A NOTE ON EQUIMULTIPLE DEFORMATIONS

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Abstract. While the tangent space to an equisingular family of curves can be discribed by the sections of a twisted ideal sheaf, this is no longer true if we only prescribe the multiplicity which a singular point should have. However, it is still possible to compute the dimension of the tangent space with the aid of the equimulitplicity ideal. In this note we consider families $\mathcal{L}_m = \{(C,p) \in |L| \times S \mid \mathrm{mult}_p(C) = m\}$ for some linear system $|L|$ on a smooth projective surface S and a fixed positive integer m, and we compute the dimension of the tangent space to $\mathcal{L}_m$ at a point (C,p) depending on whether p is a unitangential singular point of C or not. We deduce that the expected dimension of $\mathcal{L}_m$ at (C,p) in any case is just $\dim|L| - \frac{m\cdot(m+1)}{2} + 2$. The result is used in the study of triple-point defective surfaces in [ChM06a] and [ChM06b].

The paper is based on considerations about the Hilbert scheme of curves in a projective surface (see e.g. [Mum66], Lecture 22) and about local equimultiple deformations of plane curves (see [Wah74]).

Definition 1
Let T be a complex space. An *embedded family of curves in S with section* over T is a commutative diagram of morphisms

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[d, "\varphi"] & T \times S \arrow[dl] \\ T \arrow[u, bend left=70, "\sigma"] & \end{tikzcd}$$

where $\mathrm{codim}_{T\times S}(\mathcal{C}) = 1$, φ is flat and proper, and σ is a section, i.e. $\varphi \circ \sigma = \mathrm{id}_T$. Thus we have a morphism $\mathcal{O}_T \to \varphi_*\mathcal{O}_\mathcal{C} = \varphi_*\big(\mathcal{O}_{T\times S}/\mathcal{J}_\mathcal{C}\big)$ such that $\varphi_*\mathcal{O}_\mathcal{C}$ is a flat $\mathcal{O}_T$-module.

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The family is said to be *equimultiple* of multiplicity m *along the section* σ if the ideal sheaf $\mathcal{J}_{\mathcal{C}}$ of $\mathcal{C}$ in $\mathcal{O}_{T\times S}$ satisfies

$$\mathcal{J}_{\mathcal{C}} \subseteq \mathcal{J}_{\sigma(T)}^{m} \quad \text{and} \quad \mathcal{J}_{\mathcal{C}} \not\subseteq \mathcal{J}_{\sigma(T)}^{m+1},$$

where $\mathcal{J}_{\sigma(T)}$ is the ideal sheaf of $\sigma(T)$ in $\mathcal{O}_{T\times S}$.

Remark 2

Note that the above notion commutes with base change, i.e. if we have an equimultiple embedded family of curves in S over T as above and if $\alpha : T' \to T$ is a morphism, then the fibre product diagram

$$\begin{tikzcd}
T' \times S \arrow[rrr] \arrow[dr, phantom] \arrow[ddr] & & & T \times S \arrow[ddl] \\
& \mathcal{C}' \arrow[ul, hook] \arrow[r] \arrow[d, "\varphi'"] & \mathcal{C} \arrow[ur, hook'] \arrow[d, "\varphi"'] & \\
& T' \arrow[r, "\alpha"] \arrow[u, bend right, "\sigma'"'] & T \arrow[u, bend left, "\sigma"] &
\end{tikzcd}$$

gives rise to an embedded equimultiple family of curves over T' of the same multiplicity, since locally it is defined via the tensor product.

Example 3

Let us denote by $T_\varepsilon = \operatorname{Spec}(\mathbb{C}[\varepsilon])$ with $\varepsilon^2 = 0$. Then a family of curves in S over T_ε is just a Cartier divisor of $T_\varepsilon \times S$, that is, it is given on a suitable open covering $S = \bigcup_{\lambda\in\Lambda} U_\lambda$ by equations

$$f_\lambda + \varepsilon \cdot g_\lambda \in \mathbb{C}[\varepsilon] \otimes_{\mathbb{C}} \Gamma(U_\lambda, \mathcal{O}_S) = \Gamma(U_\lambda, \mathcal{O}_{T\times S}),$$

which glue together to give a global section $\left\{\frac{g_\lambda}{f_\lambda}\right\}_{\lambda\in\Lambda}$ in $H^0\big(C, \mathcal{O}_C(C)\big)$, where C is the curve defined locally by the f_λ (see e.g. [Mum66], Lecture 22).

A section of the family through p is locally in p given as $(x, y) \mapsto (x_a, y_b) = (x + \varepsilon \cdot a, y + \varepsilon \cdot b)$ for some $a, b \in \mathbb{C}\{x, y\} = \mathcal{O}_{S,p}$.

Example 4

Let H be a connected component of the Hilbert scheme Hilb_S of curves in S, then H comes with a universal family

$$\pi : \mathcal{H} \longrightarrow H : (C, p) \mapsto C. \tag{1}$$

Let us now fix a positive integer m and set

$$\mathcal{H}_m = \{(C,p) \in H \times S \mid C \in H, \mathrm{mult}_p(C) = m\}.$$

Then $\mathcal{H}_m$ is a locally closed subvariety of $H \times S$, and (1) induces via base change a flat and proper family $\mathcal{F}_m = \{(C_p, q) \in \mathcal{H}_m \times S \mid C_p = (C,p) \in \mathcal{H}_m, q \in C\}$ which has a distinguished section σ

$$\begin{tikzcd} \mathcal{F}_m \arrow[r, hook] \arrow[d] & \mathcal{H}_m \times S \arrow[dl] \\ \mathcal{H}_m \arrow[u, bend left, "\sigma"] & \end{tikzcd} \tag{2}$$

sending $C_p = (C,p)$ to $(C_p, p) \in \mathcal{F}_m$. Moreover, this family is equimultiple along σ of multiplicity m by construction.

Example 5

Similarly, if $|L|$ is a linear system on S, then it induces a universal family

$$\pi : \mathcal{L} = \{(C,p) \in |L| \times S \mid p \in C\} \longrightarrow |L| : (C,p) \mapsto C. \tag{3}$$

If we now fix a positive integer m and set

$$\mathcal{L}_m = \{(C,p) \in |L| \times S \mid C \in |L|, \mathrm{mult}_p(C) = m\}.$$

Then $\mathcal{L}_m$ is a locally closed subvariety of $|L| \times S$, and (3) induces via base change a flat and proper family $\mathcal{G}_m = \{(C_p, q) \in \mathcal{L}_m \times S \mid C_p = (C,p) \in \mathcal{L}_m, q \in C\}$ which has a distinguished section σ

$$\begin{tikzcd} \mathcal{G}_m \arrow[r, hook] \arrow[d] & \mathcal{L}_m \times S \arrow[dl] \\ \mathcal{L}_m \arrow[u, bend left, "\sigma"] & \end{tikzcd} \tag{4}$$

sending $C_p = (C,p)$ to $(C_p, p) \in \mathcal{G}_m$. Moreover, this family is equimultiple along σ of multiplicity m by construction.

We may interpret $\mathcal{L}_m$ as the family of curves in $|L|$ with m-fold points together with a section which distinguishes the m-fold point. This is important if the m-fold point is not isolated or if it splits in a neighbourhood into several simpler m-fold points.

Of course, since (3) can be viewed as a subfamily of (1) we may view (4) in the same way as a subfamily of (2).

Definition 6
Let $t_0 \in T$ be a pointed complex space, $C \subset S$ a curve, and $p \in C$ a point of multiplicity m. Then an *embedded (equimultiple) deformation of C in S over $t_0 \in T$ with section σ through p* is a commutative diagram of morphisms

$$\begin{tikzcd}
 & S \arrow[rr, hook] \arrow[ddl] & & T \times S \arrow[ddl] \\
p \in C \arrow[ur, hook] \arrow[rr, hook] \arrow[d] & & \mathcal{C} \arrow[ur, hook] \arrow[d, "\varphi"] & \\
t_0 \arrow[rr, mapsto] \arrow[u, bend left, "\sigma"] & & T \arrow[u, bend left, "\sigma"] &
\end{tikzcd}$$

where the right hand part of the diagram is an embedded (equimultiple) family of curves in S over T with section σ. Sometimes we will simply write (φ, σ) to denote a deformation as above.
Given two deformations, say (φ, σ) and (φ', σ'), of C over $t_0 \in T$ as above, a morphism of these deformations is a morphism $\psi : \mathcal{C}' \to \mathcal{C}$ which makes the obvious diagram commute:

$$\begin{tikzcd}
T \times S \arrow[rrrr, equal] \arrow[ddddr] & & & & T \times S \arrow[ddddl] \\
 & \mathcal{C}' \arrow[ul, hook] \arrow[rr, "\psi"] \arrow[ddd, "\varphi'"'] & & \mathcal{C} \arrow[ur, hook] \arrow[ddd, "\varphi"] & \\
 & & C \arrow[ul, hook] \arrow[ur, hook] \arrow[d] & & \\
 & & t_0 \arrow[dl, mapsto] \arrow[dr, mapsto] & & \\
 & T \arrow[rr, equal] \arrow[uuu, bend left, "\sigma'"] & & T \arrow[uuu, bend right, "\sigma"'] &
\end{tikzcd}$$

This gives rise to the deformation functor

$$\underline{\mathrm{Def}}^{sec,em}_{p \in C/S} : (\text{pointed complex spaces}) \to (\text{sets})$$

of embedded equimultiple deformations of C with section through p from the category of pointed complex spaces into the category of sets, where for a pointed complex space $t_0 \in T$

$$\begin{aligned}\underline{\mathrm{Def}}^{sec,em}_{p\in C/S}(t_0 \in T) = \{&\text{isomorphism classes of embedded equimultiple}\\ &\text{deformations } (\varphi,\sigma) \text{ of } C \text{ in } S \text{ over } t_0 \in T\\ &\text{with section through } p\}.\end{aligned}$$

Moreover, forgetting the section we have a natural transformation

$$\underline{\mathrm{Def}}^{sec,em}_{p\in C/\Sigma} \longrightarrow \underline{\mathrm{Def}}_{C/\Sigma}, \tag{5}$$

where the latter is the deformation functor

$$\underline{\mathrm{Def}}_{C/\Sigma} : (\text{pointed complex spaces}) \to (\text{sets})$$

of embedded deformations of C in S given by

$$\begin{aligned}\underline{\mathrm{Def}}_{C/S}(t_0 \in T) = \{&\text{isomorphism classes of embedded deformations}\\ &\text{of } C \text{ in } S \text{ over } t_0 \in T\ \}.\end{aligned}$$

Example 7
According to Example 3 a deformation of C in S over T_ε along a section through p is given by

- local equations $f + \varepsilon \cdot g$ such that f is a local equation for C and the $\frac{g}{f}$ glue to a global section of $\mathcal{O}_C(C)$,
- together with a section which in local coordinates in p is given as $\sigma : (x,y) \mapsto (x_a, y_b) = (x+\varepsilon\cdot a, y+\varepsilon\cdot b)$ for some $a, b \in \mathbb{C}\{x,y\}$.

If we forget the section it is well known (see e.g. [Mum66], Lecture 22) that two such deformations are isomorphic if and only if they induce the same global section of $\mathcal{O}_C(C)$ and this one-to-one correspondence is functorial so that we have an isomorphism of vector spaces

$$\underline{\mathrm{Def}}_{C/S}(T_\varepsilon) \xrightarrow{\cong} H^0\big(C, \mathcal{O}_C(C)\big).$$

Considering the natural transformation from (5) we may now ask what the image of $\underline{\mathrm{Def}}^{sec,em}_{p\in C/S}(T_\varepsilon)$ in $H^0\big(C,\mathcal{O}_C(C)\big)$ is. These are, of course, the sections which allow a section σ through p along which the deformation is equimultiple, and according to Lemma 8 we thus have an epimorphism

$$\underline{\mathrm{Def}}^{sec,em}_{p\in C/S}(T_\varepsilon) \longrightarrow\!\!\!\!\!\rightarrow H^0\big(C, \mathcal{J}_{Z/C}(C)\big),$$

where $\mathcal{J}_{Z/C}$ is the restriction to C of the ideal sheaf $\mathcal{J}_Z$ on S given by

$$\mathcal{J}_{Z,q} = \begin{cases} \mathcal{O}_{S,q}, & \text{if } q \neq p, \\ \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle + \langle x, y\rangle^m, & \text{if } q = p, \end{cases} \tag{6}$$

here f is a local equation for C in local coordinates x and y in p.
It remains the question what the dimension of the kernel of this map is, that is, how many different sections such an isomorphism class of embedded deformations of C in S over T_ε through p can admit.
J. Wahl showed in [Wah74], Proposition 1.9, that locally the equimultiple deformation admits a unique section if and only if C in p is not unitangential. If C is unitangential we may assume that locally in p it is given by $f = y^m + h.o.t.$. If we have an embedded deformation of C in S which along some section is equimultiple of multiplicity m, then locally it looks like

$$f + \varepsilon \cdot \left(a \cdot \tfrac{\partial f}{\partial x} + b \cdot \tfrac{\partial f}{\partial y} + h \right)$$

with $h \in \langle x, y\rangle^m$. However, since $\frac{\partial f}{\partial x} \in \langle x, y\rangle^m$ the deformation is equimultiple along the sections $(x, y) \mapsto (x + \varepsilon \cdot (c + a), y + \varepsilon \cdot b)$ for all $c \in \mathbb{C}$. Thus in this case the kernel turns out to be one-dimensional, i.e. there is a one-dimensional vector space $\mathcal{K}$ such that the following sequence is exact:

$$0 \to \mathcal{K} \to \underline{\mathrm{Def}}^{sec,em}_{p \in C/S}(T_\varepsilon) \to H^0\big(C, \mathcal{J}_{Z/C}(C)\big) \to 0. \tag{7}$$

Lemma 8
Let $f + \varepsilon \cdot g$ be a first-order infinitesimal deformation of $f \in \mathbb{C}\{x, y\}$, $m = \mathrm{ord}(f)$, $a, b \in \mathbb{C}\{x, y\}$, and $x_a = x + \varepsilon \cdot a$, $y_b = y + \varepsilon \cdot b$.
Then $f + \varepsilon \cdot g$ is equimultiple along the section $(x, y) \mapsto (x_a, y_b)$ if and only if

$$g - a \cdot \frac{\partial f}{\partial x} - b \cdot \frac{\partial f}{\partial y} \in \langle x, y\rangle^m.$$

In particular, $f + \varepsilon \cdot g$ is equimultiple along some section if and only if

$$g \in \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle + \langle x, y\rangle^m.$$

Proof: If $a, b \in \mathbb{C}\{x, y\}$ and $h \in \langle x, y\rangle^m$ then by Taylor expansion and since $\varepsilon^2 = 0$ we have

$$f + \varepsilon \cdot \left(a \cdot \tfrac{\partial f}{\partial x} + b \cdot \tfrac{\partial f}{\partial y} + h\right) = f(x_a, y_b) + \varepsilon \cdot h(x_a, y_b),$$

where $f(x_a, y_b), h(x_a, y_b) \in \langle x_a, y_b\rangle^m$, i.e. the infinitesimal deformation $f + \varepsilon \cdot \left(a \cdot \frac{\partial f}{\partial x} + b \cdot \frac{\partial f}{\partial y} + h\right)$ is equimultiple along $(x, y) \mapsto (x_a, y_b)$. Conversely, if $f + \varepsilon \cdot g$ is equimultiple along $(x, y) \mapsto (x_a, y_b)$ then

$$f(x, y) + \varepsilon \cdot g(x, y) = F(x_a, y_b) + \varepsilon \cdot G(x_a, y_b)$$

with $F(x_a, y_b), G(x_a, y_b) \in \langle x_a, y_b\rangle^m$. Again, by Taylor expansion and since $\varepsilon^2 = 0$ we have

$$f(x, y) = f(x_a, y_b) - \varepsilon \cdot \left(a \cdot \tfrac{\partial f}{\partial x}(x_a, y_b) + b \cdot \tfrac{\partial f}{\partial y}(x_a, y_b)\right)$$

and

$$\varepsilon \cdot g(x, y) = \varepsilon \cdot g(x_a, y_b).$$

Thus

$$F(x_a, y_b) = f(x_a, y_b)$$

and

$$\langle x_a, y_b\rangle^m \ni G(x_a, y_b) = g(x_a, y_b) - a \cdot \tfrac{\partial f}{\partial x}(x_a, y_b) - b \cdot \tfrac{\partial f}{\partial y}(x_a, y_b).$$

□

Example 9
If we fix a curve $C \subset S$ and a point $p \in C$ such that $\mathrm{mult}_p(C) = m$, i.e. if using the notation of Example 4 we fix a point $C_p = (C, p) \in \mathcal{H}_m$, then the diagram

$$\begin{tikzcd}
S \arrow[r, "\cong"] & \{C_p\} \times S \arrow[rr, hook] & & \mathcal{H}_m \times S \arrow[dl] \\
C \arrow[u, hook] \arrow[d] \arrow[r, "\cong"] & \{(C_p, q) \mid q \in C\} \arrow[u, hook] \arrow[d] \arrow[r, hook] & \mathcal{F}_m \arrow[ur, hook] \arrow[d] & \\
t_0 \arrow[r, equal] & C_p \arrow[r, mapsto] & \mathcal{H}_m \arrow[u, bend left, "\sigma"] &
\end{tikzcd} \tag{8}$$

is an embedded equimultiple deformation of C in S along the section σ through p. Moreover, any embedded equimultiple deformation of C in S with section through p as a family is up to isomorphism induced via

(1) in a unique way and thus factors obviously uniquely through (8). This means that every equimultiple deformation of C in S through p is induced up to isomorphism in a unique way from (8).
We now want to examine the tangent space to $\mathcal{H}_m$ at a point $C_p = (C,p)$, which is just

$$T_{C_p}(\mathcal{H}_m) = \mathrm{Hom}_{loc-K-Alg}\big(\mathcal{O}_{\mathcal{H}_m,C_p}, \mathbb{C}[\varepsilon]\big) = \mathrm{Hom}\big(T_\varepsilon, (\mathcal{H}_m, C_p)\big),$$

where $(\mathcal{H}_m, C_p)$ denotes the germ of $\mathcal{H}_m$ at C_p. However, a morphism

$$\psi : T_\varepsilon \longrightarrow (\mathcal{H}_m, C_p)$$

gives rise to a commutative fibre product diagram

$$\begin{tikzcd} T_\varepsilon \times_{\mathcal{H}_m} \mathcal{F}_m \arrow[r] \arrow[d, "\varphi'"] & \mathcal{F}_m \arrow[d] \\ T_\varepsilon \arrow[r, "\psi"] \arrow[u, bend left, "\sigma'"] & \mathcal{H}_m \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

sending the closed point of T_ε to C. Thus $(\varphi', \sigma') \in \underline{\mathrm{Def}}^{sec,em}_{p\in C/S}(T_\varepsilon)$ is an *embedded equimultiple deformation of C in S with section through p.* The universality of (8) then implies that up to isomorphism each one is of this form for a unique φ', and this construction is functorial. We thus have

$$T_{C_p}(\mathcal{H}_m) \cong \underline{\mathrm{Def}}^{sec,em}_{p\in C/S}(T_\varepsilon),$$

and hence (7) gives the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow T_{C_p}(\mathcal{H}_m) \longrightarrow H^0\big(C, \mathcal{J}_{Z/C}(C)\big) \longrightarrow 0.$$

In particular,

$$\dim_{\mathbb{C}}\big(T_{C_p}(\mathcal{H}_m)\big) = \begin{cases} \dim_{\mathbb{C}} H^0\big(C, \mathcal{J}_{Z/C}(C)\big) - 2, & \text{if } C \text{ is unitangential,} \\ \dim_{\mathbb{C}} H^0\big(C, \mathcal{J}_{Z/C}(C)\big) - 1, & \text{else.} \end{cases}$$

Example 10
If we do the same constructions replacing in (8) the family (2) by (4) we get for the tangent space to $\mathcal{L}_m$ at $C_p = (C,p)$ the diagram of exact

sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & T_{C_p}(\mathcal{H}_m) & \longrightarrow & H^0\big(C, \mathcal{J}_{Z/C}(C)\big) & \longrightarrow & 0 \\
 & & \Big\| & & \big\uparrow & & \big\uparrow & & \\
0 & \longrightarrow & \mathcal{K} & \longrightarrow & T_{C_p}(\mathcal{L}_m) & \longrightarrow & H^0\big(S, \mathcal{J}_Z(C)\big)/H^0(\mathcal{O}_S) & \longrightarrow & 0.
\end{array}$$

In order to see this consider the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0$$

induced from the structure sequence of C. This sequence shows that the tangent space to $|L|$ at C considered as a subspace of the tangent space $H^0\big(C, \mathcal{O}_C(C)\big)$ of H at C is just $H^0\big(S, \mathcal{O}_S(C)\big)/H^0(S, \mathcal{O}_S)$ – that is, a global section of $\mathcal{O}_C(C)$ gives rise to an embedded deformation of C in S which is actually a deformation in the linear system $|L|$ if and only if it comes from a global section of $\mathcal{O}_S(C)$, and the constant sections induce the trivial deformations. This construction carries over to the families (2) and (4).

In particular we get the following proposition.

Proposition 11
Using the notation from above let C be a curve in the linear system $|L|$ on S and suppose that $p \in C$ such that $\mathrm{mult}_p(C) = m$.
Then the tangent space of $\mathcal{L}_m$ at $C_p = (C, p)$ satisfies

$$\dim_{\mathbb{C}}\big(T_{C_p}(\mathcal{L}_m)\big) = \begin{cases} \dim_{\mathbb{C}} H^0\big(S, \mathcal{J}_Z(C)\big) - 2, & \textit{if } C \textit{ is unitangential}, \\ \dim_{\mathbb{C}} H^0\big(S, \mathcal{J}_Z(C)\big) - 1, & \textit{else.} \end{cases}$$

Moreover, the expected dimension of $T_{C_p}(\mathcal{L}_m)$ and thus of $\mathcal{L}_m$ at C_p is just

$$\mathrm{expdim}_{C_p}(\mathcal{L}_m) = \mathrm{expdim}_{\mathbb{C}}\big(T_{C_p}(\mathcal{L}_m)\big) = \dim|L| - \frac{(m+1)\cdot m}{2} + 2.$$

For the last statement on the expected dimension just consider the exact sequence

$$0 \to H^0\big(S, \mathcal{J}_Z(C)\big) \to H^0\big(S, \mathcal{O}_S(L)\big) \to H^0(S, \mathcal{O}_Z)$$

and note that the dimension of $H^0\big(S, \mathcal{J}_Z(C)\big)$, and hence of $T_{C_p}(C)$, attains the minimal possible value if the last map is surjective. The expected dimension of $H^0\big(S,\mathcal{J}_Z(C)\big)$ hence is

$$\operatorname{expdim}_{\mathbb{C}} H^0\big(S,\mathcal{J}_Z(C)\big)=\dim|L|+1-\deg(Z),$$

and it suffices to calculate $\deg(Z)$. If C is unitangential we may assume that C locally in p is given by $f=y^m+h.o.t.$, so that

$$\mathcal{O}_{Z,p}=\mathbb{C}\{x,y\}/\langle y^{m-1}\rangle+\langle x,y\rangle^m,$$

and hence $\deg(Z)=\frac{(m+1)\cdot m}{2}-1$. If C is not unitangential, then we may assume that it locally in p is given by an equation f such that $f_m=\operatorname{jet}_m(f)=x^\mu\cdot y^\nu\cdot g$, where x and y do not divide g, but μ and ν are at least one. Suppose now that the partial derivatives of f_m are not linearly independent, then we may assume $\frac{\partial f_m}{\partial x}\equiv\alpha\cdot\frac{\partial f_m}{\partial y}$ and thus

$$\mu yg\equiv\alpha\nu xg+\alpha xy\cdot\frac{\partial g}{\partial y}-xy\cdot\frac{\partial g}{\partial x},$$

which would imply that y divides g in contradiction to our assumption. Thus the partial derivatives of f_m are linearly independent, which shows that

$$\deg(Z)=\dim_{\mathbb{C}}\Big(\mathbb{C}\{x,y\}/\langle\tfrac{\partial f}{\partial x},\tfrac{\partial f}{\partial y}\rangle+\langle x,y\rangle^m\Big)=\frac{(m+1)\cdot m}{2}-2.$$

Example 12

Let us consider the Example 5 in the case where $S=\mathbb{P}^2$ and $L=\mathcal{O}_{\mathbb{P}^2}(d)$. We will show that $\mathcal{L}_m$ is then *smooth of the expected dimension.* Note that $\pi(\mathcal{L}_m)$ will only be smooth at C if C has an ordinary m-fold point, that is, if all tangents are different.

Given $C_p=(C,p)\in\mathcal{L}_m$ we may pass to a suitable affine chart containing p as origin and assume that $H^0\big(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(d)\big)$ is parametrised by polynomials

$$F_{\underline{a}}=f+\sum_{i+j=0}^{d}a_{i,j}\cdot x^iy^j,$$

where f is the equation of C in this chart. The closure of $\pi(\mathcal{L}_m)$ in $|L|$ locally at C is then given by several equations, say $F_1,\ldots,F_k\in\mathbb{C}[a_{i,j}|i+j=0,\ldots,d]$, in the coefficients $a_{i,j}$. We get these equations by eliminating the variables x and y from the ideal defined by

$$\left\langle\frac{\partial^{i+j}F_{\underline{a}}}{\partial x^iy^j}\;\Big|\;i+j=0,\ldots,m-1\right\rangle.$$

And $\mathcal{L}_m$ is locally in C_p described by the equations

$$F_1 = 0, \ldots, F_k = 0, \quad \frac{\partial^{i+j} F_{\underline{a}}}{\partial x^i y^j} = 0, \quad i + j = 0, \ldots, m - 1.$$

However, the Jacoby matrix of these equations with respect to the variables $x, y, a_{i,j}$ contains a diagonal submatrix of size $\frac{m\cdot(m+1)}{2}$ with ones on the diagonal, so that its rank is at least $\frac{m\cdot(m+1)}{2}$, which – taking into account that $|L| = \mathbb{P}\big(H^0\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)\big)\big)$ – implies that the tangent space to $\mathcal{L}_m$ at C_p has codimension at least $\frac{m\cdot(m+1)}{2} - 1$ in the tangent space of $\mathcal{L}$. By Proposition 11 we thus have

$$\begin{aligned}
\dim_{C_p}(\mathcal{L}_m) &\leq \dim_{\mathbb{C}} T_{C_p}(\mathcal{L}_m) \leq \dim_{\mathbb{C}} T_{C_p}(\mathcal{L}) - \frac{m \cdot (m+1)}{2} + 1 \\
&= \dim(\mathcal{L}) - \frac{m \cdot (m+1)}{2} + 1 \\
&= \dim |L| - \frac{m \cdot (m+1)}{2} + 2 \\
&= \operatorname{expdim}_{C_p}(\mathcal{L}_m) \leq \dim_{C_p}(\mathcal{L}_m),
\end{aligned}$$

which shows that $\mathcal{L}_m$ is smooth at C_p of the expected dimension.

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