INJECTIVE ANALYTIC MAPS - A COUNTEREXAMPLE TO THE PROOF

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Abstract. In [Ném93] the author translates a conjecture of Le Dung Trang on the non-existence of injective analytic maps $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ with $df(0) = 0$ into the non-existence of a hypersurface germ in $(\mathbb{C}^{n+1}, 0)$ with rather unexpected properties. However, the proof given in [Ném93] contains an apparently fatal error, as we demonstrate with an example.

In [Ném93] the author addresses the problem whether the differential $df(0)$ of an injective analytic map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ can possibly be of rank less than $n-1$. A long standing conjecture of Le Dung Trang for the case $n = 2$ states that this cannot be the case, even though it is not at all obvious how the topological fact of injectivity and the analytic datum on the rank of the derivative might relate to each other. Analysing the image $(X, 0)$ of f as an analytic subspace of $(\mathbb{C}^{n+1}, 0)$, the author claims that a counter example to Le's conjecture would have an unexpected "bad" property. More precisely, he defines what it means for $(X, 0)$ to be "good", and sets out to show that if X is good then the rank of $df(0)$ is at least $n-1$ and $(X, 0)$ is an equisingular family of plane curves. However, the proof of this theorem contains a fundamental error, which – as we are convinced after discussions with the author – cannot be repaired. We will outline the main ideas of the proof and give an example which shows that it does not work as described, and where it goes wrong. In order to keep the notation simple we restrict ourselves to the case where $n = 2$.

We would like to point out that our example is not a counter-example to the statement of the Theorem in [Ném93] nor do we know of any such. It shows merely that the proof is wrong.

Let us now recall the necessary definitions from [Ném93].

Definition: A two-dimensional subgerm $(X, 0) \subset (\mathbb{C}^3, 0)$ is called *good* if there exist coordinates (w_1, w_2, w_3) for $(\mathbb{C}^3, 0)$ and a map germ $F : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ defining $(X, 0)$, i. e. $X = F^{-1}(0)$, such that $W_0 = X \cap \{w_1 = 0\}$ is an isolated plane curve singularity, and $\frac{\partial F}{\partial w_1} \notin \left\langle w_1, \frac{\partial F}{\partial w_2}, \frac{\partial F}{\partial w_3} \right\rangle$.

Nemethi then states the following

"Theorem": *If the image $(X, 0)$ of an injective analytic map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is good, then the rank of $df(0)$ is at least one. Moreover, $(X, 0)$ is an equisingular family of plane curve singularities over the base $(\mathbb{C}, 0)$.*

The idea of the proof is to compare the two isolated plane curve singularities $V_0 = f_1^{-1}(0)$ and $W_0 = X \cap \{w_1 = 0\} = \psi(V_0)$, where $f_i = w_i \circ f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ for $i = 1, 2, 3$ and $\psi = (f_2, f_3) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. The Milnor fibre $V_t = f^{-1}(t)$ for $t \neq 0$ maps via ψ to $V_t' = \psi(V_t)$, which is in general singular. If f is injective, then the restriction of ψ to each level set of f_1 (i.e. to V_t) must also be injective. The vanishing cycles of V_t must therefore be mapped homeomorphically by ψ to non-trivial cycles in V_t'. Nemethi claims that under these circumstances, the vanishing cycles of V_t,

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mapped by ψ into V_t', together with the vanishing cycles of the singularities of V_t' (which it has acquired under the map ψ) together make up a complete set of vanishing cycles of a Milnor fibre of W_0. In V_t one can choose vanishing cycles which do not pass through the (isolated) non-immersive points of ψ. In a smoothing of the singularities of V_t', the vanishing cycles can be confined to arbitrarily small neighbourhoods (in the ambient space) of the points being smoothed, and thus the vanishing cycles coming from the singularities of V_t' have zero intersection number with the images under ψ of the vanishing cycles coming from V_t. *This implies that the Dynkin diagram of the isolated plane curve singularity W_0 is disconnected,* contradicting a well-known theorem of Lazzeri ([Laz73]).

From this Nemethi concludes that one of the two sets of vanishing cycles must be empty, and thus that either V_0 or V_t' is smooth. in the first case, the derivative at $(0,0)$ of f_1 is not zero, and so the derivative of f itself is not zero. In the second case, V_t' is a Milnor fibre for W_0, and so W_0 and V_0 have the same Milnor number, from which it follows that ψ gives an isomorphism $V_0 \to W_0$. From this Nemethi is able to show that the germ $(X,0)$ is not good.

To make this argument rigorous, Nemethi has to show that the two types of cycles together really do form a basis of vanishing cycles in a Milnor fibre of W_0. To do this he considers the deformation of V_0 induced by $f_1 : (\mathbb{C}^2,0) \to (\ell,0) = (\mathbb{C},0)$. The image of this deformation under ψ then gives a deformation of W_0 which can be induced from an $\mathcal{R}$-miniversal deformation Θ of $F_| : \{w_1=0\} \to (\mathbb{C},0)$ via base change r. The author claims then that a small perturbation of $r(\ell)$ gives rises to a Milnor fibre of W_0 in which the set of vanishing cycles splits into those coming from a Milnor fibre of V_0 and those arising from the singularities of V_t'. For this to be the case, it must be possible to deform $\ell' = r(\ell)$ in a family to $\{\ell_t'\}_{t\in\mathbb{C},0)}$ in such a way that for $t \neq 0$, ℓ_t' intersects the discriminant D in the base of the deformation Θ transversally in a finite number of points, and that $\ell_t' \cap D$ does not meet the boundary of a good representative of the deformation. The problem with the argument is that if $r(\ell)$ is *contained* in D, then this is not in general possible. And this is exactly what happens in our example, even though to see this one has to follow the constructions in the proof of the theorem very closely. For the details we refer to [Kei93].

An easy way to see that the proof must go wrong somewhere is to consider the following example.

$$f : \big(\mathbb{C}^2,0\big) \to \big(\mathbb{C}^3,0\big) : (x,y) \mapsto \big(y^3+x^2, x, y^2\big).$$

Obviously f is injective and

$$F : \big(\mathbb{C}^3,0\big) \to (\mathbb{C},0) : (w_1,w_2,w_3) \mapsto \big(w_1 - w_2^2\big)^2 - w_3^3$$

is a defining equation of $(X,0) = \big(\mathrm{im}(f),0\big)$. In this case

$$V_0 = f_1^{-1}(0) = \big\{y^3+x^2=0\big\}$$

is a cusp, hence in particular not smooth, while

$$W_0 = X \cap \{w_1=0\} = \big\{w_2^4 - w_3^3 = 0\big\}$$

is an E_6-singularity. Even though f is injective, V_0 and W_0 do not have the same Milnor number!

References

[Kei93] Thomas Keilen, *A claim on the rank of an injective map*, Master's thesis, University of Warwick, 1993, http://www.mathematik.uni-kl.de/~keilen/Documents/master.ps.gz.

[Laz73] Fulvio Lazzeri, *A theorem on the monodromy of isolated singularities*, Astérisque **7–8** (1973), 269–275.

[Ném93] András Némethi, *Injective analytic maps*, Duke Math. J. **69** (1993), no. 2, 335–347.

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