respectively $C_0 + \frac{e}{2} \cdot F$.
From (b) and (c) we deduce

$$\deg(X_0) \geq \frac{\kappa_1}{2} \cdot \beta' + \frac{\kappa_2}{2} \cdot \alpha,$$

and thus, taking (3.12) into account,

$$\deg(X_0)^2 \geq 4 \cdot \frac{\kappa_1}{2} \cdot \frac{\kappa_2}{2} \cdot \alpha \cdot \beta' = \frac{(D-K_\Sigma)^2}{2} \cdot \alpha \cdot \beta'. \tag{3.13}$$

Considering now (a) and (b) we get

$$0 < \sum_{i=1}^{s} \varepsilon_i \leq \Delta.(\Delta + K_\Sigma) = 2\alpha\beta' + (2g-2)\cdot\alpha - 2\beta' \leq \frac{\alpha\beta'}{2\gamma},$$

where the last inequality holds if $\beta' \neq 0$. We see, in particular, that $\beta' \neq 0$ if $g \leq 1$. But this together with (3.13) gives for $\beta' \neq 0$

$$\sum_{i=1}^{s} \varepsilon_i \leq \frac{\deg(X_0)^2}{\gamma \cdot (D-K_\Sigma)^2}.$$

If $\beta' = 0$, then we deduce from (a) and (b)

$$0 < \sum_{i=1}^{s} \varepsilon_i \leq (2g-2)\cdot\alpha \leq 4\cdot(g-1)\cdot A \cdot \frac{\deg(X_0)^2}{(D-K_\Sigma)^2} \leq \frac{\deg(X_0)^2}{\gamma\cdot(D-K_\Sigma)^2}.$$

Applying the Cauchy inequality, we finally get

$$\gamma \cdot (D-K_\Sigma)^2 \leq \frac{\deg(X_0)^2}{\sum_{i=1}^{s}\varepsilon_i} \leq \sum_{i=1}^{s} \frac{\deg(X_i)^2}{\varepsilon_i} \leq \sum_{i=1}^{r} \gamma_0^*(\mathcal{S}_i),$$

in contradiction to Assumption (2.4). □

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