EXISTENCE OF CURVES WITH PRESCRIBED TOPOLOGICAL SINGULARITIES

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Abstract. Throughout this paper we study the existence of irreducible curves C on smooth projective surfaces Σ with singular points of prescribed topological types $\mathcal{S}_1,\ldots,\mathcal{S}_r$. There are *necessary* conditions for the existence of the type $\sum_{i=1}^r \mu(\mathcal{S}_i) \leq \alpha C^2 + \beta C.K + \gamma$ for some fixed divisor K on Σ and suitable coefficients α, β and γ, and the main *sufficient* condition that we find is of the same type, saying it is asymptotically optimal. An important ingredient for the proof is a vanishing theorem for invertible sheaves on the blown up Σ of the form $\mathcal{O}_{\widetilde{\Sigma}}(\pi^*D - \sum_{i=1}^r m_iE_i)$, deduced from the Kawamata-Vieweg Vanishing Theorem. A large part of the paper is devoted to the investigation of our conditions on ruled surfaces, products of elliptic curves, surfaces in $\mathbb{P}^3_{\mathbb{C}}$, and K3-surfaces.

Contents

1. Introduction

General Assumptions and Notations

Throughout this paper Σ will be a smooth projective surface over $\mathbb{C}$.

Given distinct points $z_1,\ldots,z_r\in\Sigma$, we denote by $\pi:\widetilde{\Sigma}=\mathrm{Bl}_{\underline{z}}(\Sigma)\to\Sigma$ the blow up of Σ in $\underline{z}=(z_1,\ldots,z_r)$, and the exceptional divisors π^*z_i will be denoted by E_i, $i=1,\ldots,r$. We shall write $\widetilde{C}=\mathrm{Bl}_{\underline{z}}(C)$ for the strict transform of a curve $C\subset\Sigma$.

For any smooth surface S we will denote by $\mathrm{Div}(S)$ the group of divisors on S and by K_S its canonical divisor. If D is any divisor on S, $\mathcal{O}_S(D)$ shall be a corresponding invertible sheaf. $|D|_l=\mathbb{P}\big(H^0\big(S,\mathcal{O}_S(D)\big)\big)$ denotes the system of curves linearly equivalent to D, while we use the notation $|D|_a$ for the system of curves algebraically equivalent to D (cf. [Har77] Ex. V.1.7), that is the reduction of the connected component of Hilb_S, the Hilbert scheme of all curves on S, containing any curve algebraically equivalent to D (cf. [Mum66] Chapter 15). We will use the notation $\mathrm{Pic}(S)$ for the Picard group of S, that is $\mathrm{Div}(S)$ modulo linear equivalence (denoted by $\sim_l$), $\mathrm{NS}(S)$ for the Néron-Severi group, that is $\mathrm{Div}(S)$ modulo algebraic equivalence (denoted by $\sim_a$), and $\mathrm{Num}(S)$ for $\mathrm{Div}(S)$ modulo numerical equivalence (denoted by $\sim_n$). Note that for all examples of surfaces Σ which we consider in Section 6 $\mathrm{NS}(\Sigma)$ and $\mathrm{Num}(\Sigma)$ coincide.

Given a curve $C\subset\Sigma$ we will write $p_a(C)$ for its arithmetical genus and $g(C)$ for the geometrical one.

Let Y be a Zariski topological space. We say a subset $U\subseteq Y$ is *very general* if it is an at most countable intersection of open dense subsets of Y. Some statement is said to hold for points $z_1,\ldots,z_r\in Y$ (or $\underline{z}\in Y^r$) *in very general position* if there is a suitable very general subset $U\subseteq Y^r$, contained in the complement of the closed subvariety $\bigcup_{i\neq j}\{\underline{z}\in Y^r\mid z_i=z_j\}$ of Y^r, such that the statement holds for all $\underline{z}\in U$. The main results of this paper will only be valid for points in very general position.

Given distinct points $z_1,\ldots,z_r\in\Sigma$ and non-negative integers $m_1,\ldots,m_r$ we denote by $X(\underline{m};\underline{z})=X(m_1,\ldots,m_r;z_1,\ldots,z_r)$ the zero-dimensional subscheme of Σ defined by the ideal sheaf $\mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}$ with stalks

$$\mathcal{J}_{X(\underline{m};\underline{z})/\Sigma,z}=\begin{cases}\mathfrak{m}_{\Sigma,z_i}^{m_i}, & \text{if } z=z_i, i=1,\ldots,r,\\ \mathcal{O}_{\Sigma,z}, & \text{else.}\end{cases}$$

We call a scheme of the type $X(\underline{m};\underline{z})$ a *generic fat point scheme.*

For a reduced curve $C\subset\Sigma$ we define the zero-dimensional subscheme $X^{es}(C)$ of Σ via the ideal sheaf $\mathcal{J}_{X^{es}(C)/\Sigma}$ with stalks

$$\mathcal{J}_{X^{es}(C)/\Sigma,z}=I^{es}(C,z)=\{g\in\mathcal{O}_{\Sigma,z}\mid f+\varepsilon g\text{ is equisingular over }\mathbb{C}[\varepsilon]/(\varepsilon^2)\},$$

where $f\in\mathcal{O}_{\Sigma,z}$ is a local equation of C at z. $I^{es}(C,z)$ is called the *equisingularity ideal* of the singularity (C,z), and it is of course $\mathcal{O}_{\Sigma,z}$ whenever z is a smooth point. If x,y are local coordinates of Σ at z, then $I^{es}(C,z)/\big(f,\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\big)$ can be identified with the tangent space of the equisingular stratum in the semiuniversal deformation of

(C, z). (cf. [Wah74], [DH88], and Definition 5.1) We call $X^{es}(C)$ the *equisingularity scheme of* C.

If $X \subset \Sigma$ is any zero-dimensional scheme with ideal sheaf $\mathcal{J}_X$ and if $L \subset \Sigma$ is any curve with ideal sheaf $\mathcal{L}$, we define the *residue scheme* $X : L \subset \Sigma$ by the ideal sheaf $\mathcal{J}_{X:L/\Sigma} = \mathcal{J}_X : \mathcal{L}$ with stalks

$$\mathcal{J}_{X:L/\Sigma,z} = \mathcal{J}_{X,z} : l_z,$$

where $l_z \in \mathcal{O}_{\Sigma,z}$ is a local equation for L and ":" denotes the ideal quotient. This naturally leads to the definition of the *trace scheme* $X \cap L \subset L$ via the ideal sheaf $\mathcal{J}_{X\cap L/L}$ given by the exact sequence

$$0 \longrightarrow \mathcal{J}_{X:L/\Sigma}(-L) \xrightarrow{\cdot L} \mathcal{J}_{X/\Sigma} \longrightarrow \mathcal{J}_{X\cap L/L} \longrightarrow 0.$$

Given topological singularity types[1] $\mathcal{S}_1, \ldots, \mathcal{S}_r$ and a divisor $D \in \mathrm{Div}(\Sigma)$, we denote by $V_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ the locally closed subspace of $|D|_l$ of reduced curves in the linear system $|D|_l$ having precisely r singular points of types $\mathcal{S}_1, \ldots, \mathcal{S}_r$. Analogously, $V_{|D|}(m_1, \ldots, m_r) = V_{|D|}(\underline{m})$ denotes the locally closed subspace of $|D|_l$ of reduced curves having precisely r ordinary singular points of multiplicities $m_1, \ldots, m_r$. (cf. [Los98] 1.3.2)

The spaces $V = V_{|D|}(\mathcal{S}_1, \ldots, \mathcal{S}_r)$ respectively $V = V_{|D|}(\underline{m})$ are the main objects of interest of this paper. We say V is *T-smooth* at $C \in V$ if the germ (V, C) is smooth of the (expected) dimension $\dim |D|_l - \deg(X)$, where $X = X^{es}(C)$ respectively $X = X(\underline{m}; \underline{z})$ with $\mathrm{Sing}(C) = \{z_1, \ldots, z_r\}$. By [Los98] Proposition 2.1 T-smoothness of V at C is implied by the vanishing of $H^1\big(\Sigma, \mathcal{J}_{X/\Sigma}(C)\big)$.

It is the aim of this paper to give sufficient conditions for the non-emptiness of V in terms of the linear system $|D|_l$ and invariants of the imposed singularities. The results are generalisations of known results for $\mathbb{P}^2_{\mathbb{C}}$, and for an overview on these we refer to [Los98] Chapter 4.

We basically follow the ideas described in [Los98] 4.1.2. The case of ordinary singularities (Corollary 4.2) is treated by applying a vanishing theorem for generic fat point schemes (Theorem 2.1), and the more interesting case of prescribed topological types $\mathcal{S}_1, \ldots, \mathcal{S}_r$ is then dealt with by gluing local equations into a curve with ordinary singularities. Upper bounds for the minimal possible degrees of these local equations can be taken from the $\mathbb{P}^2_{\mathbb{C}}$-case (cf. [Los98] Theorem 4.2).

Thus the main results of this paper are the following theorems and their corollaries Corollary 4.2 and Corollary 5.4.

2.1 Theorem
Let $m_1 \geq \ldots \geq m_r \geq 0$ *be non-negative integers,* $\alpha \in \mathbb{R}$ *with* $\alpha > 1$*,* $k_\alpha = \max\left\{n \in \mathbb{N} \;\middle|\; n < \frac{\alpha}{\alpha-1}\right\}$ *and let* $D \in \mathrm{Div}(\Sigma)$ *be a divisor satisfying the following three conditions*

[1]For the definition of a singularity type and more information see [Los98] 1.2.

$$(D-K_\Sigma)^2 \geq \max\left\{\alpha\cdot\sum_{i=1}^r(m_i+1)^2, (k_\alpha\cdot m_1+k_\alpha)^2\right\}, \tag{2.1}$$

$$(D-K_\Sigma).B \geq k_\alpha\cdot(m_1+1) \text{ for any irreducible curve } B \text{ with } B^2=0 \text{ and } \dim|B|_a>0, \text{ and} \tag{2.2}$$

$$D-K_\Sigma \text{ is nef.} \tag{2.3}$$

Then for $z_1,\ldots,z_r\in\Sigma$ in very general position and $\nu>0$

$$H^\nu\left(\mathrm{Bl}_{\underline{z}}(\Sigma), \pi^*D-\sum_{i=1}^r m_iE_i\right)=0.$$

In particular,

$$H^\nu\big(\Sigma,\mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big)=0.$$

4.1 Theorem
Given $m_1,\ldots,m_r\in\mathbb{N}_0$, not all zero, and $z_1,\ldots,z_r\in\Sigma$, in very general position. Let $L\in\mathrm{Div}(\Sigma)$ be very ample over $\mathbb{C}$, and let $D\in\mathrm{Div}(\Sigma)$ be such that

$$h^1\big(\Sigma,\mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D-L)\big)=0, \text{ and} \tag{4.1}$$

$$D.L-2g(L)\geq m_i+m_j \text{ for all } i,j. \tag{4.2}$$

Then there exists a curve $C\in|D|_l$ with ordinary singular points of multiplicity m_i at z_i for $i=1,\ldots,r$ and no other singular points. Furthermore,

$$h^1\big(\Sigma,\mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big)=0,$$

and in particular, $V_{|D|}(\underline{m})$ is T-smooth at C.
If in addition (4.3) $D^2>\sum_{i=1}^r m_i^2$, *then C can be chosen to be irreducible and reduced.*

5.3 Theorem (Existence)
Let $\mathcal{S}_1,\ldots,\mathcal{S}_r$ be singularity types, and suppose there exists an irreducible curve $C\in|D|_l$ with $r+r'$ ordinary singular points $z_1,\ldots,z_{r+r'}$ of multiplicities $m_1,\ldots,m_{r+r'}$ respectively as its only singularities such that $m_i=s(\mathcal{S}_i)+1$, for $i=1,\ldots,r$, and

$$h^1\big(\Sigma,\mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big)=0.$$

Then there exists an irreducible curve $\widetilde{C}\in|D|_l$ with r singular points of types $\mathcal{S}_1,\ldots,\mathcal{S}_r$ and r' ordinary singular points of multiplicities $m_{r+1},\ldots,m_{r+r'}$ as its only singularities.[2]

Of course, combining the vanishing theorem Theorem 2.1 with the existence theorems Theorem 4.1 and Theorem 5.3 we get sufficient numerical conditions for the existence of curves with certain singularities (see Corollaries 4.2 and 5.4, and see Section 6 for special surfaces).

[2]Here, of course, $\underline{m}=(m_1,\ldots,m_{r+r'})$ and $\underline{z}=(z_1,\ldots,z_{r+r'})$. See Definition 5.1 for the definition of $s(\mathcal{S}_i)$.

Given any scheme X and any coherent sheaf $\mathcal{F}$ on X, we will often write $H^\nu(\mathcal{F})$ instead of $H^\nu(X,\mathcal{F})$ when no ambiguity can arise. Moreover, if $\mathcal{F} = \mathcal{O}_X(D)$ is the invertible sheaf corresponding to a divisor D, we will usually use the notation $H^\nu(X,D)$ instead of $H^\nu\big(X,\mathcal{O}_X(D)\big)$. Similarly when considering tensor products over the structure sheaf of some scheme X we may sometimes just write $\otimes$ instead of $\otimes_{\mathcal{O}_X}$.

Section 2 is devoted to the proof of the vanishing theorem Theorem 2.1, and Section 3 provides an important ingredient in this proof. The following sections Section 4 and Section 5 are concerned with the existence theorems Theorem 4.1 and Theorem 5.3, while in Section 6 we calculate the conditions which we have found in the case of ruled surfaces, products of elliptic curves, surfaces in $\mathbb{P}^3_{\mathbb{C}}$, and K3-surfaces. Finally, in the appendix we gather some well known respectively fairly easy facts used throughout the paper for the convenience of the reader.

2. The Vanishing Theorem

2.1 Theorem

Let $m_1 \geq \ldots \geq m_r \geq 0$ be non-negative integers, $\alpha \in \mathbb{R}$ with $\alpha > 1$, $k_\alpha = \max\big\{n \in \mathbb{N} \;\big|\; n < \frac{\alpha}{\alpha-1}\big\}$ and let $D \in \operatorname{Div}(\Sigma)$ be a divisor satisfying the following three conditions

$$(D-K_\Sigma)^2 \geq \max\left\{\alpha \cdot \sum_{i=1}^r (m_i+1)^2, (k_\alpha \cdot m_1 + k_\alpha)^2\right\}, \tag{2.1}$$

$$(D-K_\Sigma).B \geq k_\alpha \cdot (m_1+1) \text{ for any irreducible curve } B \text{ with } B^2=0 \text{ and } \dim|B|_a > 0, \text{ and} \tag{2.2}$$

$$D - K_\Sigma \text{ is nef.} \tag{2.3}$$

Then for $z_1,\ldots,z_r \in \Sigma$ in very general position and $\nu > 0$

$$H^\nu\left(\operatorname{Bl}_{\underline{z}}(\Sigma), \pi^*D - \sum_{i=1}^r m_i E_i\right) = 0.$$

In particular,

$$H^\nu\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big) = 0.$$

Proof: By the Kawamata–Viehweg Vanishing Theorem (cf. [Kaw82] and [Vie82]) it suffices to show that $A = \big(\pi^*D - \sum_{i=1}^r m_iE_i\big) - K_{\widetilde{\Sigma}}$ is big and nef, i. e. we have to show:

(a) $A^2 > 0$, and

(b) $A.B' \geq 0$ for any irreducible curve B' in $\widetilde{\Sigma} = \operatorname{Bl}_{\underline{z}}(\Sigma)$.

Note that $A = \pi^*(D-K_\Sigma) - \sum_{i=1}^r (m_i+1)E_i$, and thus by Hypothesis (2.4) we have

$$A^2 = (D-K_\Sigma)^2 - \sum_{i=1}^r (m_i+1)^2 > 0,$$

which gives condition (a).

For condition (b) we observe that an irreducible curve B' on $\widetilde{\Sigma}$ is either the strict transform of an irreducible curve B in Σ or is one of the exceptional curves E_i. In the latter case we have

$$A.B' = A.E_i = m_i + 1 > 0.$$

We may, therefore, assume that $B' = \widetilde{B}$ is the strict transform of an irreducible curve B on Σ having multiplicity $\operatorname{mult}_{z_i}(B) = n_i$ at z_i, $i = 1, \ldots, r$. Then

$$A.B' = (D - K_\Sigma).B - \sum_{i=1}^{r}(m_i + 1)n_i,$$

and thus condition (b) is equivalent to

(b') $(D - K_\Sigma).B \geq \sum_{i=1}^{r}(m_i + 1)n_i$.

Since $\underline{z}$ is in very general position Lemma 3.1 applies in view of Corollary A.2. Using the Hodge Index Theorem, Hypothesis (2.4), Lemma 3.1, and the Cauchy-Schwarz Inequality we get the following sequence of inequalities:

$$\begin{aligned}
\big((D - K_\Sigma).B\big)^2 \;&\geq\; (D - K_\Sigma)^2 \cdot B^2 \;\geq\; \alpha \cdot \left(\textstyle\sum_{i=1}^{r}(m_i+1)^2\right) \cdot \left(\textstyle\sum_{i=1}^{r} n_i^2 - n_{i_0}\right) \\
&= \textstyle\sum_{i=1}^{r}(m_i+1)^2 \cdot \sum_{i=1}^{r} n_i^2 + (\alpha - 1) \cdot \Big(\sum_{i=1}^{r}(m_i+1)^2 \cdot \big(\sum_{i=1}^{r} n_i^2 - \frac{\alpha}{\alpha-1} \cdot n_{i_0}\big)\Big) \\
&\geq \Big(\textstyle\sum_{i=1}^{r}(m_i+1) \cdot n_i\Big)^2 + (\alpha - 1) \cdot \Big(\sum_{i=1}^{r}(m_i+1)^2 \cdot \big(\sum_{i=1}^{r} n_i^2 - \frac{\alpha}{\alpha-1} \cdot n_{i_0}\big)\Big),
\end{aligned}$$

where $i_0 \in \{1, \ldots, r\}$ is such that $n_{i_0} = \min\{n_i \mid n_i \neq 0\}$. Since $D - K_\Sigma$ is nef, condition (b') is satisfied as soon as we have

$$\sum_{i=1}^{r} n_i^2 \geq \tfrac{\alpha}{\alpha-1} \cdot n_{i_0}.$$

If this is not fulfilled, then $n_i < \frac{\alpha}{\alpha-1}$ for all $i = 1, \ldots, r$, and thus

$$\sum_{i=1}^{r}(m_i + 1) \cdot n_i \leq k_\alpha \cdot (m_1 + 1).$$

Hence, for the remaining considerations (b') may be replaced by the worst case

$$(D - K_\Sigma).B \geq k_\alpha \cdot (m_1 + 1).$$

Note that since the z_i are in very general position and $z_{i_0} \in B$ we have that $B^2 \geq 0$ and $\dim |B|_a > 0$ (cf. Corollary A.3). If $B^2 > 0$ then we are done by the Hodge Index Theorem and Hypothesis (2.4), since $D - K_\Sigma$ is nef:

$$(D - K_\Sigma).B \geq \sqrt{(D - K_\Sigma)^2} \geq \sqrt{(k_\alpha \cdot m_1 + k_\alpha)^2} \geq k_\alpha \cdot (m_1 + 1).$$

It remains to consider the case $B^2 = 0$ which is covered by Hypothesis (2.5).

For the "in particular" part we just note that

$$H^\nu\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big) = H^\nu\left(\Sigma, \bigotimes_{i=1}^{r} \mathcal{J}_{X(m_i;z_i)/\Sigma} \otimes \mathcal{O}_\Sigma(D)\right) =$$

$$H^\nu\left(\widetilde{\Sigma}, \pi^*D - \sum_{i=1}^{r} m_i E_i\right).$$

□

Choosing the constant $\alpha = 2$ in Theorem 2.1, then $\frac{\alpha}{\alpha-1} = 2$ and thus $k_\alpha = 1$. We therefore get the following corollary, which has the advantage that the conditions look simpler, and that the hypotheses on the "exceptional" curves are not too hard.

2.2 Corollary

Let $m_1, \ldots, m_r \in \mathbb{N}_0$, and $D \in \operatorname{Div}(\Sigma)$ be a divisor satisfying the following three conditions

(2.4) $$(D - K_\Sigma)^2 \geq 2 \cdot \sum_{i=1}^{r} (m_i + 1)^2,$$

(2.5) $(D - K_\Sigma).B > \max\{m_i \mid i = 1, \ldots, r\}$ *for any irreducible curve B with $B^2 = 0$ and* $\dim |B|_a > 0$, *and*

(2.6) $D - K_\Sigma$ *is nef.*

Then for $z_1, \ldots, z_r \in \Sigma$ in very general position and $\nu > 0$

$$H^\nu\left(\mathrm{Bl}_{\underline{z}}(\Sigma), \pi^*D - \sum_{i=1}^{r} m_i E_i\right) = 0.$$

In particular,

$$H^\nu\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big) = 0.$$

2.3 Remark

Condition (2.3) respectively Condition (2.6) are in several respects "expectable". First, Theorem 2.1 is a corollary of the Kawamata–Viehweg Vanishing Theorem, and if we take all m_i to be zero, our assumptions should basically be the same, i. e. $D - K_\Sigma$ nef and big. The latter is more or less just (2.1) respectively (2.4). Secondly, we want to apply the theorem to an existence problem. A divisor being nef means it is somehow close to being effective, or better its linear system is close to being non-empty. If we want that some linear system $|D|_l$ contains a curve with certain properties, then it seems not to be so unreasonable to restrict to systems where already $|D - K_\Sigma|_l$, or even $|D - L - K_\Sigma|_l$ with L some fixed divisor, is of positive dimension, thus nef.

In many interesting examples, such as $\mathbb{P}^2_{\mathbb{C}}$, Condition (2.2) respectively (2.5) turn out to be obsolete or easy to handle. So finally the most restrictive obstruction seems to be (2.1) respectively (2.4).

If we consider the situation where the largest multiplicity m_1 occurs in a large number, more precisely, if $m_1 = \ldots = m_{l_\alpha}$ with $l_\alpha = \min\{n \in \mathbb{N} \mid \alpha \cdot n \geq k_\alpha^2\}$, then Condition (2.1) comes down to

$$(D - K_\Sigma)^2 \geq \alpha \cdot \sum_{i=1}^{r} (m_i + 1)^2. \tag{2.1'}$$

2.4 Remark

Even though we said that Condition (2.4) was the really restrictive condition we would like to understand better what Condition (2.5) means. We therefore show in Appendix B that an algebraic system $|B|_a$ of dimension greater than zero with B irreducible and $B^2 = 0$ gives rise to a fibration $f : \Sigma \to H$ of Σ over a smooth projective curve H whose fibres are just the elements of $|B|_a$.

3. The Lemma of Geng Xu

3.1 Lemma

Let $\underline{z} = (z_1, \ldots, z_r) \in \Sigma^r$ be in very general position, $\underline{n} \in \mathbb{N}_0^r$, and let $B \subset \Sigma$ be an irreducible curve with $\operatorname{mult}_{z_i}(B) \geq n_i$*, then*

$$B^2 \geq \sum_{i=1}^{r} n_i^2 - \min\{n_i \mid n_i \neq 0\}.$$

3.2 Remark

(i) A proof for the above lemma in the case $\Sigma = \mathbb{P}_{\mathbb{C}}^2$ can be found in [Xu94] and in the case $r = 1$ in [EL93]. Here we just extend the arguments given there to the slightly more general situation.

(ii) For better estimates of the self intersection number of curves where one has some knowledge on equisingular deformations inside the algebraic system see [GS84].

(iii) With the notation of Lemma A.1 respectively Corollary A.2 the assumption in Lemma 3.1 could be formulated more precisely as "let $B \subset \Sigma \subseteq \mathbb{P}_{\mathbb{C}}^N$ be an irreducible curve such that $V_{B,\underline{n}} = \Sigma^r$", or "let $\underline{z} \in \Sigma^r \setminus V$".

(iv) Note, that one cannot expect to get rid of the "$-\min\{n_i \mid n_i \neq 0\}$". E. g. $\Sigma = \operatorname{Bl}_z(\mathbb{P}_{\mathbb{C}}^2)$, the projective plane blown up in a point z, and $B \subset \Sigma$ the strict transform of a line through z. Let now $r = 1$, $n_1 = 1$ and $z_1 \in \Sigma$ be any point. Then there is of course a (unique) curve $B_1 \in |B|_a$ through z_1, but $B^2 = 0 < 1 = n_1^2$.

Idea of the proof: Set $e_1 := n_1 - 1$ and $e_i := n_i$ for $i \neq 1$, where w. l. o. g. $n_1 = \min\{n_i | n_i \neq 0\}$. By assumption there is a family $\{C_t\}_{t \in \mathbb{C}}$ in $|B|_a$ satisfying the requirements of Lemma 3.3. Setting $C := C_0$ the proof is done in three steps:

Step 1: We show that $H^0\big(C, \mathcal{J}_{X(\underline{e};\underline{z})/\Sigma} \cdot \mathcal{O}_C(C)\big) \neq 0$. (Lemma 3.3)

Step 2: We deduce that $H^0\big(C, \pi_* \mathcal{O}_{\widetilde{C}}\big(-\sum_{i=1}^{r} e_i E_i\big) \otimes \mathcal{O}_C(C)\big) \neq 0$. (Lemma 3.4)

Step 3: It follows that $\deg\big(\pi_*\mathcal{O}_{\widetilde{C}}\big(-\sum_{i=1}^r e_iE_i\big)\otimes\mathcal{O}_C(C)\big)\geq 0$, but this degree is just $C^2-\sum_{i=1}^r e_in_i$.

□

3.3 Lemma

Given $e_1,\ldots,e_r\in\mathbb{N}_0$, $r\geq 1$. Let $\{C_t\}_{t\in\mathbb{C}}$ be a non-trivial family of curves in Σ such that

- $\mathbb{C}\to\Sigma\,:\,t\mapsto z_{1,t}\in C_t$ *is a smooth curve,*
- $\operatorname{mult}_{z_{1,t}}(C_t)\geq e_1+1$ *for all* $t\in\mathbb{C}$,
- $z_2,\ldots,z_r\in C_t$ *for any* $t\in\mathbb{C}$, *and*
- $\operatorname{mult}_{z_i}(C_t)\geq e_i$ *for all* $i=2,\ldots,r$ *and* $t\in\mathbb{C}$.

Then for $z_1=z_{1,0}$

$$H^0\big(C,\mathcal{J}_{X(\underline{e};\underline{z})/\Sigma}\cdot\mathcal{O}_C(C)\big)\neq 0,$$

i. e. there is a non-trivial section of the normal bundle of C, vanishing at z_i to the order of at least e_i for $i=1,\ldots,r$.

Proof: We stick to the convention $n_1=e_1+1$ and $n_i=e_i$ for $i=2,\ldots,r$, and we set $z_{i,t}:=z_i$ for $i=2,\ldots,r$ and $t\in\mathbb{C}$. Let $\Delta\subset\mathbb{C}$ be a small disc around 0 with coordinate t, and choose coordinates (x_i,y_i) on Σ around z_i such that

- $z_{i,t}=(a_i(t),b_i(t))$ for $t\in\Delta$ with $a_i,b_i\in\mathbb{C}\{t\}$,
- $z_i=(a_i(0),b_i(0))=(0,0)$, and
- $F_i(x_i,y_i,t)=f_{i,t}(x_i,y_i)\in\mathbb{C}\{x_i,y_i,t\}$, where $C_t=\{f_{i,t}=0\}$ locally at $z_{i,t}$ (for $t\in\Delta$).

We view $\{C_t\}_{t\in\Delta}$ as a non-trivial deformation of C, which implies that the image of $\frac{\partial}{\partial t}{}_{|t=0}\in T_0(\Delta)$ under the Kodaira-Spencer map is a non-zero section s of $H^0\big(C,\mathcal{O}_C(C)\big)$. s is locally at z_i given by $\frac{\partial F_i}{\partial t}{}_{|t=0}$.

Idea: Show that $\frac{\partial F_i}{\partial t}{}_{|t=0}\in(x_i,y_i)^{e_i}$, which are the stalks of $\mathcal{J}_{X(\underline{e};\underline{z})/\Sigma}\cdot\mathcal{O}_C(C)$ at the z_i, and hence s is actually a global section of the subsheaf $\mathcal{J}_{X(\underline{e};\underline{z})/\Sigma}\cdot\mathcal{O}_C(C)$.

Set $\Phi_{i,t}(x_i,y_i):=F_{i,t}(x_i+a_i(t),y_i+b_i(t),t)=\sum_{k=0}^\infty\varphi_{i,k}(x_i,y_i)\cdot t^k\in\mathbb{C}\{x_i,y_i,t\}$. By assumption for any $t\in\Delta$ the multiplicity of $\Phi_{i,t}$ at $(0,0)$ is at least n_i, i. e. $\Phi_{i,t}(x_i,y_i)\in(x_i,y_i)^{n_i}$ for every fixed complex number $t\in\Delta$. Hence, $\varphi_{i,k}(x_i,y_i)\in(x_i,y_i)^{n_i}$ for every k.[3]

On the other hand we have

$$\begin{aligned}\varphi_{i,1}(x_i,y_i)&=\frac{\partial\Phi_{i,t}(x_i,y_i)}{\partial t}{}_{|t=0}\\&=\Big\langle\Big(\frac{\partial F_i}{\partial x_i}(x_i,y_i,0),\frac{\partial F_i}{\partial y_i}(x_i,y_i,0),\frac{\partial F_i}{\partial t}(x_i,y_i,0)\Big),\big(\dot a_i(0),\dot b_i(0),1\big)\Big\rangle\\&=\frac{\partial f_{i,0}}{\partial x_i}(x_i,y_i)\cdot\dot a_i(0)+\frac{\partial f_{i,0}}{\partial y_i}(x_i,y_i)\cdot\dot b_i(0)+\frac{\partial F_i}{\partial t}(x_i,y_i,0).\end{aligned}$$

Since $f_{i,0}\in(x_i,y_i)^{n_i}$, we have $\frac{\partial f_{i,0}}{\partial x_i}(x_i,y_i),\frac{\partial f_{i,0}}{\partial y_i}(x_i,y_i)\in(x_i,y_i)^{n_i-1}$, and hence $\frac{\partial F_i}{\partial t}(x_i,y_i,0)\in(x_i,y_i)^{e_i}$. For this note that $\dot a_i(0)=\dot b_i(0)=0$, if $i\neq 1$. □

[3]See Lemma C.1.

3.4 Lemma

Given $e_1,\ldots,e_r\in\mathbb{N}_0$ and $z_1,\ldots,z_r\in\Sigma$, $r\geq 1$.

The canonical morphism $\mathcal{J}_{X(e_1;z_1)/\Sigma}\otimes\cdots\otimes\mathcal{J}_{X(e_r;z_r)/\Sigma}\otimes\mathcal{O}_C(C)\longrightarrow\mathcal{J}_{X(\underline{e};\underline{z})/\Sigma}\cdot\mathcal{O}_C(C)$ induces a surjective morphism β on the level of global sections.

If $s\in H^0\big(C,\mathcal{J}_{X(e_1;z_1)/\Sigma}\otimes\cdots\otimes\mathcal{J}_{X(e_r;z_r)/\Sigma}\otimes\mathcal{O}_C(C)\big)$, but not in $\operatorname{Ker}(\beta)$, then s induces a non-zero section $\tilde{s}$ in $H^0\big(C,\pi_*\mathcal{O}_{\widetilde{C}}\big(-\sum_{i=1}^r e_iE_i\big)\otimes_{\mathcal{O}_C}\mathcal{O}_C(C)\big)$.

Proof: Set $E:=-\sum_{i=1}^r e_iE_i$.
We start with the structure sequence for $\widetilde{C}$:

$$0\longrightarrow\mathcal{O}_{\widetilde{\Sigma}}\big(-\widetilde{C}\big)\longrightarrow\mathcal{O}_{\widetilde{\Sigma}}\longrightarrow\mathcal{O}_{\widetilde{C}}\longrightarrow 0.$$

Tensoring with the locally free sheaf $\mathcal{O}_{\widetilde{\Sigma}}(E)$ and then applying π_* we get a morphism:

$$\pi_*\mathcal{O}_{\widetilde{\Sigma}}(E)\longrightarrow\pi_*\mathcal{O}_{\widetilde{C}}(E).$$

Now tensoring by $\mathcal{O}_C(C)$ over $\mathcal{O}_\Sigma$ we have an exact sequence:

$$0\longrightarrow\operatorname{Ker}(\gamma)\longrightarrow\pi_*\mathcal{O}_{\widetilde{\Sigma}}(E)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C(C)\xrightarrow{\gamma}\pi_*\mathcal{O}_{\widetilde{C}}(E)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C(C).$$

And finally taking global sections, we end up with:

$$0\longrightarrow H^0\big(\Sigma,\operatorname{Ker}(\gamma)\big)\longrightarrow H^0\big(\Sigma,\pi_*\mathcal{O}_{\widetilde{\Sigma}}(E)\otimes\mathcal{O}_C(C)\big)\xrightarrow{\alpha}H^0\big(\Sigma,\pi_*\mathcal{O}_{\widetilde{C}}(E)\otimes\mathcal{O}_C(C)\big).$$

Since the sheaves we look at are actually $\mathcal{O}_C$-sheaves and since C is a closed subscheme of Σ, the global sections of the sheaves as sheaves on Σ and as sheaves on C coincide (cf. [Har77] III.2.10 - for more details, see Corollary C.3). Furthermore, $\pi_*\mathcal{O}_{\widetilde{\Sigma}}(E)=\bigotimes_{i=1}^r\mathcal{J}_{X(e_i;z_i)/\Sigma}$.
Thus it suffices to show that $\operatorname{Ker}(\alpha)\subseteq\operatorname{Ker}(\beta)$.
Since $\pi_|:\widetilde{\Sigma}\backslash(\bigcup_{i=1}^r E_i)\to\Sigma\backslash\{z_1,\ldots,z_r\}$ is an isomorphism, we have that $\operatorname{supp}\big(\operatorname{Ker}(\gamma)\big)\subseteq\{z_1,\ldots,z_r\}$ is finite.[4] Hence, by Lemma C.5,

$$\begin{aligned}\operatorname{Ker}(\alpha)\;&=\;H^0\big(\Sigma,\operatorname{Ker}(\gamma)\big)\;\subseteq\;H^0\Big(\Sigma,\operatorname{Tor}\big(\operatorname{Ker}(\gamma)\big)\Big)\\&\subseteq H^0\bigg(\Sigma,\operatorname{Tor}\Big(\bigotimes_{i=1}^r\mathcal{J}_{X(e_i;z_i)/\Sigma}\otimes\mathcal{O}_C(C)\Big)\bigg).\end{aligned}$$

Let now $t\in\operatorname{Ker}(\alpha)$ be given. We have to show that $\beta(t)=0$, i. e. $\beta_z(t_z)=0$ for every $z\in\Sigma$. If $z\notin\{z_1,\ldots,z_r\}$, then $t_z=0$. Thus we may assume $z=z_k$. As we have shown,

$$t_{z_k}\in\operatorname{Tor}\big(\mathfrak{m}_{\Sigma,z_k}^{e_k}\otimes_{\mathcal{O}_{\Sigma,z_k}}\mathcal{O}_{C,z_k}\big)=\operatorname{Tor}\big(\mathfrak{m}_{\Sigma,z_k}^{e_k}/f_{z_k}\mathfrak{m}_{\Sigma,z_k}^{e_k}\big)=(f_{z_k})/f_{z_k}\mathfrak{m}_{\Sigma,z_k}^{e_k},$$

where f_{z_k} is a local equation of C at z_k. Therefore, there exists a $0\neq g_{z_k}\in\mathcal{O}_{\Sigma,z_k}$ such that $t_{z_k}=f_{z_k}g_{z_k}\ \big(\mathrm{mod}\ f_{z_k}\mathfrak{m}_{\Sigma,z_k}^{e_k}\big)\equiv f_{z_k}\otimes g_{z_k}$ (note that $f_{z_k}\in\mathfrak{m}_{\Sigma,z_k}^{n_k}\subseteq\mathfrak{m}_{\Sigma,z_k}^{e_k}$!). But then $\beta_{z_k}(t_{z_k})$ is just the residue class of $f_{z_k}g_{z_k}$ in $\mathfrak{m}_{\Sigma,z_k}^{e_k}\mathcal{O}_{C,z_k}=\mathfrak{m}_{\Sigma,z_k}^{e_k}/(f_{z_k})$, and is thus zero. $\square$

[4] See Lemma C.4.

Proof of Lemma 3.1: Using the notation of the idea of the proof given on page 8, we have, by Lemma 3.3, a non-zero section $s \in H^0\big(C, \mathcal{J}_{X(\underline{e};\underline{z})/\Sigma} \cdot \mathcal{O}_C(C)\big)$. This lifts under the surjection β to a section $s' \in H^0\big(C, \bigotimes_{i=1}^r \mathcal{J}_{X(e_i;z_i)/\Sigma} \otimes \mathcal{O}_C(C)\big)$ which is not in the kernel of β. Again setting $E := -\sum_{i=1}^r e_i E_i$, by Lemma 3.4, we have a non-zero section $\tilde{s} \in H^0\big(C, \pi_* \mathcal{O}_{\widetilde{C}}(E) \otimes_{\mathcal{O}_C} \mathcal{O}_C(C)\big)$, where by the projection formula the latter is just $H^0\big(C, \pi_*\big(\mathcal{O}_{\widetilde{C}}(E) \otimes_{\mathcal{O}_{\widetilde{C}}} \pi^* \mathcal{O}_C(C)\big)\big) =_{\text{def}} H^0\big(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(E) \otimes_{\mathcal{O}_{\widetilde{C}}} \pi^* \mathcal{O}_C(C)\big)$. Since $\mathcal{O}_{\widetilde{C}}(E) \otimes_{\mathcal{O}_{\widetilde{C}}} \pi^* \mathcal{O}_C(C)$ has a global section and since $\widetilde{C}$ is irreducible and reduced, we get by Lemma D.2:

$$0 \leq \deg\big(\mathcal{O}_{\widetilde{C}}(E) \otimes_{\mathcal{O}_{\widetilde{C}}} \pi^* \mathcal{O}_C(C)\big) = E.\widetilde{C} + \deg\big(\mathcal{O}_C(C)\big) = \sum_{i=1}^r -e_i n_i + C^2.$$

□

4. Existence Theorem for Generic Fat Point Schemes

4.1 Theorem

Given $m_1, \ldots, m_r \in \mathbb{N}_0$, not all zero, and $z_1, \ldots, z_r \in \Sigma$, in very general position. Let $L \in \operatorname{Div}(\Sigma)$ be very ample over $\mathbb{C}$, and let $D \in \operatorname{Div}(\Sigma)$ be such that

$$h^1\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D - L)\big) = 0, \text{ and} \tag{4.1}$$

$$D.L - 2g(L) \geq m_i + m_j \text{ for all } i, j. \tag{4.2}$$

Then there exists a curve $C \in |D|_l$ with ordinary singular points of multiplicity m_i at z_i for $i = 1, \ldots, r$ and no other singular points. Furthermore,

$$h^1\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big) = 0,$$

and in particular, $V_{|D|}(\underline{m})$ is T-smooth at C.

If in addition (4.3) $D^2 > \sum_{i=1}^r m_i^2$, then C can be chosen to be irreducible and reduced.

Idea of the proof: For each z_j find a curve $C_j \in \big|H^0\big(\mathcal{J}_{X(\underline{m},\underline{z})/\Sigma}(D)\big)\big|_l$ with an ordinary singular point of multiplicity m_j and show that this linear system has no other base points than $z_1, \ldots, z_r$. Then the generic element is smooth outside $z_1, \ldots, z_r$ and has an ordinary singularity of multiplicity m_j in z_j.

Proof: W. l. o. g. $m_i \geq 1$ for all $i = 1 \ldots, r$. For the convenience of notation we set $z_{r+1} := z_1$ and $m_{r+1} := m_1$. Since L is very ample, we may choose smooth curves $L_j \in |L|_l$ through z_j and z_{j+1} for $j = 1, \ldots, r$ (cf. Lemma E.3). Writing X for $X(\underline{m}; \underline{z})$ we introduce zero-dimensional schemes X_j for $j = 1, \ldots, r$ by

$$\mathcal{J}_{X_j/\Sigma,z} = \begin{cases} \mathcal{J}_{X/\Sigma,z}, & \text{if } z \neq z_j, \\ \mathfrak{m}_{\Sigma,z_j} \cdot \mathcal{J}_{X/\Sigma,z_j}, & \text{if } z = z_j. \end{cases}$$

Step 1: $h^1\big(\mathcal{J}_{X_j/\Sigma}(D)\big) = 0$.

By Condition (4.2) we get

$$\deg\big(X_j \cap L_j\big) \le m_j + m_{j+1} + 1 \le D.L + 1 - 2g(L), \tag{4.4}$$

and the exact sequence

$$0 \longrightarrow \mathcal{J}_{X/\Sigma} \longrightarrow \mathcal{J}_{X_j:L_j/\Sigma} \longrightarrow \mathfrak{m}_{\Sigma,z_{j+1}}^{m_{j+1}-1}/\mathfrak{m}_{\Sigma,z_{j+1}}^{m_{j+1}} \longrightarrow 0$$

implies with the aid of (4.1)

$$h^1\big(\mathcal{J}_{X_j:L_j/\Sigma}(D-L)\big) = 0. \tag{4.5}$$

(4.4) and (4.5) allow us to apply Lemma E.5 in order to obtain

$$h^1\big(\mathcal{J}_{X_j/\Sigma}(D)\big) = 0.$$

Step 2: For each $j = 1, \ldots, r$ there exists a curve $C_j \in |D|_l$ with an ordinary singular point of multiplicity m_j at z_j and with $\operatorname{mult}_{z_i}(C_j) \ge m_i$ for $i \ne j$.

Consider the exact sequence

$$0 \longrightarrow \mathcal{J}_{X_j/\Sigma} \longrightarrow \mathcal{J}_{X/\Sigma} \longrightarrow \mathfrak{m}_{\Sigma,z_j}^{m_j}/\mathfrak{m}_{\Sigma,z_j}^{m_j+1} \longrightarrow 0$$

twisted by D and the corresponding long exact cohomology sequence

$$H^0\big(\mathcal{J}_{X/\Sigma}(D)\big) \to \mathfrak{m}_{\Sigma,z_j}^{m_j}/\mathfrak{m}_{\Sigma,z_j}^{m_j+1} \to \underset{\substack{\| \text{Step 1} \\ 0}}{H^1\big(\mathcal{J}_{X_j/\Sigma}(D)\big)} \to H^1\big(\mathcal{J}_{X/\Sigma}(D)\big) \to 0. \tag{4.6}$$

Thus we may choose the C_j to be given by a section in $H^0\big(\mathcal{J}_{X/\Sigma}(D)\big)$ where the m_j tangent directions at z_j are all different.

Step 3: The base locus of $\mathbb{P}\big(H^0\big(\mathcal{J}_{X/\Sigma}(D)\big)\big)$ is $\{z_1, \ldots, z_r\}$.

Suppose $w \in \Sigma$ was an additional base point and define the zero-dimensional scheme $X \cup \{w\}$ by

$$\mathcal{J}_{X\cup\{w\}/\Sigma,z} = \begin{cases} \mathcal{J}_{X/\Sigma,z}, & \text{if } z \ne w, \\ \mathfrak{m}_{\Sigma,w} \cdot \mathcal{J}_{X/\Sigma,w}, & \text{if } z = w. \end{cases}$$

Choosing a generic, and thus smooth, curve $L_w \in |L|_l$ through w we may deduce as in Step 1

$$h^1\big(\mathcal{J}_{X\cup\{w\}/\Sigma}(D)\big) = 0,$$

and thus as in Step 2

$$h^0\big(\mathcal{J}_{X/\Sigma}(D)\big) = h^0\big(\mathcal{J}_{X\cup\{w\}/\Sigma}(D)\big) + 1.$$

But by assumption w is a base point, and thus

$$h^0\big(\mathcal{J}_{X/\Sigma}(D)\big) = h^0\big(\mathcal{J}_{X\cup\{w\}/\Sigma}(D)\big),$$

which gives us the desired contradiction.

Step 4: $\exists\, C \in \mathbb{P}\big(H^0\big(\mathcal{J}_{X/\Sigma}(D)\big)\big) \subseteq |D|_l$ with an ordinary singular point of multiplicity m_i at z_i for $i = 1, \ldots, r$ and no other singular points.

Because of Step 2 the generic element in $\mathbb{P}\big(H^0\big(\mathcal{J}_{X/\Sigma}(D)\big)\big)$ has an ordinary singular point of multiplicity m_i at z_i and is by Bertini's Theorem (cf. [Har77] III.10.9.2) smooth outside its base locus.

For two generic curves $C, C' \in \mathbb{P}\big(H^0\big(\mathcal{J}_{X/\Sigma}(D)\big)\big)$ the intersection multiplicity in z_i is $i(C, C'; z_i) = m_i^2$. Thus, if Condition (4.3) is fulfilled then C and C' have an additional intersection point outside the base locus of $\mathbb{P}\big(H^0\big(\mathcal{J}_{X/\Sigma}(D)\big)\big)$, and Bertini's Theorem (cf. [Wae73] §47, Satz 4) implies that the generic curve in $\mathbb{P}\big(H^0\big(\mathcal{J}_{X/\Sigma}(D)\big)\big)$ is irreducible.

Step 5: $h^1\big(\mathcal{J}_{X/\Sigma}(D)\big) = 0$, by Equation (4.6).

Step 6: $V_{|D|}(\underline{m})$ is T-smooth at C.

By [GLS98b], Lemma 2.7, we have

$$\mathcal{J}_{X/\Sigma} \subseteq \mathcal{J}_{X^{es}(C)/\Sigma},$$

and thus by Step 5

$$h^1\big(\mathcal{J}_{X^{es}(C)/\Sigma}(D)\big) = 0,$$

which proves the claim. □

4.2 Corollary

Let $m_1, \ldots, m_r \in \mathbb{N}_0$, not all zero, $r \geq 1$, and let $L \in \mathrm{Div}(\Sigma)$ be very ample over $\mathbb{C}$. Suppose $D \in \mathrm{Div}(\Sigma)$ such that

(4.7) $\quad (D - L - K_\Sigma)^2 \geq 2\sum_{i=1}^r (m_i + 1)^2,$

(4.8) $\quad (D - L - K_\Sigma).B > \max\{m_1, \ldots, m_r\}$ *for any irreducible curve $B \subset \Sigma$ with $B^2 = 0$ and $\dim |B|_a \geq 1$,*

(4.9) $\quad D - L - K_\Sigma$ *is nef, and*

(4.10) $\quad D.L - 2g(L) \geq m_i + m_j$ *for all i, j.*

Then for $z_1, \ldots, z_r \in \Sigma$ in very general position there exists a curve $C \in |D|_l$ with ordinary singular points of multiplicity m_i at z_i for $i = 1, \ldots, r$ and no other singular points. Furthermore,

$$h^1\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big) = 0,$$

and in particular, $V_{|D|}(\underline{m})$ is T-smooth in C.

If in addition (4.11) $D^2 > \sum_{i=1}^r m_i^2$, *then C can be chosen to be irreducible and reduced.*

Proof: Follows from Theorem 4.1 and Corollary 2.2. □

4.3 Remark

In view of Condition (4.7) Condition (4.11) is satisfied if the following condition is fulfilled:

$$D^2 + (2D - L - K_\Sigma).(L + K_\Sigma) + 4\sum_{i=1}^r m_i + 2r > 0 \tag{4.12}$$

Proof: Suppose (4.11) was not satisfied, then

$$2\sum_{i=1}^{r} m_i^2 \geq 2D^2 = D^2 + (D-L-K_\Sigma)^2 + (2D-L-K_\Sigma).(L+K_\Sigma)$$

$$\geq D^2 + 2\sum_{i=1}^{r} m_i^2 + 4\sum_{i=1}^{r} m_i + 2r + (2D-L-K_\Sigma).(L+K_\Sigma).$$

Hence,

$$D^2 + (2D-L-K_\Sigma).(L+K_\Sigma) + 4\sum_{i=1}^{r} m_i + 2r \leq 0,$$

which implies that (4.12) is sufficient. □

5. Existence Theorem for General Equisingularity Schemes

Notation

In the following we will denote by $\mathbb{C}[x,y]_d$, respectively by $\mathbb{C}[x,y]_{\leq d}$ the $\mathbb{C}$-vector spaces of polynomials of degree d, respectively of degree at most d. If $f \in \mathbb{C}[x,y]_{\leq d}$ we denote by $f_k \in \mathbb{C}[x,y]_k$ for $k=0,\ldots,d$ the homogeneous part of degree k of f, and thus $f=\sum_{k=0}^{d} f_k$. By $\underline{a}=(a_{k,l}|0\leq k+l\leq d)$ we will denote the coordinates of $\mathbb{C}[x,y]_{\leq d}$ with respect to the basis $\big\{x^k y^l|0\leq k+l\leq d\big\}$.

For any $f \in \mathbb{C}[x,y]_{\leq d}$ the tautological family

$$\mathbb{C}[x,y]_{\leq d}\times\mathbb{C}^2 \;\supset \bigcup_{g\in\mathbb{C}[x,y]_{\leq d}} \{g\}\times g^{-1}(0) \longrightarrow \mathbb{C}[x,y]_{\leq d}$$

induces a deformation of the plane curve singularity $\big(f^{-1}(0),0\big)$ whose base space is the germ $\big(\mathbb{C}[x,y]_{\leq d},f\big)$ of $\mathbb{C}[x,y]_{\leq d}$ at f. Given any deformation $(X,x)\hookrightarrow(\mathcal{X},x)\to(S,s)$ of a plane curve singularity (X,x), we will denote by $S^{es}=(S^{es},s)$ the germ of the equisingular stratum of (S,s). Thus, fixed an $f\in\mathbb{C}[x,y]_{\leq d}$, $\mathbb{C}[x,y]_{\leq d}^{es}=\big(\mathbb{C}[x,y]_{\leq d}^{es},f\big)$ is the (local) equisingular stratum of $\mathbb{C}[x,y]_{\leq d}$ at f.

5.1 Definition

(i) We say the family $\mathbb{C}[x,y]_{\leq d}$ is *T-smooth* at $f\in\mathbb{C}[x,y]_{\leq d}$ if for any $e\geq d$ there exists a $\Lambda\subset\big\{(k,l)\in\mathbb{N}_0^2 \;\big|\; 0\leq k+l\leq d\big\}$ with $\#\Lambda=\tau^{es}$ such that $\mathbb{C}[x,y]_{\leq e}^{es}$ is given by equations

$$a_{k,l}=\phi_{k,l}\big(\underline{a}_{(1)},\underline{a}_{(2)}\big),\quad (k,l)\in\Lambda,$$

with $\phi_{k,l}\in\mathbb{C}\big\{\underline{a}_{(1)},\underline{a}_{(2)}\big\}$ where $\underline{a}_{(0)}=(a_{k,l}|(k,l)\in\Lambda)$, $\underline{a}_{(1)}=(a_{k,l}\mid 0\leq k+l\leq d,(k,l)\notin\Lambda)$, and $\underline{a}_{(2)}=(a_{k,l}\mid d+1\leq k+l\leq e)$, and where $\tau^{es}=\dim_{\mathbb{C}}\big(\mathbb{C}\{x,y\}/I^{es}\big(f^{-1}(0),0\big)\big)$ is the codimension of the equisingular stratum in the base space of the semiuniversal deformation of $\big(f^{-1}(0),0\big)$.

(ii) A polynomial $f\in\mathbb{C}[x,y]_{\leq d}$ is said to be a *good representative* of the singularity type $\mathcal{S}$ in $\mathbb{C}[x,y]_{\leq d}$ if it meets the following conditions:

(a) $\operatorname{Sing}\big(f^{-1}(0)\big)=\Big\{p\in\mathbb{C}^2\mid f(p)=0,\frac{\partial f}{\partial x}(p)=0,\frac{\partial f}{\partial y}(p)=0\Big\}=\{0\}$,

(b) $\big(f^{-1}(0),0\big) \sim_t \mathcal{S}$,

(c) f_d is reduced, and

(d) $\mathbb{C}[x,y]_{\leq d}$ is T-smooth at f.

(iii) Given a singularity type $\mathcal{S}$ we define $s(\mathcal{S})$ to be the minimal number d such that $\mathcal{S}$ has a good representative of degree d.

5.2 Remark

(i) The condition for T-smoothness just means that for any $e \geq d$ the equisingular stratum $\mathbb{C}[x,y]^{es}_{\leq e}$ is smooth at the point f of the expected codimension in $\big(\mathbb{C}[x,y]_{\leq e}, f\big)$.

(ii) Note that for a polynomial of degree d the highest homogeneous part f_d defines the normal cone, i. e. the intersection of the curve $\{\hat{f} = 0\}$ with the line at infinity in $\mathbb{P}^2_{\mathbb{C}}$, where $\hat{f}$ is the homogenisation of f. Thus the condition "f_d reduced" in the definition of a good representative just means that the line at infinity intersects the curve transversally in d different points.

(iii) If $f \in \mathbb{C}[x,y,z]_d$ is an irreducible polynomial such that $(0:0:1)$ is the only singular point of the plane curve $\{f=0\} \subset \mathbb{P}^2_{\mathbb{C}}$, then a linear change of coordinates of the type $(x,y,z) \mapsto (x,y,z+ax+by)$ will ensure that the dehomogenisation $\check{f}$ of f satisfies "$\check{f}_d$ reduced". Note for this that the coordinate change corresponds to choosing a line in $\mathbb{P}^2_{\mathbb{C}}$, not passing through $(0:0:1)$ and meeting the curve in d distinct points. Therefore, the bounds for $s(\mathcal{S})$ given in [Los98] Theorem 4.2 and Remark 4.3 do apply here.

(iv) For refined results using the techniques of the following proof we refer to [Shu99].

5.3 Theorem (Existence)
Let $\mathcal{S}_1,\dots,\mathcal{S}_r$ be singularity types, and suppose there exists an irreducible curve $C \in |D|_l$ with $r+r'$ ordinary singular points $z_1,\dots,z_{r+r'}$ of multiplicities $m_1,\dots,m_{r+r'}$ respectively as its only singularities such that $m_i = s(\mathcal{S}_i)+1$, for $i=1,\dots,r$, and

$$h^1\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big) = 0.$$

Then there exists an irreducible curve $\widetilde{C} \in |D|_l$ with r singular points of types $\mathcal{S}_1,\dots,\mathcal{S}_r$ and r' ordinary singular points of multiplicities $m_{r+1},\dots,m_{r+r'}$ as its only singularities.[5]

Idea of the proof: The basic idea is to glue locally at the z_i equations of good representatives for the $\mathcal{S}_i$ into the curve C. Let us now explain more detailed what we mean by this.
If $g_i = \sum_{k+l=0}^{m_i-1} a_{k,l}^{i,fix} x_i^k y_i^l$, $i=1,\dots,r$, are good representatives of the $\mathcal{S}_i$, then we are looking for a family F_t, $t \in (\mathbb{C},0)$, in $H^0\big(\Sigma, \mathcal{O}_\Sigma(D)\big)$ which in local coordinates

[5] Here, of course, $\underline{m} = (m_1,\dots,m_{r+r'})$ and $\underline{z} = (z_1,\dots,z_{r+r'})$.

x_i, y_i at z_i looks like

$$F_t^i = \sum_{k+l=0}^{m_i-1} t^{m_i-1-k-l} \tilde{a}_{k,l}^i(t) x_i^k y_i^l + h.o.t.,$$

where the $\tilde{a}_{k,l}^i(t)$ should be convergent power series in t with $\tilde{a}_{k,l}^i(0) = a_{k,l}^{i,fix}$. Replacing g_i by some arbitrarily small multiple $\lambda_i g_i$ the curve defined by F_0 is an arbitrarily small deformation of C inside some suitable linear system, thus it is smooth outside $z_1, \ldots, z_{r+r'}$ and has ordinary singular points in $z_1, \ldots, z_{r+r'}$. For $t \neq 0$, on the other hand, F_t^i can be transformed, by $(x_i, y_i) \mapsto (tx_i, ty_i)$, into a member of some family

$$\tilde{F}_t^i = \sum_{k+l=0}^{m_i-1} \tilde{a}_{k,l}^i(t) x_i^k y_i^l + h.o.t., \quad t \in \mathbb{C},$$

with

$$\tilde{F}_0^i = g_i.$$

Using now the T-smoothness property of g_i, $i = 1, \ldots, r$, we can choose the $\tilde{a}_{k,l}^i(t)$ such that this family is equisingular. Hence, for small $t \neq 0$, the curve given by F_t will have the right singularities at the z_i. Finally, the knowledge on the singularities of the curve defined by F_0 and the conservation of Milnor numbers will ensure that the curve given by F_t has no further singularities, for $t \neq 0$ sufficiently small.

The proof will be done in several steps. First of all we are going to fix some notation by choosing a basis of $H^0\big(\Sigma, \mathcal{O}_\Sigma(D)\big)$ which reflects the "independence" of the coordinates at the different z_i ensured by $h^1\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big) = 0$ (Step 1.1), and by choosing good representatives for the $\mathcal{S}_i$ (Step 1.2). In a second step we are making an "Ansatz" for the family F_t, and, for the local investigation of the singularity type, we are switching to some other families $\tilde{F}_t^i$, $i = 1, \ldots, r$ (Step 2.1). We, then, reduce the problem of F_t, for $t \neq 0$ small, having the right singularities to a question about the equisingular strata of some families of polynomials (Step 2.2), which in Step 2.3 will be solved. The final step serves to show that the curves F_t have only the singularities which we controlled in the previous steps.

Proof:

Step 1.1: Parametrise $|D|_l = \mathbb{P}\big(H^0\big(\mathcal{O}_\Sigma(D)\big)\big)$.

Consider the following exact sequence:

$$0 \longrightarrow \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D) \longrightarrow \mathcal{O}_\Sigma(D) \longrightarrow \bigoplus_{i=1}^{r+r'} \mathcal{O}_{\Sigma,z_i}/\mathfrak{m}_{\Sigma,z_i}^{m_i} \longrightarrow 0.$$

Since $h^1\big(\mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big) = 0$, the long exact cohomology sequence gives

$$H^0\big(\mathcal{O}_\Sigma(D)\big) = \bigoplus_{i=1}^{r+r'} \mathbb{C}\{x_i, y_i\}/(x_i, y_i)^{m_i} \oplus H^0\big(\mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big),$$

where x_i, y_i are local coordinates of (Σ, z_i).

We, therefore, can find a basis $\big\{s_{k,l}^i, s_j \;\big|\; j=1,\ldots,e,\ i=1,\ldots,r+r',\ 0\leq k+l\leq m_i-1\big\}$ of $H^0\big(\mathcal{O}_\Sigma(D)\big)$, with $e=h^0\big(\mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(D)\big)$, such that[6]

- C is the curve defined by s_1,
- $(s_j)_{z_i}=\sum_{|\alpha|\geq m_i} B_\alpha^{j,i} x_i^{\alpha_1} y_i^{\alpha_2}$ for $j=1,\ldots,e$, $i=1,\ldots,r+r'$,
- $\big(s_{k,l}^j\big)_{z_i}=\begin{cases} x_i^k y_i^l + \sum\limits_{|\alpha|\geq m_i} A_{\alpha,k,l}^{i,i} x_i^{\alpha_1} y_i^{\alpha_2}, & \text{if } i=j,\\ \sum\limits_{|\alpha|\geq m_i} A_{\alpha,k,l}^{j,i} x_i^{\alpha_1} y_i^{\alpha_2}, & \text{if } i\neq j.\end{cases}$

Let us now denote the coordinates of $H^0\big(\mathcal{O}_\Sigma(D)\big)$ w. r. t. this basis by $(\underline{a},\underline{b})=\big(\underline{a}^1,\ldots,\underline{a}^{r+r'},\underline{b}\big)$ with $\underline{a}^i=\big(a_{k,l}^i \mid 0\leq k+l\leq m_i-1\big)$ and $\underline{b}=(b_j \mid j=1,\ldots,e)$. Thus the family

$$F_{(\underline{a},\underline{b})}=\sum_{i=1}^{r+r'}\sum_{k+l=0}^{m_i-1} a_{k,l}^i s_{k,l}^i+\sum_{j=1}^{e} b_j s_j,\quad (\underline{a},\underline{b})\in\mathbb{C}^N \text{ with } N=e+\sum_{i=1}^{r+r'}\binom{m_i+1}{2},$$

parametrises $H^0\big(\mathcal{O}_\Sigma(D)\big)$.

Step 1.2: By the definition of $s(\mathcal{S}_i)$ and since $s(\mathcal{S}_i)=m_i-1$, we may choose good representatives

$$g_i=\sum_{k+l=0}^{m_i-1} a_{k,l}^{i,fix} x_i^k y_i^l\in\mathbb{C}[x_i,y_i]_{\leq m_i-1}$$

for the $\mathcal{S}_i$, $i=1,\ldots,r$. Let $\underline{a}^{i,fix}=\big(a_{k,l}^{i,fix} \;\big|\; 0\leq k+l\leq m_i-1\big)$ and $\underline{a}^{fix}=\big(\underline{a}^{1,fix},\ldots,\underline{a}^{r,fix}\big)$. We should remark here that for any $\lambda_i\neq 0$ the polynomial $\lambda_i g_i$ is also a good representative, and thus, replacing g_i by $\lambda_i g_i$, we may assume that the $a_{k,l}^{i,fix}$ are arbitrarily close to 0.

Step 2: We are going to glue the good representatives for the $\mathcal{S}_i$ into the curve C. More precisely, we are constructing a subfamily F_t, $t\in(\mathbb{C},0)$, in $H^0\big(\mathcal{O}_\Sigma(D)\big)$ such that, if $C_t\in|D|_l$ denotes the curve defined by F_t,

(1) $z_1,\ldots,z_{r+r'}$ are the only singular points of the irreducible reduced curve C_0, and they are ordinary singularities of multiplicities m_i-1, for $i=1,\ldots,r$, and m_i, for $i=r+1,\ldots,r+r'$ respectively,

(2) locally in z_i, $i=1,\ldots,r$, the F_t, for small $t\neq 0$, can be transformed into members of a fixed $\mathcal{S}_i$-equisingular family,

(3) while for $i=r+1,\ldots,r+r'$ and $t\neq 0$ small C_t has an ordinary singularity of multiplicity m_i in z_i.

Step 2.1: "Ansatz" and first reduction for a local investigation.

Let us make the following "Ansatz":

$$b_1=1,\ b_2=\ldots=b_e=0,\ \underline{a}^i=0,\ \text{ for } i=r+1,\ldots,r+r',$$
$$a_{k,l}^i=t^{m_i-1-k-l}\cdot\tilde{a}_{k,l}^i,\ \text{ for } i=1,\ldots,r,\ 0\leq k+l\leq m_i-1.$$

[6]Throughout this proof we will use the multi index notation $\alpha=(\alpha_1,\alpha_2)\in\mathbb{N}^2$ and $|\alpha|=\alpha_1+\alpha_2$.

This gives rise to a family

$$F_{(t,\underline{\tilde{a}})} = s_1 + \sum_{i=1}^{r} \sum_{k+l=0}^{m_i-1} t^{m_i-1-k-l} \tilde{a}_{k,l}^i s_{k,l}^i \in H^0\big(\mathcal{O}_\Sigma(D)\big)$$

with $t \in \mathbb{C}$ and $\underline{\tilde{a}} = \big(\underline{\tilde{a}}^1, \ldots, \underline{\tilde{a}}^r\big)$ where $\underline{\tilde{a}}^i = \big(\tilde{a}_{k,l}^i \mid 0 \leq k+l \leq m_i - 1\big) \in \mathbb{C}^{N_i}$ with $N_i = \binom{m_i+1}{2}$.

Fixing $i \in \{1, \ldots, r\}$, in local coordinates at z_i the family looks like

$$F_{(t,\underline{\tilde{a}})}^i := \big(F_{(t,\underline{\tilde{a}})}\big)_{z_i} = \sum_{k+l=0}^{m_i-1} t^{m_i-1-k-l} \tilde{a}_{k,l}^i x_i^k y_i^l + \sum_{|\alpha| \geq m_i} \varphi_\alpha^i(t, \underline{\tilde{a}})\, x_i^{\alpha_1} y_i^{\alpha_2},$$

with

$$\varphi_\alpha^i(t, \underline{\tilde{a}}) = B_\alpha^{1,i} + \sum_{j=1}^{r} \sum_{k+l=0}^{m_j-1} t^{m_j-1-k-l} \tilde{a}_{k,l}^j A_{\alpha,k,l}^{j,i}.$$

For $t \neq 0$ the transformation $\psi_t^i : (x_i, y_i) \mapsto (tx_i, ty_i)$ is indeed a coordinate transformation, and thus $F_{(t,\underline{\tilde{a}})}^i$ is contact equivalent[7] to

$$\tilde{F}_{(t,\underline{\tilde{a}})}^i := t^{-m_i+1} \cdot F_{(t,\underline{\tilde{a}})}^i(tx_i, ty_i) = \sum_{k+l=0}^{m_i-1} \tilde{a}_{k,l}^i x_i^k y_i^l + \sum_{|\alpha| \geq m_i} t^{1+|\alpha|-m_i} \varphi_\alpha^i(t, \underline{\tilde{a}})\, x_i^{\alpha_1} y_i^{\alpha_2}.$$

Note that for this new family in $\mathbb{C}\{x_i, y_i\}$ we have

$$\tilde{F}_{(0,\underline{a}^{fix})}^i = \sum_{k+l=0}^{m_i-1} a_{k,l}^{i,fix} x_i^k y_i^l = g_i,$$

and hence it gives rise to a deformation of $\big(g_i^{-1}(0), 0\big)$.

Step 2.2: Reduction to the investigation of the equisingular strata of certain families of polynomials.

It is basically our aim to verify the $\underline{\tilde{a}}$ as convergent power series in t such that the corresponding family is equisingular. However, since the $\tilde{F}_{(t,\underline{\tilde{a}})}^i$ are power series in x_i and y_i, we cannot right away apply the T-smoothness property of g_i, but we rather have to reduce to polynomials. For this let e_i be the determinacy bound[8] of $\mathcal{S}_i$ and

[7]Let $f, g \in \mathcal{O}_n = \mathbb{C}\{x_1, \ldots, x_n\}$ be two convergent power series in n indeterminates. We call f and g *contact equivalent*, if $\mathcal{O}_n/(f) \cong \mathcal{O}_n/(g)$, and we write in this case $f \sim_c g$. Equivalently, we could ask the germs $\big(V(f), 0\big)$ and $\big(V(g), 0\big)$ to be isomorphic, that is, ask the singularities to be *analytically equivalent*. C. f. [DP00] Definition 9.1.1 and Definition 3.4.19.

[8]A power series $f \in \mathcal{O}_n = \mathbb{C}\{x_1, \ldots, x_n\}$ (respectively the singularity $\big(V(f), 0\big)$ defined by f) is said to be *finitely determined* with respect to some equivalence relation $\sim$ if there exists some positive integer e such that $f \sim g$ whenever f and g have the same e-jet. If f is finitely determined, the smallest possible e is called the *determinacy bound*. Isolated singularities are finitely determined with respect to analytical equivalence and hence, for $n = 2$, as well with respect to topological equivalence. C. f. [DP00] Theorem 9.1.3 and Footnote 9.

define

$$\hat{F}^i_{(t,\underline{\tilde{a}})} := \sum_{k+l=0}^{m_i-1} \tilde{a}^i_{k,l} x_i^k y_i^l + \sum_{|\alpha|=m_i}^{e_i} t^{1+|\alpha|-m_i} \varphi^i_\alpha(t,\underline{\tilde{a}})\, x_i^{\alpha_1} y_i^{\alpha_2} \equiv \tilde{F}^i_{(t,\underline{\tilde{a}})} \big(\text{mod } (x_i,y_i)^{e_i+1}\big).$$

Thus $\hat{F}^i_{(t,\underline{\tilde{a}})}$ is a family in $\mathbb{C}[x_i,y_i]_{\leq e_i}$, and still

$$\hat{F}^i_{(0,\underline{a}^{fix})} = \tilde{F}^i_{(0,\underline{a}^{fix})} = g_i.$$

We claim that it suffices to find $\underline{\tilde{a}}(t) \in \mathbb{C}\{t\}$ with $\underline{\tilde{a}}(0) = \big(a^{i,fix}_{k,l} \mid i=1,\ldots,r,\ 0 \leq k+l \leq m_i-1\big)$, such that the families $\hat{F}^i_t := \hat{F}^i_{(t,\underline{\tilde{a}}(t))}$, $t \in (\mathbb{C},0)$, are in the equisingular strata $\mathbb{C}[x_i,y_i]^{es}_{\leq e_i}$, for $i=1,\ldots,r$.

Since then we have, for small[9] $t \neq 0$,

$$g_i = \hat{F}^i_0 \sim \hat{F}^i_t \sim \tilde{F}^i_{(t,\underline{\tilde{a}}(t))} \sim F^i_{(t,\underline{\tilde{a}}(t))} = \big(F_{(t,\underline{\tilde{a}}(t))}\big)_{z_i(t)},$$

by the e_i-determinacy and since ψ^i_t is a coordinate change for $t \neq 0$, which proves condition (2). Note that the singular points z_i will move with t.

It remains to verify conditions (1) and (3). Setting $F_t := F_{(t,\underline{\tilde{a}}(t))} \in H^0\big(\mathcal{O}_\Sigma(D)\big)$, for $t \in (\mathbb{C},0)$, we find that

$$F_0 = s_1 + \sum_{j=1}^{r} \sum_{k+l=m_j-1} a^{j,fix}_{k,l} s^j_{k,l}$$

is an element inside the linear system $\mathcal{D} = \{\lambda_0 s_1 + \sum_{j=1}^r \lambda_j s^j \mid (\lambda_0 : \ldots : \lambda_r) \in \mathbb{P}^r_{\mathbb{C}}\}$, where $s^j = \sum_{k+l=m_j-1} a^{j,fix}_{k,l} s^j_{k,l}$. Locally at z_i, $i=1,\ldots,r+r'$, $\mathcal{D}$ induces a deformation of (C,z_i) with equations

$$\lambda_i \cdot \big(g_i\big)_{m_i-1} + h.o.t., \quad \text{if } i=1,\ldots,r,$$

and

$$\lambda_0 \cdot \left(\sum_{|\alpha|=m_i} B^{1,i}_\alpha x_i^{\alpha_1} y_i^{\alpha_2}\right) + \sum_{j=1}^r \lambda_j \cdot \left(\sum_{k+l=m_j-1} a^{j,fix}_{k,l} \sum_{|\alpha|=m_i} A^{j,i}_{\alpha,k,l} x_i^{\alpha_1} y_i^{\alpha_2}\right) + h.o.t.,$$

$$\text{if } i=r+1,\ldots,r+r',$$

respectively. Thus any element of $\mathcal{D}$ has ordinary singularities of multiplicity m_i-1 at z_i for $i=1,\ldots,r$, and since s_1 has an ordinary singularity of multiplicity m_i at z_i for $i=r+1,\ldots,r+r'$, so has a generic element of $\mathcal{D}$. Moreover, a generic element of $\mathcal{D}$ has not more singular points than the special element s_1 and has thus singularities precisely in $\{z_1,\ldots,z_{r+r'}\}$. Replacing the g_i by some suitable multiples, we may assume that the curve defined by F_0 is a generic element of $\mathcal{D}$, which proves

[9] Here $f \sim_t g$, for two convergent power series $f,g \in \mathcal{O}_2 = \mathbb{C}\{x,y\}$, means that the singularities $\big(V(f),0\big)$ and $\big(V(g),0\big)$ are *topologically equivalent*, that is, there exists a homeomorphism $\Phi : (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ with $\Phi\big(V(f),0\big) = \big(V(g),0\big)$, which of course means, that this is correct for suitably chosen representatives. Note that if f and g are contact equivalent, then there exists even an analytical coordinate change Φ, that is, $f \sim_c g$ implies $f \sim_t g$.

(1). Similarly, we note that F_t in local coordinates at z_i, for $i=r+1,\ldots,r+r'$, looks like

$$\sum_{|\alpha|=m_i} B^{1,i}_\alpha x_i^{\alpha_1}y_i^{\alpha_2}+\sum_{j=1}^r\sum_{k+l=0}^{m_j-1} t^{m_j-1-k-l}\tilde{a}^j_{k,l}(t)\sum_{|\alpha|=m_i} A^{j,i}_{\alpha,k,l}x_i^{\alpha_1}y_i^{\alpha_2}+h.o.t.$$

$$=\sum_{|\alpha|=m_i}\left(B^{1,i}_\alpha+\sum_{j=1}^r\sum_{k+l=0}^{m_j-1} t^{m_j-1-k-l}\tilde{a}^j_{k,l}(t)A^{j,i}_{\alpha,k,l}\right)\,x_i^{\alpha_1}y_i^{\alpha_2}+h.o.t.,$$

and thus, for $t\neq 0$ sufficiently small, the singularity of F_t at z_i will be an ordinary singularity of multiplicity m_i, which gives (3).

Step 2.3: Find $\underline{\tilde{a}}(t)\in\mathbb{C}\{t\}^n$ with $\underline{\tilde{a}}(0)=\big(a^{i,fix}_{k,l},\ i=1,\ldots,r,\ 0\leq k+l\leq m_i-1\big)$, $n=\sum_{i=1}^r\binom{m_i+1}{2}$, such that the families $\hat{F}^i_t=\hat{F}^i_{(t,\underline{\tilde{a}}(t))}$, $t\in(\mathbb{C},0)$, are in the equisingular strata $\mathbb{C}[x_i,y_i]^{es}_{\leq e_i}$, for $i=1,\ldots r$.

In the sequel we adopt the notation of definition 5.1 adding indices i in the obvious way.

Since $\mathbb{C}[x_i,y_i]_{\leq m_i-1}$ is T-smooth at g_i, for $i=1,\ldots,r$, there exist $\Lambda_i\subseteq\{(k,l)\mid 0\leq k+l\leq m_i-1\}$ and power series $\phi^i_{k,l}\in\mathbb{C}\big\{\underline{\tilde{a}}^i_{(1)},\underline{\tilde{a}}^i_{(2)}\big\}$, for $(k,l)\in\Lambda_i$, such that the equisingular stratum $\mathbb{C}[x_i,y_i]^{es}_{\leq e_i}$ is given by the $\tau^{es,i}=\#\Lambda_i$ equations

$$\tilde{a}^i_{k,l}=\phi^i_{k,l}\big(\underline{\tilde{a}}^i_{(1)},\underline{\tilde{a}}^i_{(2)}\big),\quad\text{for }(k,l)\in\Lambda_i.$$

Setting $\Lambda=\bigcup_{j=1}^r\{j\}\times\Lambda_j$ we use the notation $\underline{\tilde{a}}_{(0)}=\big(\underline{\tilde{a}}^1_{(0)},\ldots,\underline{\tilde{a}}^r_{(0)}\big)=\big(\tilde{a}^i_{k,l}\bigm|(i,k,l)\in\Lambda\big)$ and, similarly $\underline{\tilde{a}}_{(1)}$, $\underline{\tilde{a}}_{(2)}$, $\underline{a}^{i,fix}_{(1)}$, $\underline{a}^{fix}_{(0)}$, and $\underline{a}^{fix}_{(1)}$. Moreover, setting $\tilde{\varphi}^i\big(t,\underline{\tilde{a}}_{(0)}\big)=\big(t^{|\alpha|-m_i}\varphi^i_\alpha\big(t,\underline{\tilde{a}}_{(0)},\underline{a}^{fix}_{(1)}\big)\bigm| m_i\leq|\alpha|\leq e_i\big)$, we define an analytical map germ

$$\Phi:\Big(\mathbb{C}\times\mathbb{C}^{\tau^{es,1}}\times\cdots\times\mathbb{C}^{\tau^{es,r}},\big(0,\underline{a}^{fix}_{(0)}\big)\Big)\to\Big(\mathbb{C}^{\tau^{es,1}}\times\cdots\times\mathbb{C}^{\tau^{es,r}},0\Big)$$

by

$$\Phi^i_{k,l}\big(t,\underline{\tilde{a}}_{(0)}\big)=\tilde{a}^i_{k,l}-\phi^i_{k,l}\Big(\underline{a}^{i,fix}_{(1)},t\cdot\tilde{\varphi}^i\big(t,\underline{\tilde{a}}_{(0)}\big)\Big),\quad\text{for }(i,k,l)\in\Lambda,$$

and we consider the system of equations

$$\Phi^i_{k,l}\big(t,\underline{\tilde{a}}_{(0)}\big)=0,\quad\text{for }(i,k,l)\in\Lambda.$$

One easily verifies that

$$\left(\frac{\partial\Phi^i_{k,l}}{\partial\tilde{a}^j_{\kappa,\lambda}}\big(0,\underline{a}^{fix}_{(0)}\big)\right)_{(i,k,l),(j,\kappa,\lambda)\in\Lambda}=\mathrm{id}_{\mathbb{C}^n}\,.$$

Thus by the Inverse Function Theorem there exist $\tilde{a}^i_{k,l}(t)\in\mathbb{C}\{t\}$ with $\tilde{a}^i_{k,l}(0)=a^{i,fix}_{k,l}$ such that

$$\tilde{a}^i_{k,l}(t)=\phi^i_{k,l}\Big(\underline{a}^{i,fix}_{(1)},t\cdot\tilde{\varphi}^i\big(t,\underline{\tilde{a}}_{(0)}(t)\big)\Big),\quad(i,k,l)\in\Lambda.$$

Now, setting $\underline{\tilde{a}}_{(1)}(t)\equiv\underline{a}^{fix}_{(1)}$, the families $\hat{F}^i_t=\hat{F}^i_{(t,\underline{\tilde{a}}(t))}$ are in the equisingular strata $\mathbb{C}[x_i,y_i]^{es}_{\leq e_i}$, for $i=1,\ldots,r$.

Step 3: It finally remains to show that F_t, for small $t \neq 0$, has no other singular points than $z_1(t), \ldots, z_r(t), z_{r+1}, \ldots, z_{r+r'}$.

Since for any $i = 1, \ldots, r + r'$ the family F_t, $t \in (\mathbb{C}, 0)$, induces a deformation of the singularity (C_0, z_i) there are, by the Conservation of Milnor Numbers[10] (cf. [DP00], Chapter 6), (Euclidean) open neighbourhoods $U(z_i) \subset \Sigma$ and $V(0) \subset \mathbb{C}$ such that for any $t \in V(0)$

(5.1) $\quad \mathrm{Sing}(C_t) \subset \bigcup_{i=1}^{r+r'} U(z_i)$, i. e. singular points of C_t come from singular points of C_0,

(5.2) $\quad \mu(C_0, z_i) = \sum_{z \in \mathrm{Sing}(F_t^i) \cap U(z_i)} \mu\big(F_t^i, z\big), \quad i = 1, \ldots, r + r'.$

For $i = r + 1, \ldots, r + r'$ condition (5.2) implies

$$(m_i - 1)^2 = \mu(C_0, z_i) \geq \mu\big(F_t^i, z_i\big) = (m_i - 1)^2,$$

and thus z_i must be the only critical point of F_t^i in $U(z_i)$, in particular,

$$\mathrm{Sing}(C_t) \cap U(z_i) = \{z_i\}.$$

Let now $i \in \{1, \ldots, r\}$. For $t \neq 0$ fixed, we consider the transformation defined by the coordinate change ψ_t^i,

$$\begin{array}{ccccccc} \mathbb{C}^2 & \supset & U(z_i) & \longrightarrow & U_t(z_i) & \subset & \mathbb{C}^2 \\ & & \cup\!\!\!\!| & & \cup\!\!\!\!| & & \\ & & (x_i, y_i) & \mapsto & \big(\frac{1}{t}x_i, \frac{1}{t}y_i\big), & & \end{array}$$

and the transformed equations

$$\tilde{F}_t^i(x_i, y_i) = t^{-m_i+1} F_t^i(tx_i, ty_i) = 0.$$

Condition (5.2) then implies,

$$(m_i - 2)^2 = \mu(C_0, z_i) = \sum_{z \in \mathrm{Sing}(F_t^i) \cap U(z_i)} \mu\big(F_t^i, z\big) = \sum_{z \in \mathrm{Sing}(\tilde{F}_t^i) \cap U_t(z_i)} \mu\big(\tilde{F}_t^i, z\big).$$

For $t \neq 0$ very small $U_t(z_i)$ becomes very large, so that, by shrinking $V(0)$ we may suppose that for any $0 \neq t \in V(0)$

$$\mathrm{Sing}(g_i) \subset U_t(z_i),$$

and that for any $z \in \mathrm{Sing}(g_i)$ there is an open neighbourhood $U(z) \subset U_t(z_i)$ such that

$$\mu(g_i, z) = \sum_{z' \in \mathrm{Sing}(\tilde{F}_t^i) \cap U(z)} \mu\big(\tilde{F}_t^i, z'\big).$$

[10]Recall the definition of the Milnor number of a holomorphic map $f \in \mathcal{O}(U)$ respectively of $f^{-1}\big(f(z)\big)$ at a point $z \in U \subset \mathbb{C}^2$: $\mu(f, z) = \mu\big(f^{-1}\big(f(z)\big), z\big) = \dim_{\mathbb{C}} \Big(\mathcal{O}_{U,z} / \big(\frac{\partial f}{\partial x}(z), \frac{\partial f}{\partial y}(z)\big)\Big)$.

If we now take into account that g_i has precisely one critical point, z_i, on its zero level, and that the critical points on the zero level of $\tilde{F}_t^i$ all contribute to the Milnor number $\mu(g_i, z_i)$, then we get the following sequence of inequalities:

$$\begin{aligned}
(m_i-2)^2-\mu(\mathcal{S}_i) &= \sum_{z\in \mathrm{Sing}(g_i)} \mu(g_i,z) - \sum_{z\in\mathrm{Sing}(g_i^{-1}(0))} \mu(g_i,z)\\
&\leq \sum_{z\in\mathrm{Sing}(\tilde{F}_t^i)\cap U_t(z_i)} \mu\big(\tilde{F}_t^i,z\big) - \sum_{z\in\mathrm{Sing}((\tilde{F}_t^i)^{-1}(0))\cap U_t(z_i)} \mu\big(\tilde{F}_t^i,z\big)\\
&= \sum_{z\in\mathrm{Sing}(F_t^i)\cap U(z_i)} \mu(F_t^i,z) - \sum_{z\in\mathrm{Sing}((F_t^i)^{-1}(0))\cap U(z_i)} \mu(F_t^i,z)\\
&\leq \mu(C_0,z_i)-\mu\big(F_t^i,z_i\big) = (m_i-2)^2-\mu(\mathcal{S}_i).
\end{aligned}$$

Hence all inequalities must have been equalities, and, in particular,

$$\mathrm{Sing}(C_t)\cap U(z_i) = \mathrm{Sing}\left((F_t^i)^{-1}(0)\right)\cap U(z_i) = \{z_i\},$$

which in view of Condition (5.1) finishes the proof.

Note that C_t, being a small deformation of the irreducible reduced curve C_0, will again be irreducible and reduced. $\square$

5.4 Corollary

Let $L\in \mathrm{Div}(\Sigma)$ be very ample over $\mathbb{C}$. Suppose that $D\in\mathrm{Div}(\Sigma)$ and $\mathcal{S}_1,\ldots,\mathcal{S}_r$ are topological singularity types with $\mu(\mathcal{S}_1)\geq\ldots\geq\mu(\mathcal{S}_r)$ such that

$$(D-L-K_\Sigma)^2 \geq \tfrac{414}{5}\sum_{\mu(\mathcal{S}_i)\leq 38}\mu(\mathcal{S}_i) + 58\sum_{\mu(\mathcal{S}_i)\geq 39}\left(\sqrt{\mu(\mathcal{S}_i)}+\tfrac{13}{2\sqrt{29}}\right)^2, \tag{5.3}$$

$$(D-L-K_\Sigma).B > \begin{cases} \sqrt{\tfrac{207}{5}}\sqrt{\mu(\mathcal{S}_1)}-1, & \text{if } \mu(\mathcal{S}_1)\leq 38,\\ \sqrt{29}\sqrt{\mu(\mathcal{S}_1)}+\tfrac{11}{2}, & \text{if } \mu(\mathcal{S}_1)\geq 39,\end{cases} \tag{5.4}$$

for any irreducible curve B with $B^2=0$ and $\dim|B|_a>0$,

$$D-L-K_\Sigma \text{ is nef,} \tag{5.5}$$

$$D.L-2g(L)\geq \begin{cases} \sqrt{\tfrac{207}{5}}\left(\sqrt{\mu(\mathcal{S}_1)}+\sqrt{\mu(\mathcal{S}_2)}\right)-2, & \text{if } \mu(\mathcal{S}_1)\leq 38,\\ \sqrt{\tfrac{207}{5}}\sqrt{\mu(\mathcal{S}_1)}+\sqrt{29}\sqrt{\mu(\mathcal{S}_1)}+\tfrac{9}{2}, & \text{if } \mu(\mathcal{S}_1)\geq 39\\ & \quad\&\ \mu(\mathcal{S}_2)\leq 38,\\ \sqrt{29}\left(\sqrt{\mu(\mathcal{S}_1)}+\sqrt{\mu(\mathcal{S}_2)}\right)+11, & \text{if } \mu(\mathcal{S}_2)\geq 39,\end{cases} \tag{5.6}$$

$$D^2\geq \tfrac{207}{5}\sum_{\mu(\mathcal{S}_i)\leq 38}\left(\sqrt{\mu(\mathcal{S}_i)}-\sqrt{\tfrac{5}{207}}\right)^2 + 29\sum_{\mu(\mathcal{S}_i)\geq 39}\left(\sqrt{\mu(\mathcal{S}_i)}+\tfrac{11}{2\sqrt{29}}\right)^2, \tag{5.7}$$

then there is an irreducible reduced curve C in $|D|_l$ with r singular points of topological types $\mathcal{S}_1,\ldots,\mathcal{S}_r$ as its only singularities.

Proof: This follows right away from Corollary 4.2, Theorem 5.3, and [Los98] Theorem 4.2. □

5.5 Remark
One could easily simplify the above formulae by not distinguishing the cases $\mu(\mathcal{S}_i) \geq 39$ and $\mu(\mathcal{S}_i) \leq 38$. However, one would loose information.

On the other hand, knowing something more about the singularity type one could achieve much better results, applying the corresponding bounds for the $s(\mathcal{S}_i)$. We leave it to the reader to apply the bounds. (Cf. [Los98] Remarks 4.3, 4.8, and 4.15)

As we have already mentioned earlier the most restrictive of the above *sufficient* conditions is (5.3), which could be characterised as a condition of the type

$$\sum_{i=1}^{r} \mu(\mathcal{S}_i) \leq \alpha D^2 + \beta D.K + \gamma,$$

where K is some fixed divisor class, α, β and γ are some constants.

There are also *necessary* conditions of this type, e. g.

$$\sum_{i=1}^{r} \mu(\mathcal{S}_i) \leq D^2 + D.K_\Sigma + 2,$$

which follows from the genus formula.[11]

See [Los98] Section 4.1 for considerations on the asymptotical optimality of the constant α.

6. Examples

In this section we are going to examine the conditions in the vanishing theorem (Corollary 2.2) and in the corresponding existence results for various types of surfaces. Unless otherwise stated, $r \geq 1$ is a positive integer, and $m_1, \ldots, m_r \in \mathbb{N}_0$ are non-negative, while at least one m_i is positive whenever we consider conditions for existence theorems.

[11]If D is an irreducible curve with precisely r singular points of types $\mathcal{S}_1, \ldots, \mathcal{S}_r$ and $\nu : \widetilde{D} \to D$ its normalisation, then $p_a(D) = g\big(\widetilde{D}\big) + \delta(D) \geq \delta(D)$, where $\delta(D) = \dim_{\mathbb{C}}\big(\nu_*\mathcal{O}_{\widetilde{D}}/\mathcal{O}_D\big)$ is the delta-invariant of D (cf. [BPV84] II.11). Moreover, by definition $\delta(D) = \sum_{z\in\mathrm{Sing}(D)} \delta(D,z)$ where $\delta(D,z) = \dim_{\mathbb{C}}\big(\nu_*\mathcal{O}_{\widetilde{D}}/\mathcal{O}_D\big)_z$ is the local delta-invariant at z, and it is well known that $2\delta(D,z) = \mu(D,z) + r(D,z) - 1 \geq \mu(D,z)$, where $r(D,z)$ is the number of branches of the curve singularity (D,z) and $\mu(D,z)$ is its Milnor number (cf. [Mil68] Chapter 10). Using now the genus formula we get

$$D^2 + D.K_\Sigma + 2 = 2p_a(D) \geq 2\sum_{i=1}^{r} \delta(\mathcal{S}_i) \geq \sum_{i=1}^{r} \mu(\mathcal{S}_i).$$

6.a. **The Classical Case - $\Sigma = \mathbb{P}^2_{\mathbb{C}}$.** Since in $\mathbb{P}^2_{\mathbb{C}}$ there are no irreducible curves of self-intersection number zero, Condition (2.5) is redundant. Moreover, Condition (2.6) takes in view of (2.4) the form $d + 3 \geq \sqrt{2}$. Corollary 2.2 thus takes the following form, where $L \in |\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)|_l$ is a generic line.

2.2a Corollary
Let d be any integer such that

(2.4a) $\quad (d+3)^2 \geq 2\sum_{i=1}^r (m_i+1)^2,$

(2.6a) $\quad d \geq -1.$

Then for $z_1, \ldots, z_r \in \mathbb{P}^2_{\mathbb{C}}$ in very general position and $\nu > 0$

$$H^\nu\left(\mathrm{Bl}_{\underline{z}}\left(\mathbb{P}^2_{\mathbb{C}}\right), d\pi^*L - \sum_{i=1}^r m_i E_i\right) = 0.$$

Now turning to the existence theorem Corollary 4.2 for generic fat point schemes, we, of course, find that Condition (4.8) is obsolete, and so is (4.9), taking into account that (4.10) implies $d > 0$. But then Conditions (4.10) and (4.11) become also redundant in view of Condition (4.7) and equation (4.12).

Thus the conditions in Corollary 4.2 reduce to $d > 0$ and

(4.7a) $\quad (d+2)^2 \geq 2\sum_{i=1}^r (m_i+1)^2,$

and, similarly, the conditions in Corollary 5.4 reduce to $d > 7$ and

(5.3a) $\quad (d+2)^2 \geq \frac{414}{5} \sum_{\mu(\mathcal{S}_i)\leq 38} \mu(\mathcal{S}_i) + 58 \sum_{\mu(\mathcal{S}_i)\geq 39} \left(\sqrt{\mu(\mathcal{S}_i)} + \frac{13}{2\sqrt{29}}\right)^2.$

These results are much weaker than the previously known ones (e. g. [Los98] Proposition 4.11, where the factor 2 is replaced by $\frac{10}{9}$) which use the Vanishing Theorem of Geng Xu (cf. [Xu95] Theorem 3), particularly designed for $\mathbb{P}^2_{\mathbb{C}}$. – Using $L \in |\mathcal{O}_\Sigma(l)|_l$ with $l > 1$ instead of $\mathcal{O}_\Sigma(1)$ in Corollary 4.2 does not improve the conditions.

6.b. **Geometrically Ruled Surfaces.** Let $\Sigma = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} C$ be a geometrically ruled surface with normalised bundle $\mathcal{E}$ (in the sense of [Har77] V.2.8.1). The Néron-Severi group of Σ is

$$\mathrm{NS}(\Sigma) = C_0\mathbb{Z} \oplus F\mathbb{Z},$$

with intersection matrix

$$\begin{pmatrix} -e & 1 \\ 1 & 0 \end{pmatrix},$$

where $F \cong \mathbb{P}^1_{\mathbb{C}}$ is a fibre of π, C_0 a section of π with $\mathcal{O}_\Sigma(C_0) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, and $e = -\deg(\Lambda^2\mathcal{E}) \geq -g$.[12] For the canonical divisor we have

$$K_\Sigma \sim_a -2C_0 + (2g-2-e)F,$$

[12]By [Nag70] Theorem 1 there is some section $D \sim_a C_0 + bF$ with $g \geq D^2 = 2b - e$. Since D is irreducible, by [Har77] V.2.20/21 $b \geq 0$, and thus $-g \leq e$.

where $g = g(C)$ is the genus of the base curve C.

In order to understand Condition (2.5) we have to examine special irreducible curves on Σ.

6.2 Lemma

Let $B \in |aC_0 + bF|_a$ be an irreducible curve with $B^2 = 0$ and $\dim |B|_a \geq 1$. Then we are in one of the following cases

$$a = 0,\ b = 1,\ \text{and } B \sim_a F, \tag{6.1}$$

$$e = 0,\ a \geq 1,\ b = 0,\ \text{and } B \sim_a aC_0,\ \text{or} \tag{6.2}$$

$$e < 0,\ a \geq 2,\ b = \tfrac{a}{2}e < 0,\ \text{and } B \sim_a aC_0 + \tfrac{a}{2}eF. \tag{6.3}$$

Moreover, if $a = 1$, then $\Sigma \cong C_0 \times \mathbb{P}^1_{\mathbb{C}}$.

Proof: Since B is irreducible, we have

$$0 \leq B.F = a \quad \text{and} \quad 0 \leq B.C_0 = b - ae. \tag{6.4}$$

If $a = 0$, then $|B|_a = |bF|_a$, but since the general element of $|B|_a$ is irreducible, b has to be one, and we are in case (6.1).

We, therefore, may assume that $a \geq 1$. Since $B^2 = 0$ we have

$$0 = B^2 = 2a\left(b - \tfrac{a}{2}e\right), \quad \text{hence} \quad b = \tfrac{a}{2}e. \tag{6.5}$$

Combining this with (6.4) we get $e \leq 0$.

Moreover, if $e = 0$, then of course $b = 0$, while, if $e < 0$, then $a \geq 2$ by [Har77] V.2.21, since otherwise b would have to be non-negative. This brings us down to the cases (6.2) and (6.3).

It remains to show, that $B.F = a = 1$ implies $\Sigma \cong C_0 \times \mathbb{P}^1_{\mathbb{C}}$. But by assumption the elements of $|B|_a$ are disjoint sections of the fibration π. Thus, by Lemma 6.3, $\Sigma \cong C \times \mathbb{P}^1_{\mathbb{C}}$. $\square$

6.3 Lemma

If $\pi : \Sigma \to C$ has three disjoint sections, then Σ is isomorphic to $C \times \mathbb{P}^1_{\mathbb{C}}$ as a ruled surface, i. e. there is an isomorphism $\alpha : \Sigma \to C \times \mathbb{P}^1_{\mathbb{C}}$ such that the following diagram is commutative:

$$\begin{tikzcd} \Sigma \arrow[rr, "\alpha"] \arrow[dr, "\pi"'] & & C \times \mathbb{P}^1_{\mathbb{C}} \arrow[dl, "\mathrm{pr}"] \\ & C. & \end{tikzcd}$$

Proof: See [IS96] p. 229.

π is a locally trivial $\mathbb{P}^1_{\mathbb{C}}$-bundle, thus C is covered by a finite number of open affine subsets $U_i \subset C$ with trivialisations

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\cong", "\varphi_i"'] \arrow[dr, "\pi"'] & & U_i \times \mathbb{P}^1_{\mathbb{C}} \arrow[dl, "\mathrm{pr}"] \\ & U_i, & \end{tikzcd}$$

which are linear on the fibres.

The three disjoint sections on Σ, say S_0, S_1, and S_∞, give rise to three sections S_0^i, S_1^i, and S_∞^i on $U_i \times \mathbb{P}^1_{\mathbb{C}}$. For each point $z \in U_i$ there is a unique linear projectivity on the fibre $\{z\} \times \mathbb{P}^1_{\mathbb{C}}$ mapping the three points $p_{0,z} = S_0^i \cap \big(\{z\} \times \mathbb{P}^1_{\mathbb{C}}\big)$, $p_{1,z} = S_1^i \cap \big(\{z\} \times \mathbb{P}^1_{\mathbb{C}}\big)$, and $p_{\infty,z} = S_\infty^i \cap \big(\{z\} \times \mathbb{P}^1_{\mathbb{C}}\big)$ to the standard basis $0 \equiv \big(z, (1:0)\big)$, $1 \equiv \big(z, (1:1)\big)$, and $\infty \equiv \big(z, (0:1)\big)$ of $\mathbb{P}^1_{\mathbb{C}} \cong \{z\} \times \mathbb{P}^1_{\mathbb{C}}$. If $p_{0,z} = \big(z, (x_0 : y_0)\big)$, $p_{1,z} = \big(z, (x_1 : y_1)\big)$, and $p_{\infty,z} = \big(z, (x_\infty : y_\infty)\big)$, the projectivity is given by the matrix

$$A = \begin{pmatrix} \frac{(x_0y_1 - y_0x_1)y_\infty}{y_0y_1x_\infty^2 - y_0x_1x_\infty y_\infty - x_0y_1x_\infty y_\infty + x_0x_1y_\infty} & \frac{(x_0y_1 - y_0x_1)x_\infty}{y_0y_1x_\infty^2 - y_0x_1x_\infty y_\infty - x_0y_1x_\infty y_\infty + x_0x_1y_\infty} \\ \frac{y_0}{x_0y_\infty - y_0x_\infty} & \frac{x_0}{x_0y_\infty - y_0x_\infty} \end{pmatrix},$$

whose entries are rational functions in the coordinates of $p_{0,z}$, $p_{1,z}$, and $p_{\infty,z}$. Inserting for the coordinates local equations of the sections, A finally gives rise to an isomorphism of $\mathbb{P}^1_{\mathbb{C}}$-bundles

$$\alpha_i : U_i \times \mathbb{P}^1_{\mathbb{C}} \to U_i \times \mathbb{P}^1_{\mathbb{C}}$$

mapping the sections S_0^i, S_1^i, and S_∞^i to the trivial sections.

The transition maps

$$U_{ij} \times \mathbb{P}^1_{\mathbb{C}} \xrightarrow{\alpha_{i|}^{-1}} U_{ij} \times \mathbb{P}^1_{\mathbb{C}} \xrightarrow{\varphi_{i|}^{-1}} \pi^{-1}(U_{ij}) \xrightarrow{\varphi_{j|}} U_{ij} \times \mathbb{P}^1_{\mathbb{C}} \xrightarrow{\alpha_{j|}} U_{ij} \times \mathbb{P}^1_{\mathbb{C}},$$

with $U_{ij} = U_i \cap U_j$, are linear on the fibres and fix the three trivial sections. Thus they must be the identity maps, which implies that the $\alpha_i \circ \varphi_i$, $i = 1, \ldots, r$, glue together to an isomorphism of ruled surfaces:

$$\begin{tikzcd} \Sigma \arrow[rr, "\alpha"] \arrow[dr, "\pi"'] & & C \times \mathbb{P}^1_{\mathbb{C}} \arrow[dl, "\mathrm{pr}"] \\ & C. & \end{tikzcd}$$

□

Knowing the algebraic equivalence classes of irreducible curves in Σ which satisfy the assumptions in Condition (2.5) we can give a better formulation of the vanishing theorem in the case of geometrically ruled surfaces.

In order to do the same for the existence theorems, we have to study very ample divisors on Σ. These, however, depend very much on the structure of the base curve C, and the general results which we give may be not the best possible. Only in the case $C = \mathbb{P}^1_{\mathbb{C}}$ we can give a complete investigation.

The geometrically ruled surfaces with base curve $\mathbb{P}^1_{\mathbb{C}}$ are, up to isomorphism, just the Hirzebruch surfaces $\mathbb{F}_e = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}} \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-e)\big)$, $e \geq 0$. Note that $\text{Pic}(\mathbb{F}_e) = \text{NS}(\mathbb{F}_e)$, that is, algebraic equivalence and linear equivalence coincide. Moreover, by [Har77] V.2.18 a divisor class $L = \alpha C_0 + \beta F$ is very ample over $\mathbb{C}$ if and only if $\alpha > 0$ and $\beta > \alpha e$. The conditions throughout the existence theorems turn out to be optimal if we work with $L = C_0 + (e+1)F$, while for other choices of L they become more restrictive.[13]

In the case $C \not\cong \mathbb{P}^1_{\mathbb{C}}$, we may choose an integer $l \geq \max\{e+1, 2\}$ such that the algebraic equivalence class $|C_0 + lF|_a$ contains a very ample divisor L, e. g. $l = e+3$ will do, if C is an elliptic curve.[14] In particular, $l \geq 2$ as soon as $\Sigma \not\cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$. With the above choice of L we have $g(L) = 1 + \frac{L^2 + L.K_\Sigma}{2} = 1 + \frac{(-e+2l)+(e-2l+2g-2)}{2} = g$, and hence the generic curve in $|L|_l$ is a smooth curve whose genus equals the genus of the base curve.

2.2b Corollary

Given two integers $a, b \in \mathbb{Z}$ satisfying

(2.4b) $\quad a\big(b - \big(\frac{a}{2} - 1\big)e\big) \geq \sum_{i=1}^r (m_i + 1)^2,$

(2.5b.i) $\quad a > \max\{m_i \mid i = 1, \ldots, r\},$

(2.5b.ii) $\quad b > \max\{m_i \mid i = 1, \ldots, r\}$, *if* $e = 0$,

(2.5b.iii) $\quad 2\big(b - \big(\frac{a}{2} - 1\big)e\big) > \max\{m_i \mid i = 1, \ldots, r\}$, *if* $e < 0$, *and*

(2.6b) $\quad b \geq (a-1)e$, *if* $e > 0$.

For $z_1, \ldots, z_r \in \Sigma$ in very general position and $\nu > 0$

$$H^\nu\left(\text{Bl}_{\underline{z}}(\Sigma), (a-2) \cdot \pi^* C_0 + (b-2+2g)\pi^* F - \sum_{i=1}^r m_i E_i\right) = 0.$$

Proof: Note that if the invariant e is non-positive, then $\big(b - \big(\frac{a}{2} - 1\big)e\big) > 0$ implies

$$b \geq (a-1)e, \tag{6.6}$$

so that this inequality is fulfilled for any choice of e.

---

[13] Let $L' = \alpha C_0 + \beta F$, then $D - L' - K_{\mathbb{F}_e} = (a+1-\alpha)C_0 + (b+1+e-\beta)F$, and thus the optimality of the conditions follows from

(4.7b.i/ii) $\quad (D - L' - K_{\mathbb{F}_e})^2 = (a+1-\alpha)\big(2(b+1+e-\beta) - (a+1-\alpha)e\big) \leq a\big((2b-ae) + (\alpha e + e + 2 - 2\beta)\big) \leq a(2b-ae) = (D - L - K_{\mathbb{F}_e})^2,$

(4.8b.i/ii) $\quad (D - L' - K_{\mathbb{F}_e}).F = a+1-\alpha \leq a = (D - L - K_{\mathbb{F}_e}).F$, and for $e = 0$, $(D - L' - K_{\mathbb{F}_e}).C_0 = b+1-\beta \leq b = (D - L - K_{\mathbb{F}_e}).C_0$, and

(4.9b.ii) $\quad b+1+e-\beta \geq e(a+1-\alpha)$ implies $b \geq b + e\alpha + 1 - \beta \geq ae$.

[14] l will be the degree of a suitable very ample divisor $\mathfrak{d}$ on C. Now $\mathfrak{d}$ defines an embedding of C into some $\mathbb{P}^N_{\mathbb{C}}$ such that the degree of the image C' is just $\deg(\mathfrak{d})$. Therefore $\deg(\mathfrak{d}) \geq 2$, unless C' is linear, which implies $C \cong \mathbb{P}^1_{\mathbb{C}}$.

Setting $D=(a-2)C_0+(b-2+2g)F$ we have

$$(D-K_\Sigma)^2=\big(aC_0+(b+e)F\big)^2=2a\bigg(b-\Big(\frac{a}{2}-1\Big)e\bigg)\geq 2\sum_{i=1}^r(m_i+1)^2,$$

which is just (2.4). Similarly, by (2.5b.i/ii/iii) and Lemma 6.2 Condition (2.5) is satisfied.[15] Finally Condition (2.6b) implies that $D-K_\Sigma$ is nef.

In order to see the last statement, we have to consider two cases.

Case 1: $e\geq 0$.

If $B\in|a'C_0+b'F|_a$ is irreducible, then we are in one of the following situations, by [Har77] V.2.20:

(i) $a'=0$ and $b'=1$, which, considering (2.5b.i), implies

$$(D-K_\Sigma).B=a>0.$$

(ii) $a'=1$ and $b'=0$, which by (6.6) leads to

$$(D-K_\Sigma).B=b-(a-1)e\geq 0.$$

(iii) $a'>0$ and $b'\geq a'e$, which in view of (2.5b.i), (6.6), and $e\geq 0$ gives

$$(D-K_\Sigma).B=-aa'e+ab'+(b+e)a'\geq(b+e)a'\geq 0.$$

Hence, $D-K_\Sigma$ is nef.

Case 2: $e<0$.

In this case we may apply [Har77] V.2.21 and find that if $B\in|a'C_0+b'F|_a$ is irreducible, then we are in one of the following situations:

(i) $a'=0$ and $b'=1$, which is treated as in Case 1.

(ii) $a'=1$ and $b'\geq 0$, which, considering (2.5b.i) and (6.6), implies

$$(D-K_\Sigma).B=b-(a-1)e+ab'\geq 0.$$

(iii) $a'\geq 2$ and $b'\geq\frac{1}{2}a'e$, which in view of (2.5b.iii) leads to

$$(D-K_\Sigma).B=-aa'e+ab'+(b+e)a'\geq-\frac{1}{2}aa'e+(b+e)a'$$

$$=\bigg(b-\Big(\frac{a}{2}-1\Big)e\bigg)a'>0.$$

Hence, $D-K_\Sigma$ is nef. $\square$

[15]To see this, let $B\sim_a a'C_0+b'F$ be an irreducible curve with $B^2=0$. Then by Lemma 6.2 either $a'=0$ and $b'=1$, or $e=0$, $a'\geq 1$ and $b'=0$, or $e<0$, $a'\geq 2$, and $b'=\frac{a'}{2}e<0$. In the first case, $(D-K_\Sigma).B=a>\max\{m_i\mid i=1,\ldots,r\}$ by (2.5b.i). In the second case, $(D-K_\Sigma).B=ba'\geq b>\max\{m_i\mid i=1,\ldots,r\}$ by (2.5b.ii). And finally, in the third case, we have $(D-K_\Sigma).B=a'\cdot\big(b-\big(\frac{a}{2}-1\big)e\big)\big)>\max\{m_i\mid i=1,\ldots,r\}$ by (2.5b.iii).

In order to obtain nice formulae we considered $D = (a-2)C_0 + (b-2+2g)F$ in the formulation of the vanishing theorem. For the existence theorems it turns out that the formulae look best if we work with $D = (a-1)C_0 + (b+l+2g-2-e)F$ instead. In the case of Hirzebruch surfaces this is just $D = (a-1)C_0 + (b-1)F$.

4.2b Corollary

Given integers $a, b \in \mathbb{Z}$ satisfying

(4.7b) $\quad a\big(b - \frac{a}{2}e\big) \geq \sum_{i=1}^{r}(m_i+1)^2$,

(4.8b.i) $\quad a > \max\{m_i \mid i = 1, \ldots, r\}$,

(4.8b.ii) $\quad b > \max\{m_i \mid i = 1, \ldots, r\}$, *if* $e = 0$,

(4.8b.iii) $\quad 2\big(b - \frac{a}{2}e\big) > \max\{m_i \mid i = 1, \ldots, r\}$, *if* $e < 0$, *and*

(4.9b) $\quad b \geq ae$, *if* $e > 0$,

then for $z_1, \ldots, z_r \in \Sigma$ in very general position there is an irreducible reduced curve $C \in |(a-1)C_0 + (b+l+2g-2-e)F|_a$ with ordinary singularities of multiplicities m_i at the z_i as only singularities. Moreover, $V_{|C|}(\underline{m})$ is T-smooth at C.

Proof: Note that by (4.7b) and (4.8b.i) $b > \frac{a}{2}e \geq ae$, if $e \leq 0$, and thus the inequality

$$b \geq ae, \tag{6.7}$$

is fulfilled no matter what e is.

Noting that $D - L - K_\Sigma \sim_a aC_0 + bF$, it is in view of Lemma 6.2 clear, that the Conditions (4.7) and (4.8) take the form (4.7b) respectively (4.8b). It, therefore, remains to show that (4.10) and (4.11) are obsolete, and that (4.9) takes the form (4.9b), which in particular means that it is obsolete in the case $\Sigma \cong C \times \mathbb{P}^1_{\mathbb{C}}$.

Step 1: (4.10) is obsolete.

If $\Sigma \not\cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, then $l \geq 2$. Since, moreover, $g(L) = g$ and $D.L = a(l-e)+b+2g-2$, Condition (4.8b.i) and (6.7) imply (4.10), i. e. for all i, j

$$D.L - 2g(L) = a(l-e) + b - 2 \geq \begin{cases} a + b - 2 \geq m_i + m_j, & \text{if } \Sigma \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}, \\ 2a + (b-ae) - 2 \geq m_i + m_j, & \text{else.} \end{cases}$$

Step 2: (4.9) takes the form (4.9b).

We have to consider two cases.

Case 1: $e \geq 0$.

If $B \in |a'C_0 + b'F|_a$ is irreducible, then we are in one of the following situations, by [Har77] V.2.20:

(i) $a' = 0$ and $b' = 1$, which, considering (4.8b.i), implies

$$(D - L - K_\Sigma).B = a > 0.$$

(ii) $a' = 1$ and $b' = 0$, which by (6.7) leads to

$$(D - L - K_\Sigma).B = b - ae \geq 0.$$

(iii) $a' > 0$ and $b' \geq a'e$, which in view of (4.8b.i), (6.7), and $e \geq 0$ gives

$$(D - L - K_\Sigma).B = -aa'e + ab' + ba' \geq ba' \geq 0.$$

Hence, $D - L - K_\Sigma$ is nef.

Case 2: $e < 0$.

In this case we may apply [Har77] V.2.21 and find that if $B \in |a'C_0 + b'F|_a$ is irreducible, then we are in one of the following situations:

(i) $a' = 0$ and $b' = 1$, which is treated as in Case 1.

(ii) $a' = 1$ and $b' \geq 0$, which, considering (4.8b.i) and (6.7), implies

$$(D - L - K_\Sigma).B = b - ae + ab' \geq 0.$$

(iii) $a' \geq 2$ and $b' \geq \frac{1}{2}a'e$, which in view of (4.8b.iii) leads to

$$(D - L - K_\Sigma).B = -aa'e + ab' + ba' \geq -\frac{1}{2}aa'e + ba' = \left(b - \frac{a}{2}e\right)a' > 0.$$

Hence, $D - L - K_\Sigma$ is nef.

Step 3: (4.12) is satisfied, and thus (4.11) is obsolete.

We have

$$D^2 = -e(a-1)^2 + 2(a-1)(b + l + 2g - 2 - e),$$

and

$$(2D - L - K_\Sigma).(L + K_\Sigma) = e + 2al + 4ag + 4 - 2b - 4a - 4g - 2l.$$

Hence Condition (4.12) is equivalent to

$$4b + 8a + 4l + a^2e + 8g < 2ab + 4al + 2e + 8ag + 8 + 4\sum_{i=1}^{r} m_i + 2r. \tag{6.8}$$

If $\Sigma \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, then the situation is symmetric and we may w. l. o. g. assume that $b \geq a$. Since by (4.8b.i) $a \geq 2$ we have to consider the following cases:

$a \geq 4$: $g = 0,\ e = 0$: Then $\Sigma \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, and by assumption $b \geq a \geq 4$ and $l = e + 1 = 1$. We thus have $2ab + 4al = ab + ab + 4a \geq 4b + 8a$ and $8 > 4l$, which implies (6.8).

$g = 0,\ e > 0$ or $g \geq 1,\ e \geq 0$: By (6.7) we get $2ab \geq 4b + ab \geq 4b + a^2e$.

$g = 0,\ e \in \{1, 2\}$: Then $l = e + 1$, and hence $4al \geq 8a$ and $8 + 2e \geq 4l$.

$g = 0,\ e \geq 3$: Thus $l = e + 1 \geq 4$, which implies $2al \geq 8a$ and $2al \geq 4l$.

$g \geq 1$: Then $l \geq 2$, and thus $2al + 4ag \geq 8a$, $2al \geq 4l$, and $4ag \geq 8g$.

In any of the above cases (6.8) is satisfied.

$g \geq 1,\ e < 0$: Then $l \geq 2$ and $2ag \geq 8g$. We therefore consider the following cases:

$b \geq 0$: Thus $2ab \geq 4b$, $2al + 4ag \geq 8a$, $2al \geq 4l$, and $2e \geq a^2e$.

$b<0$: By (4.8b.i) and (4.8b.iii) $2ab \geq a^2e$, and of course $0 > 4b$. Moreover, since $e \geq -g$, we have $ag + 2e \geq 0$. And finally, $3al + 5ag \geq 8a$ and $al \geq 4l$.

These considerations together ensure that (6.8) is fulfilled.

$a = 3$: In this case (6.8) comes down to

$$16 + 7e < 2b + 8l + 16g + 4\sum_{i=1}^{r} m_i + 2r. \tag{6.9}$$

$e > 0$, or $e = 0$ and $g = 0$: Then $b \geq a = 3$. Thus $2b + 8l + 4\sum_{i=1}^r m_i \geq 6 + 8(e+1) + 4 > 16 + 7e$, so that the inequality (6.9) is certainly satisfied.

$e < 0$, or $e = 0$ and $g \geq 1$: Then $g \geq 1$ and $16g \geq 16 + 7e$, so that again the inequality (6.9) is fulfilled.

$a = 2$: (6.8) reads just

$$8 + 2e < 4l + 8g + 4\sum_{i=1}^{r} m_i + 2r. \tag{6.10}$$

$e < 0$: Then $g \geq 1$, and thus $8g \geq 8 + 2e$, which implies (6.10).

$e \geq 0$: Then $4l + 4\sum_{i=1}^r m_i \geq 4(e+1) + 4 \geq 8 + 2e$, and hence (6.10) is fulfilled.

□

With the same D and L as above the conditions in the existence theorem Corollary 5.4 reduce to

(5.3b) $$a\left(b - \tfrac{a}{2}e\right) \geq \tfrac{207}{5}\sum_{\mu(\mathcal{S}_i)\leq 38} \mu(\mathcal{S}_i) + 29\sum_{\mu(\mathcal{S}_i)\geq 39}\left(\sqrt{\mu(\mathcal{S}_i)} + \tfrac{13}{2\sqrt{29}}\right)^2,$$

(5.4b.i) $$a > \begin{cases} \sqrt{\tfrac{207}{5}}\sqrt{\mu(\mathcal{S}_1)} - 1, & \text{if } \mu(\mathcal{S}_1) \leq 38,\\ \sqrt{29}\sqrt{\mu(\mathcal{S}_1)} + \tfrac{11}{2}, & \text{if } \mu(\mathcal{S}_1) \geq 39,\end{cases}$$

(5.4b.ii) $$b > \begin{cases} \sqrt{\tfrac{207}{5}}\sqrt{\mu(\mathcal{S}_1)} - 1, & \text{if } \mu(\mathcal{S}_1) \leq 38,\\ \sqrt{29}\sqrt{\mu(\mathcal{S}_1)} + \tfrac{11}{2}, & \text{if } \mu(\mathcal{S}_1) \geq 39,\end{cases} \quad \text{if } e = 0,$$

(5.4b.iii) $$2\left(b - \tfrac{a}{2}e\right) > \begin{cases} \sqrt{\tfrac{207}{5}}\sqrt{\mu(\mathcal{S}_1)} - 1, & \text{if } \mu(\mathcal{S}_1) \leq 38,\\ \sqrt{29}\sqrt{\mu(\mathcal{S}_1)} + \tfrac{11}{2}, & \text{if } \mu(\mathcal{S}_1) \geq 39,\end{cases} \quad \text{if } e < 0, \text{ and}$$

(5.5b) $b \geq ae$, if $e > 0$.

6.c. **Products of Curves.** Let C_1 and C_2 be two smooth projective curves of genuses $g_1 \geq 1$ and $g_2 \geq 1$ respectively. The surface $\Sigma = C_1 \times C_2$ is naturally equipped with two fibrations $\mathrm{pr}_i : \Sigma \to C_i$, $i = 1, 2$, and by abuse of notation we denote two generic fibres $\mathrm{pr}_2^{-1}(p_2) = C_1 \times \{p_2\}$ resp. $\mathrm{pr}_1^{-1}(p_1) = \{p_1\} \times C_2$ again by C_1 resp. C_2.

One can show that for a generic choice of the curves C_1 and C_2 the Neron-Severi group $\mathrm{NS}(\Sigma) = C_1\mathbb{Z} \oplus C_2\mathbb{Z}$ of Σ is two-dimensional[16] with intersection matrix

$$(C_i.C_j)_{i,j} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus, the only irreducible curves $B \subset \Sigma$ with selfintersection $B^2 = 0$ are the fibres C_1 and C_2, and for any irreducible curve $B \sim_a aC_1 + bC_2$ the coefficients a and b must be non-negative. Taking into account that $K_\Sigma \sim_a (2g_2 - 2)C_1 + (2g_1 - 2)C_2$ Corollary 2.2 comes down to the following.

2.2c Corollary

Let C_1 and C_2 be two generic curves with $g(C_i) = g_i \geq 1$, $i = 1, 2$, and let $a, b \in \mathbb{Z}$ be integers satisfying

(2.4c) $\quad (a - 2g_2 + 2)(b - 2g_1 + 2) \geq \sum_{i=1}^r (m_i + 1)^2$, *and*

(2.5c) $\quad (a - 2g_2 + 2), (b - 2g_1 + 2) > \max\{m_i \mid i = 1, \ldots, r\}$,

then for $z_1, \ldots, z_r \in \Sigma = C_1 \times C_2$ in very general position and $\nu > 0$

$$H^\nu\left(\mathrm{Bl}_{\underline{z}}(\Sigma),\ a\pi^*C_1 + b\pi^*C_2 - \sum_{i=1}^r m_iE_i\right) = 0.$$

We know that $C_1 + C_2$ has positive self-intersection and intersects any irreducible curve positive, is thus ample by Nakai-Moishezon. But then we may find some integer $l \geq 3$ such that $L = lC_1 + lC_2$ is very ample. We choose l minimal with this property for the existence theorem Corollary 4.2, and we claim that the Conditions (4.9), (4.10) and (4.11) become obsolete, while (4.7) and (4.8) take the form

(4.7c) $\quad (a - l - 2g_2 + 2)(b - l - 2g_1 + 2) \geq \sum_{i=1}^r (m_i + 1)^2$, and

(4.8c) $\quad (a - l - 2g_2 + 2), (b - l - 2g_1 + 2) > \max\{m_i \mid i = 1, \ldots, r\}$.

That is, under these hypotheses there is an irreducible curve in $|D|_l$, for any $D \sim_a aC_1 + bC_2$, with precisely r ordinary singular points of multiplicities $m_1, \ldots, m_r$.

(4.9) becomes redundant in view of (4.8c) and since an irreducible curve $B \sim_a a'C_1 + b'C_2$ has non-negative coefficients a' and b'. For (4.11) we look at (4.12), which in this case takes the form

$$2ab + \big((2a - l - 2g_2 + 2)(l + 2g_1 - 2) + (2b - l - 2g_1 + 2)(l + 2g_2 - 2)\big)$$
$$+4\sum_{i=1}^r m_i + 2r > 0.$$

However, in view of (4.8c) the factors and summands on the left-hand side are all positive, so that the inequality is fulfilled.

[16] In the case that C_1 and C_2 are elliptic curves, generic means precisely, that they are not isogenous - see Section 6.d. For a further investigation of the Neron-Severi group of a product of two curves we refer to Appendix F.

It remains to show that $D.L-g(L) \geq m_i+m_j$ for all i, j. However, by the adjunction formula $g(L) = 1+\frac{1}{2}(L^2+L.K_\Sigma) = 1+l\cdot(l+g_1+g_2-2)$, and by (4.8c) $D.L-g(L) > l\cdot\big((a-l-2g_2+2)+(b-l-2g_1+2)\big) > 3(m_i+m_j) \geq m_i+m_j$. Thus the claim is proved.

From these considerations we at once deduce the conditions for the existence of an irreducible curve in $|D|_l$, $D \sim_a aC_1+bC_2$, with prescribed singularities of arbitrary type, i. e. the conditions in Corollary 5.4. They come down to

$$(a-l-2g_2+2)(b-l-2g_1+2) \geq \frac{207}{5}\sum_{\mu(\mathcal{S}_i)\leq 38}\mu(\mathcal{S}_i)+29\sum_{\mu(\mathcal{S}_i)\geq 39}\left(\sqrt{\mu(\mathcal{S}_i)}+\frac{13}{2\sqrt{29}}\right)^2, \tag{5.3c}$$

and

$$(a-l-2g_2+2),(b-l-2g_1+2) > \begin{cases} \sqrt{\frac{207}{5}}\sqrt{\mu(\mathcal{S}_1)}-1, & \text{if } \mu(\mathcal{S}_1)\leq 38,\\ \sqrt{29}\sqrt{\mu(\mathcal{S}_1)}+\frac{11}{2}, & \text{if } \mu(\mathcal{S}_1)\geq 39.\end{cases} \tag{5.4c}$$

6.d. **Products of Elliptic Curves.** Let $C_1 = \mathbb{C}/\Lambda_1$ and $C_2 = \mathbb{C}/\Lambda_2$ be two elliptic curves, where $\Lambda_i = \mathbb{Z}\oplus\tau_i\mathbb{Z} \subset \mathbb{C}$ is a lattice and τ_i is in the upper half plane of $\mathbb{C}$. We denote the natural group structure on each of the C_i by $+$ and the neutral element by 0.

Our interest lies in the study of the surface $\Sigma = C_1\times C_2$. This surface is naturally equipped with two fibrations $\mathrm{pr}_i : \Sigma \to C_i$, $i = 1,2$, and by abuse of notation we denote the fibres $\mathrm{pr}_2^{-1}(0) = C_1\times\{0\}$ resp. $\mathrm{pr}_1^{-1}(0) = \{0\}\times C_2$ again by C_1 resp. C_2. The group structures on C_1 and C_2 extend to Σ so that Σ itself is an abelian variety. Moreover, for $p = (p_1,p_2)\in\Sigma$ the mapping $\tau_p : \Sigma\to\Sigma : (a,b)\mapsto(a+p_1,b+p_2)$ is an automorphism of abelian varieties. Due to these translation morphisms we know that for any curve $B\subset\Sigma$ the algebraic family of curves $|B|_a$ covers the whole of Σ, and in particular $\dim|B|_a\geq 1$. This also implies $B^2\geq 0$.

Since Σ is an abelian surface, $\mathrm{NS}(\Sigma) = \mathrm{Num}(\Sigma)$, $K_\Sigma = 0$, and the Picard number $\rho = \rho(\Sigma)\leq 4$ (cf. [LB92] 4.11.2 and Ex. 2.5). But the Néron-Severi group of Σ contains the two independent elements C_1 and C_2, so that $\rho\geq 2$. The general case[17] is indeed $\rho = 2$, however ρ might also be larger (see Example 6.8), in which case the additional generators may be chosen to be graphs of surjective morphisms from C_1 to C_2 (cf. [IS96] 3.2 Example 3). That is, $\rho(\Sigma) = 2$ if and only if C_1 and C_2 are not isogenous.

6.7 Lemma
Let $B\subset\Sigma$ be an irreducible curve, $B\not\sim_a C_k$, $k = 1,2$, and $\{i,j\} = \{1,2\}$.

(i) *If $B^2 = 0$, then B is smooth, $g(B) = 1$, and $\mathrm{pr}_{i|} : B\to C_i$ is an unramified covering of degree $B.C_j$.*

[17] The abelian surfaces with $\rho\geq 2$ possessing a principle polarisation are parametrized by a countable number of surfaces in a three-dimensional space, and the Picard number of such an abelian surface is two unless it is contained the intersection of two or three of these surfaces (cf. [IS96] 11.2). See also [GH94] p. 286 and Proposition F.1.

(ii) *If $B^2 = 0$, then $\#\big(B \cap \tau_p(C_i)\big) = B.C_j$ for any $p \in \Sigma$, and if $q, q' \in B$, then $\tau_{q-q'}(B) = B$.*

(iii) *If $B^2 = 0$, then the base curve H in the fibration $\pi : \Sigma \to H$ with fibre B, which exists according to Proposition B.1, is an elliptic curve.*

(iv) *If $B.C_i = 1$, then $B^2 = 0$ and $C_j \cong B$.*

(v) *If $B.C_i = 1 = B.C_j$, then $C_1 \cong C_2$.*

(vi) *If B is the graph of a morphism $\alpha : C_i \to C_j$, then $B.C_j = 1$ and $B^2 = 0$.*

Proof:

(i) The adjunction formula gives

$$p_a(B) = 1 + \frac{B^2 + K_\Sigma.B}{2} = 1.$$

Since $|C_2|_a$ covers the whole of Σ and $B \not\sim_a C_2$, the two irreducible curves B and C_2 must intersect properly, that is, B is not a fibre of pr_1. But then the mapping $\mathrm{pr}_{1|} : B \to C_1$ is a finite surjective morphism of degree $B.C_2$. If B was a singular curve its normalisation would have to have arithmetical genus 0 and the composition of the normalisation with $\mathrm{pr}_{1|}$ would give rise to a surjective morphism from $\mathbb{P}^1_{\mathbb{C}}$ to an elliptic curve, contradicting Hurwitz's formula. Hence, B is smooth and $g(B) = p_a(B) = 1$. We thus may apply the formula of Hurwitz to $\mathrm{pr}_{1|}$ and the degree of the ramification divisor R turns out to be

$$\deg(R) = 2\big(g(B) - 1 + (g(C_1) - 1)\deg(\mathrm{pr}_{1|})\big) = 0.$$

The remaining case is treated analogously.

(ii) W. l. o. g. $i = 2$. For $p = (p_1, p_2) \in \Sigma$ we have $\tau_p(C_2) = \mathrm{pr}_1^{-1}(p_1)$ is a fibre of pr_1, and since $\mathrm{pr}_{1|}$ is unramified, $\#\big(B \cap \tau_p(C_2)\big) = \deg(\mathrm{pr}_{1|}) = B.C_2$.
Suppose $q, q' \in B$ with $\tau_{q-q'}(B) \neq B$. Then $q = \tau_{q-q'}(q') \in B \cap \tau_{q-q'}(B)$, and hence $B^2 = B.\tau_{q-q'}(B) > 0$, which contradicts the assumption $B^2 = 0$.

(iii) Since $\chi(\Sigma) = 0$, [FM94] Lemma I.3.18 and Proposition I.3.22 imply that $g(H) = p_g(\Sigma) = h^0(\Sigma, K_\Sigma) = 1$.

(iv) W. l. o. g. $B.C_2 = 1$. Let $0 \neq p \in C_2$. We claim that $B \cap \tau_p(B) = \emptyset$, and hence $B^2 = B.\tau_p(B) = 0$.
Suppose $(a, b) \in B \cap \tau_p(B)$, then there is an $(a', b') \in B$ such that $(a, b) = \tau_p(a', b') = (a', b' + p)$, i. e. $a = a'$ and $b = b' + p$. Hence, $(0, b), (0, b') \in \tau_{-a}(B) \cap C_2$. But, $\tau_{-a}(B).C_2 = B.C_2 = 1$, and thus $b' = b = b' + p$ in contradiction to the choice of p.
$C_1 \cong B$ via $\mathrm{pr}_{1|}$ follows from (i).

(v) By (iv) we have $C_1 \cong B \cong C_2$.

(vi) $\mathrm{pr}_{i|} : B \to C_i$ is an isomorphism, and has thus degree one. But $\deg(\mathrm{pr}_{i|}) = B.C_j$. Thus we are done with (iv).

□

6.8 Example

(i) Let $C_1 = C_2 = C = \mathbb{C}/\Lambda$ with $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$, and $\Sigma = C_1 \times C_2 = C \times C$. The Picard number $\rho(\Sigma)$ is then either three or four, depending on whether the group $\mathrm{End}_0(C)$ of endomorphisms of C fixing 0 is just $\mathbb{Z}$ or larger. Using [Har77] Theorem IV.4.19 and Exercise IV.4.11 we find the following classification.

Case 1: $\exists\, d \in \mathbb{N}$ such that $\tau \in \mathbb{Q}[\sqrt{-d}]$, i. e. $\mathbb{Z} \subsetneqq \mathrm{End}_0(C)$.

Then $\rho(\Sigma) = 4$ and $\mathrm{NS}(\Sigma) = C_1\mathbb{Z} \oplus C_2\mathbb{Z} \oplus C_3\mathbb{Z} \oplus C_4\mathbb{Z}$ where C_3 is the diagonal in Σ and C_4 is the graph of the morphism $\alpha : C \to C : p \mapsto (b\tau) \cdot p$ of degree $|b\tau|^2$, where $0 \neq b \in \mathbb{N}$ minimal with $b(\tau + \overline{\tau}) \in \mathbb{Z}$ and $b\tau\overline{\tau} \in \mathbb{Z}$. Setting $a := C_3.C_4 \geq 1$, the intersection matrix is

$$(C_j.C_k)_{j,k=1,\ldots,4} = \begin{pmatrix} 0 & 1 & 1 & |b\tau|^2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & a \\ |b\tau|^2 & 1 & a & 0 \end{pmatrix}.$$

If e. g. $\tau = i$, then $C_4 = \big\{(c, ic) \mid c \in C\big\}$ and

$$(C_j.C_k)_{j,k=1,\ldots,4} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Case 2: $\nexists d \in \mathbb{N}$ such that $\tau \in \mathbb{Q}[\sqrt{-d}]$, i. e. $\mathbb{Z} = \mathrm{End}_0(C)$.

Then $\rho(\Sigma) = 3$ and $\mathrm{NS}(\Sigma) = C_1\mathbb{Z} \oplus C_2\mathbb{Z} \oplus C_3\mathbb{Z}$ where again C_3 is the diagonal in Σ. The intersection matrix in this case is

$$(C_j.C_k)_{j,k=1,2,3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

(ii) Let $C_1 = \mathbb{C}/\Lambda_1$ and $C_2 = \mathbb{C}/\Lambda_2$ with $\Lambda_1 = \mathbb{Z} \oplus \tau_1\mathbb{Z}$, $\tau_1 = i$, and $\Lambda_2 = \mathbb{Z} \oplus \tau_2\mathbb{Z}$, $\tau_2 = \frac{1}{2}i$. Then $C_1 \not\cong C_2$.
We consider the surjective morphisms $\alpha_j : C_1 \to C_2$, $j = 3, 4$, induced by multiplication with the complex numbers $\alpha_3 = 1$ and $\alpha_4 = i$ respectively. Denoting by C_j the graph of α_j, we claim, $C_1.C_3 = \deg(\alpha_3) = 2$ and $C_1.C_4 = \deg(\alpha_4) = 2$. α_j being an unramified covering, we can calculate its degree by counting the preimages of 0. If $p = [a + ib] \in \mathbb{C}/\Lambda_1 = C_1$ with $0 \leq a, b < 1$, then

$$\begin{aligned} \alpha_3(p) = 0 &\Leftrightarrow a + ib = \alpha_3 \cdot (a + ib) \in \Lambda_2 \\ &\Leftrightarrow \exists\, r, s \in \mathbb{Z} \ :\ a = r \text{ and } b = \tfrac{1}{2}s \\ &\Leftrightarrow a = 0 \text{ and } b \in \big\{0, \tfrac{1}{2}\big\}. \end{aligned}$$

and

$$\begin{aligned}\alpha_4(p)=0 &\Leftrightarrow ia-b=\alpha_4\cdot(a+ib)\in\Lambda_2\\ &\Leftrightarrow \exists\, r,s\in\mathbb{Z}\;:\; -b=r \text{ and } a=\tfrac{1}{2}s\\ &\Leftrightarrow b=0 \text{ and } a\in\left\{0,\tfrac{1}{2}\right\}.\end{aligned}$$

Moreover, the graphs C_3 and C_4 intersect only in the point $(0,0)$ and the intersection is obviously transversal, so $C_3.C_4=1$.

Thus $\Sigma=C_1\times C_2$ is an example for a product of non-isomorphic elliptic curves with $\rho(\Sigma)=4$, $\mathrm{NS}(\Sigma)=C_1\mathbb{Z}\oplus C_2\mathbb{Z}\oplus C_3\mathbb{Z}\oplus C_4\mathbb{Z}$, and intersection matrix

$$(C_j.C_k)_{j,k=1,\ldots,4}=\begin{pmatrix}0&1&2&2\\1&0&1&1\\2&1&0&1\\2&1&1&0\end{pmatrix}.$$

(iii) See [HR98] p. 4 for examples $\Sigma=C_1\times C_2$ with $\rho(\Sigma)=3$ and intersection matrix

$$\begin{pmatrix}0&1&a\\1&0&1\\a&1&0\end{pmatrix},\quad a\neq 1.$$

6.9 Remark

(i) If C_1 and C_2 are isogenous, then there are irreducible curves $B\subset\Sigma$ with $B.C_i$ arbitrarily large.
For this just note, that we have a curve $\Gamma\subset\Sigma$ which is the graph of an isogeny $\alpha: C_1\to C_2$. Denoting by $n_{C_2}:C_2\to C_2$ the morphism induced by the multiplication with $n\in\mathbb{N}$, we have a morphism $n_{C_2}\circ\alpha$ whose degree is just $n^2\deg(\alpha)$. But the degree is the intersection number of the graph with C_1. The dual morphism of $n_{C_2}\circ\alpha$ has the the same degree, which then is the intersection multiplicity of its graph with C_2. (cf. [Har77] Ex. IV.4.7)

(ii) If C_1 and C_2 are isogenous, then Σ might very well contain smooth irreducible elliptic curves B which are neither isomorphic to C_1 nor to C_2, and hence cannot be the graph of an isogeny between C_1 and C_2. But being an elliptic curve we have $B^2=0$ by the adjunction formula. If now $\mathrm{NS}(\Sigma)=\bigoplus_{i=1}^{\rho(\Sigma)}C_i\mathbb{Z}$, where the additional generators are graphs, then $B\sim_a\sum_{i=1}^{\rho(\Sigma)}n_iC_i$ with some $n_i<0$. (cf. [LB92] Ex. 10.6)

Throughout the remaining part of the subsection we will restrict our attention to the general case, that is that C_1 and C_2 are *not* isogenous. This makes the formulae look much nicer, since then $\mathrm{NS}(\Sigma)=C_1\mathbb{Z}\oplus C_2\mathbb{Z}$.

6.10 Lemma

Let C_1 and C_2 be non-isogenous elliptic curves, $D\in\mathrm{Div}(\Sigma)$ with $D\sim_a aC_1+bC_2$.

(i) $D^2=0$ *if and only if* $a=0$ *or* $b=0$.

(ii) *If D is an irreducible curve, then we are in one of the following cases:*
  (1) $a=0$ *and* $b=1$,

(2) *$a = 1$ and $b = 0$,*

(3) *$a, b > 0$,*

and if we are in one of these cases, then there is an irreducible curve algebraically equivalent to D.

(iii) *If D is an irreducible curve and $D^2 = 0$, then either $D \sim_a C_1$ or $D \sim_a C_2$.*

(iv) *D is nef if and only if $a, b \geq 0$.*

(v) *D is ample if and only if $a, b > 0$.*

(vi) *D is very ample if and only if $a, b \geq 3$.*

Proof:

(i) $0 = D^2 = 2ab$ if and only if $a = 0$ or $b = 0$.

(ii) Let us first consider the case that D is irreducible.
If $a = 0$ or $b = 0$, then D is algebraically equivalent to a multiple of a fibre of one of the projections pr_i, $i = 1, 2$. In this situation $D^2 = 0$ and thus the irreducible curve D does not intersect any of the fibres properly. Hence it must be a union of several fibres, and being irreducible it must be a fibre. That is we are in one of the first two cases.
Suppose now that $a, b \neq 0$. Thus D intersects C_i properly, and $0 < D.C_1 = b$ and $0 < D.C_2 = a$.
It now remains to show that the mentioned algebraic systems contain irreducible curves, which is clear for the first two of them. Let therefore a and b be positive. Then obviously the linear system $|aC_1 + bC_2|_l$ contains no fixed component, and being ample by (v) its general element is irreducible according to [LB92] Theorem 4.3.5.

(iii) Follows from (i) and (ii).

(iv) By definition D is nef if and only if $D.D' \geq 0$ for every irreducible curve $D' \subset \Sigma$. Thus the claim is an immediate consequence of (ii).

(v) Since by the Nakai-Moishezon-Criterion ampleness depends only on the numerical class of a divisor, we may assume that $D = aC_1 + bC_2$. Moreover, by [LB92] Proposition 4.5.2 D is ample if and only if $D^2 > 0$ and $|D|_l \neq \emptyset$.
If $a, b > 0$, then $D^2 = 2ab > 0$ and the effective divisor $aC_1 + bC_2 \in |D|_l$, thus D is ample. Conversely, if D is ample, then $0 < D^2 = 2ab$ and $0 < D.C_1 = b$, thus $a, b > 0$.

(vi) By [LB92] Corollary 4.5.3 and (v) $L = 3C_1 + 3C_2$ is very ample. If $a, b \geq 3$, then the system $|(a-3)C_1 + (b-3)C_2|_l$ is basepoint free, which is an immediate consequence of the existence of the translation morphisms τ_p, $p \in \Sigma$. But then $L' = (a-3)C_1 + (b-3)C_2$ is globally generated and $D = L + L'$ is very ample. Conversely, if $a < 3$, then $D \cap C_2$ is a divisor of degree $D.C_2 = a < 3$ on the elliptic curve C_2 and hence not very ample (cf. [Har77] Example IV.3.3.3). But then D is not very ample. Analogously if $b < 3$.

$\square$

In view of (2.5d) and Lemma 6.10 (iv) the Condition (2.6) becomes obsolete, and Corollary 2.2 has the following form, taking Lemma 6.10 (iii) and $K_\Sigma = 0$ into account.

2.2d Corollary
Let C_1 and C_2 be two non-isogenous elliptic curves, $a, b \in \mathbb{Z}$ be integers satisfying

(2.4d) $\quad ab \geq \sum_{i=1}^r (m_i+1)^2$, *and*

(2.5d) $\quad a, b > \max\{m_i \mid i = 1, \ldots, r\}$,

then for $z_1, \ldots, z_r \in \Sigma = C_1 \times C_2$ in very general position and $\nu > 0$

$$H^\nu\left(\operatorname{Bl}_{\underline{z}}(\Sigma),\ a\pi^*C_1 + b\pi^*C_2 - \sum_{i=1}^r m_iE_i\right) = 0.$$

As for the existence theorem Corollary 4.2 we work with the very ample divisor class $L = 3C_1 + 3C_2$, and we claim that the Conditions (4.9), (4.10) and (4.11) become obsolete, while, in view of Lemma 6.10 (iii), (4.7) and (4.8) take the form

(4.7d) $\quad (a-3)(b-3) \geq \sum_{i=1}^r (m_i+1)^2$, and

(4.8d) $\quad (a-3), (b-3) > \max\{m_i \mid i = 1, \ldots, r\}$.

That is, under these hypotheses there is an irreducible curve in $|D|_l$, for any $D \sim_a aC_1 + bC_2$, with precisely r ordinary singular points of multiplicities $m_1, \ldots, m_r$.

(4.9) becomes redundant in view of (4.8d) and Lemma 6.10 (iv), while (4.11) is fulfilled in view of (4.12) and $K_\Sigma = 0$. It remains to show that $D.L - g(L) \geq m_i + m_j$ for all i, j. However, by the adjunction formula $g(L) = 1 + \frac{1}{2}L^2 = 10$, and by (4.8d) $D.L - g(L) > 3(a-3+b-3) > 3(m_i+m_j) \geq m_i + m_j$. Thus the claim is proved.

From these considerations we at once deduce the conditions for the existence of an irreducible curve in $|D|_l$, $D \sim_a aC_1 + bC_2$, with prescribed singularities of arbitrary type, i. e. the conditions in Corollary 5.4. They come down to

(5.3d) $\quad (a-3)(b-3) \geq \frac{207}{5} \sum_{\mu(\mathcal{S}_i) \leq 38} \mu(\mathcal{S}_i) + 29 \sum_{\mu(\mathcal{S}_i) \geq 39} \left(\sqrt{\mu(\mathcal{S}_i)} + \frac{13}{2\sqrt{29}}\right)^2$, and

(5.4d) $\quad (a-3), (b-3) > \begin{cases} \sqrt{\frac{207}{5}}\sqrt{\mu(\mathcal{S}_1)} - 1, & \text{if } \mu(\mathcal{S}_1) \leq 38, \\ \sqrt{29}\sqrt{\mu(\mathcal{S}_1)} + \frac{11}{2}, & \text{if } \mu(\mathcal{S}_1) \geq 39. \end{cases}$

6.e. **Surfaces in $\mathbb{P}^3_{\mathbb{C}}$.** A smooth projective surface Σ in $\mathbb{P}^3_{\mathbb{C}}$ is given by a single equation $f = 0$ with $f \in \mathbb{C}[w, x, y, z]$ homogeneous, and by definition the degree of Σ, say n, is just the degree of f. For $n = 1$, $\Sigma \cong \mathbb{P}^2_{\mathbb{C}}$, for $n = 2$, $\Sigma \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, and for $n = 3$, Σ is isomorphic to $\mathbb{P}^2_{\mathbb{C}}$ blown up in six points in general position. Thus the Picard number $\rho(\Sigma)$, i. e. the rank of the Néron-Severi group, in these cases is 1, 2, and 7 respectively. Note that these are also precisely the cases where Σ is rational.

In general the Picard number $\rho(\Sigma)$ of a surface in $\mathbb{P}^3_{\mathbb{C}}$ may be arbitrarily large,[18] but the Néron-Severi group always contains a very special member, namely the class $H \in \mathrm{NS}(\Sigma)$ of a hyperplane section with $H^2 = n$. And the class of the canonical divisor is then just $(n-4)H$. Moreover, if the degree of Σ is at least four, that is, if Σ is not rational, then it is likely that $\mathrm{NS}(\Sigma) = H\mathbb{Z}$. More precisely, if $n \geq 4$, Noether's Theorem says that $\{\Sigma \mid \rho(\Sigma) = 1, \deg(\Sigma) = n\}$ is a very general subset of the projective space of projective surfaces in $\mathbb{P}^3_{\mathbb{C}}$ of fixed degree n, i. e. it's complement is an at most countable union of lower dimensional subvarieties. (cf. [Har75] Corollary 3.5 or [IS96] p. 146)

Since we consider the case of rational surfaces separately the following considerations thus give a full answer for the "general case" of a surface in $\mathbb{P}^3_{\mathbb{C}}$.

2.2e Corollary

Let $\Sigma \subset \mathbb{P}^3_{\mathbb{C}}$ be a surface in $\mathbb{P}^3_{\mathbb{C}}$ of degree n, $H \in \mathrm{NS}(\Sigma)$ be the algebraic class of a hyperplane section, and d an integer satisfying

(2.4e) $\quad n(d-n+4)^2 \geq 2\sum_{i=1}^r (m_i+1)^2$, *and*

(2.5e) $\quad (d-n+4)\cdot H.B > \max\{m_i \mid i = 1,\ldots,r\}$ *for any irreducible curve B with $B^2 = 0$ and* $\dim |B|_a \geq 1$, *and*

(2.6e) $\quad d \geq n-4$,

then for $z_1,\ldots,z_r \in \Sigma$ in very general position and $\nu > 0$

$$H^\nu\left(\mathrm{Bl}_{\underline{z}}(\Sigma), d\pi^*H - \sum_{i=1}^r m_i E_i\right) = 0.$$

6.13 Remark

(i) If $\mathrm{NS}(\Sigma) = H\mathbb{Z}$, then (2.5e) is redundant, since there are no irreducible curves B with $B^2 = 0$. Otherwise we would have $B \sim_a kH$ for some $k \in \mathbb{Z}$ and $k^2 n = B^2 = 0$ would imply $k = 0$, but then $H.B = 0$ in contradiction to H being ample.

(ii) However, a quadric in $\mathbb{P}^3_{\mathbb{C}}$ or the K3-surface given by $w^4 + x^4 + y^4 + z^4 = 0$ contain irreducible curves of self-intersection zero.

(iii) If $\sum_{i=1}^r (m_i+1)^2 > \frac{n}{2} m_i^2$ for all $i = 1,\ldots,r$ then again (2.5e) becomes obsolete in view of (2.4e), since $H.B > 0$ anyway. The above inequality is, for instance, fulfilled if the highest multiplicity occurs at least $\frac{n}{2}$ times.

(iv) In the existence theorems the condition depending on curves of self-intersection will vanish in any case.

As for Corollary 4.2 we claim that if $\mathrm{NS}(\Sigma) = H\mathbb{Z}$, then

[18] E. g. the n-th Fermat surface, given by $w^n + x^n + y^n + z^n = 0$ has Picard number $\rho \geq 3(n-1)(n-2)+1$, with equality if $\gcd(n,6) = 1$. (cf. [Shi82] Theorem 7, see also [AS83] pp. 1f. and [IS96] p. 146)

(4.7e) $\qquad n(d-n+3)^2 \geq 2\sum_{i=1}^r (m_i+1)^2,$

ensures the existence of an irreducible curve $C \sim_a dH$ with precisely r ordinary singular points of multiplicities $m_1,\ldots,m_r$ and $h^1\big(\Sigma, \mathcal{J}_{X(\underline{m};\underline{z})/\Sigma}(dH)\big) = 0$.

The role of the very ample divisor L is filled by a hyperplane section, and thus $g(L) = 1 + \frac{L^2+L.K_\Sigma}{2} = \binom{n-1}{2}$. Therefore, (4.7e) obviously implies (4.7), and (4.10) takes the form

$$n \cdot (d-n+3) > m_i + 2 \text{ for all } i = 1,\ldots,r. \tag{6.11}$$

However, from (4.7e) we deduce for any $i \in \{1,\ldots,r\}$

$$n \cdot (d-n+3) \geq \sqrt{n} \cdot \sqrt{2} \cdot (m_i+1) \geq m_i + 2,$$

unless $n = r = m_1 = 1$, in which case we are done by the assumption $d \geq 3$. Thus (4.10) is redundant.

Moreover, there are no curves of self-intersection zero on Σ, and it thus remains to verify (4.9), which in this situation takes the form

$$d \geq n-3,$$

and follows at once from (6.11).

With the aid of this result the conditions of Corollary 5.4 for the existence of an irreducible curve $C \sim_a dH$ with prescribed singularities $\mathcal{S}_i$ in this situation therefore reduce to

(5.3e) $\qquad n(d-n+3)^2 \geq \frac{414}{5} \sum\limits_{\mu(\mathcal{S}_i)\leq 38} \mu(\mathcal{S}_i) + 58 \sum\limits_{\mu(\mathcal{S}_i)\geq 39} \left(\sqrt{\mu(\mathcal{S}_i)} + \frac{13}{2\sqrt{29}}\right)^2$, and

6.f. **K3-Surfaces.** We note that if Σ is a K3-surface then the Néron-Severi group $\mathrm{NS}(\Sigma)$ and the Picard group $\mathrm{Pic}(\Sigma)$ of Σ coincide, i. e. $|D|_a = |D|_l$ for every divisor D on Σ. Moreover, an irreducible curve B has self-intersection $B^2 = 0$ if and only if the arithmetical genus of B is one. In that case $|B|_l$ is a pencil of elliptic curves without base points endowing Σ with the structure of an elliptic fibration over $\mathbb{P}^1_{\mathbb{C}}$. (cf. [Mér85] or Proposition B.1) We, therefore, distinguish two cases.

6.f.i. *Generic K3-Surfaces.* Since a generic K3-surface does not possess an elliptic fibration the following version of Corollary 2.2 applies for generic K3-surfaces. (cf. [FM94] I.1.3.7)

2.2f.i Corollary

Let Σ be a K3-surface which is not elliptic, and let D a divisor on Σ satisfying

(2.4f) $\qquad D^2 \geq 2\sum_{i=1}^r (m_i+1)^2$, *and*

(2.6f) $\qquad D$ *nef,*

then for $z_1,\ldots,z_r \in \Sigma$ in very general position and $\nu > 0$

$$H^\nu\left(\mathrm{Bl}_{\underline{z}}(\Sigma), \pi^*D - \sum_{i=1}^r m_i E_i\right) = 0.$$

In view of equation (4.12) the conditions in Corollary 4.2 reduce to

(4.7f) $(D-L)^2 \geq 2\sum_{i=1}^r (m_i+1)^2$,

(4.9f) $D-L$ nef, and

(4.10f) $D.L - 2g(L) \geq m_i + m_j$ for all i, j,

and, analogously, the conditions in Corollary 5.4 reduce to (5.6),

(5.3f) $(D-L)^2 \geq \frac{414}{5} \sum_{\mu(\mathcal{S}_i)\leq 38} \mu(\mathcal{S}_i) + 58 \sum_{\mu(\mathcal{S}_i)\geq 39} \left(\sqrt{\mu(\mathcal{S}_i)} + \frac{13}{2\sqrt{29}}\right)^2$, and

(5.5f) $D-L$ nef.

6.f.ii. *K3-Surfaces with an Elliptic Structure.* The hypersurface in $\mathbb{P}^3_{\mathbb{C}}$ given by the equation $x^4+y^4+z^4+u^4=0$ is an example of a K3-surface which is endowed with an elliptic fibration. Among the elliptic K3-surfaces the general one will possess a unique elliptic fibration while there are examples with infinitely many different such fibrations. (cf. [FM94] I.1.3.7)

2.2f.ii Corollary

Let Σ be a K3-surface which possesses an elliptic fibration, and let D be a divisor on Σ satisfying

(2.4f) $D^2 \geq 2\sum_{i=1}^r (m_i+1)^2$,

(2.5f) $D.B > \max\{m_i \mid i=1,\dots,r\}$ *for any irreducible curve B with $B^2 = 0$, and*

(2.6f) D *nef,*

then for $z_1,\dots,z_r \in \Sigma$ in very general position and $\nu > 0$

$$H^\nu\left(\mathrm{Bl}_{\underline{z}}(\Sigma), \pi^*D - \sum_{i=1}^r m_i E_i\right) = 0.$$

6.16 Remark

If Σ is generic among the elliptic K3-surfaces, i. e. admits exactly one elliptic fibration, then Condition (2.5f) means that a curve in $|D|_l$ meets a general fibre in at least $k = \max\{m_i \mid i=1,\dots,r\}$ distinct points.

The conditions in Corollary 4.2 then reduce to (4.7f), (4.9f), (4.10f), and

(4.8f) $(D-L).B > \max\{m_i \mid i=1,\dots,r\}$ for any curve B with $B^2=0$.

Similarly, the conditions in Corollary 5.4 reduce to (5.3f), (5.5f), (5.6), and

(5.4f) $(D-L).B > \begin{cases} \sqrt{\frac{207}{5}}\sqrt{\mu(\mathcal{S}_1)} - 1, & \text{if } \mu(\mathcal{S}_1) \leq 38, \\ \sqrt{29}\sqrt{\mu(\mathcal{S}_1)} + \frac{11}{2}, & \text{if } \mu(\mathcal{S}_1) \geq 39, \end{cases}$

for any irreducible curve B with $B^2 = 0$.

Appendix A. Very General Position

It is our first aim to show that if there is a curve passing through points $z_1, \ldots, z_r \in \Sigma$ in very general position with multiplicities $n_1, \ldots, n_r$ then it can be equimultiply deformed in its algebraic system in a good way - i. e. suitable for Lemma 3.3.

A.1 Lemma
Let $B \subset \Sigma$ be a curve, and $\underline{n} \in \mathbb{N}_0^r$. Then

$$V_{B,\underline{n}} = \left\{\underline{z} \in \Sigma^r \;\middle|\; \exists\, C \in |B|_a \;:\; \operatorname{mult}_{z_i}(C) \geq n_i \;\forall i = 1, \ldots, r\right\}$$

is a closed subset of Σ^r.

Proof:

Step 1: Show first that for $n \in \mathbb{N}_0$

$$X_{B,n} := \left\{(C, z) \in H \times \Sigma \;\middle|\; \operatorname{mult}_z(C) \geq n\right\}$$

is a closed subset of $H \times \Sigma$, where $H := |B|_a$.

Being the reduction of a connected component of the Hilbert scheme $\operatorname{Hilb}_\Sigma$, H is a projective variety endowed with a universal family of curves, giving rise to the following diagram of morphisms

$$\begin{tikzcd}
\mathcal{C} = \bigcup_{C \in H} \{C\} \times C \arrow[r, hook] \arrow[dr] & H \times \Sigma \arrow[r, two heads, "\mathrm{pr}_\Sigma"'] \arrow[d, two heads, "\mathrm{pr}_H"] & \Sigma \\
& H, &
\end{tikzcd}$$

where $\mathcal{C}$ is an effective Cartier divisor on $H \times \Sigma$ with $\mathcal{C}_{|\{C\} \times \Sigma} = C$.
Let $s \in H^0\big(H \times \Sigma, \mathcal{O}_{H \times \Sigma}(\mathcal{C})\big)$ be a global section defining $\mathcal{C}$. Then

$$X_{B,n} = \left\{\eta = (C, z) \in H \times \Sigma \;\middle|\; s_\eta \in (\mathfrak{m}_{\Sigma,z}^n + \mathfrak{m}_{H,C}) \cdot \mathcal{O}_{H \times \Sigma,\eta}\right\}.$$

We may consider a finite open affine covering of $H \times \Sigma$ of the form $\{H_i \times U_j \mid i \in I, j \in J\}$, $H_i \subset H$ and $U_j \subset \Sigma$ open, such that $\mathcal{C}$ is locally on $H_i \times U_j$ given by one polynomial equation, say

$$s_{i,j}(\underline{a}, \underline{b}) = 0, \text{ for } \underline{a} \in H_i,\ \underline{b} \in U_j.$$

It suffices to show that $X_{B,n} \cap (H_i \times U_j)$ is closed in $H_i \times U_j$ for all i, j.
However, for $\eta = (C, z) = (\underline{a}, \underline{b}) \in H_i \times U_j$ we have

$$s_\eta \in \left(\mathfrak{m}_{\Sigma,z}^n + \mathfrak{m}_{H,C}\right) \cdot \mathcal{O}_{H \times \Sigma,\eta}$$

if and only if

$$s_{i,j}(\underline{a}, \underline{b}) = 0 \text{ and}$$

$$\frac{\partial^\alpha s_{i,j}}{\partial \underline{b}^\alpha}(\underline{a}, \underline{b}) = 0, \text{ for all } |\alpha| \leq n - 1.$$

Thus,

$$X_{B,n} \cap (H_i \times U_j) = \left\{ (\underline{a}, \underline{b}) \in H_i \times U_j \;\middle|\; s_{i,j}(\underline{a}, \underline{b}) = 0 = \frac{\partial^\alpha s_{i,j}}{\partial \underline{b}^\alpha}(\underline{a}, \underline{b}),\ \forall\, |\alpha| \leq n-1 \right\}$$

is a closed subvariety of $H_i \times U_j$.

Step 2: $V_{B,\underline{n}}$ is a closed subset of Σ^r.

By Step 1 for $i = 1, \ldots, r$ the set

$$X_{B,\underline{n},i} := \left\{ (\underline{z}, C) \in \Sigma^r \times H \;\middle|\; \operatorname{mult}_{z_i}(C) \geq n_i \right\} \cong \Sigma^{r-1} \times X_{B,n_i}$$

is a closed subset of $\Sigma^r \times H$. Considering now

$$\begin{tikzcd} X_{B,\underline{n}} := \bigcap_{i=1}^{r} X_{B,\underline{n},i} \arrow[r, hook] \arrow[dr, "\rho"'] & \Sigma^r \times H \arrow[d, two heads] \\ & \Sigma^r, \end{tikzcd}$$

we find that $V_{B,\underline{n}} = \rho(X_{B,\underline{n}})$, being the image of a closed subset under a morphism between projective varieties, is a closed subset of Σ^r (cf. [Har77] Ex. II.4.4). $\square$

A.2 Corollary

Then the complement of the set

$$V = \bigcup_{B \in \operatorname{Hilb}_\Sigma} \bigcup_{\underline{n} \in \mathbb{N}_0^r} \left\{ V_{B,\underline{n}} \mid V_{B,\underline{n}} \neq \Sigma^r \right\}$$

is very general, where $\operatorname{Hilb}_\Sigma$ *is the Hilbert scheme of curves on* Σ.

In particular, there is a very general subset $U \subseteq \Sigma^r$ *such that if for some* $\underline{z} \in U$ *there is a curve* $B \subset \Sigma$ *with* $\operatorname{mult}_{z_i}(B) = n_i$ *for* $i = 1, \ldots, r$*, then for any* $\underline{z}' \in U$ *there is a curve* $B' \in |B|_a$ *with* $\operatorname{mult}_{z_i'}(B') \geq n_i$.

Proof: Fixing some embedding $\Sigma \subseteq \mathbb{P}^n_{\mathbb{C}}$ and $h \in \mathbb{Q}[x]$, $\operatorname{Hilb}^h_\Sigma$ is a projective variety and has thus only finitely many connected components. Thus the Hilbert scheme $\operatorname{Hilb}_\Sigma$ has only a countable number of connected components, and we have only a countable number of different $V_{B,\underline{n}}$, where B runs through $\operatorname{Hilb}_\Sigma$ and $\underline{n}$ through $\mathbb{N}^r$. By Lemma A.1 the sets $V_{B,\underline{n}}$ are closed, hence their complements $\Sigma^r \setminus V_{B,\underline{n}}$ are open. But then

$$U = \Sigma^r \setminus V = \bigcap_{B \in \operatorname{Hilb}_\Sigma} \bigcap_{\underline{n} \in \mathbb{N}_0^r} \left\{ \Sigma^r \setminus V_{B,\underline{n}} \mid V_{B,\underline{n}} \neq \Sigma^r \right\}$$

is an at most countable intersection of open dense subsets of Σ^r, and is hence very general. $\square$

In the proof of Theorem 2.1 we use at some place the result of Corollary A.3. We could instead use Corollary A.2. However, since the results are quite nice and simple to prove we just give them.

A.3 Corollary

(i) *The number of curves B in Σ with* $\dim |B|_a = 0$ *is at most countable.*

(ii) *The number of exceptional curves in Σ (i. e. curves with negative self intersection) is at most countable.*

(iii) *There is a very general subset U of Σ^r, $r \geq 1$, such that for $\underline{z} \in U$ no z_i belongs to a curve $B \subset \Sigma$ with* $\dim |B|_a = 0$, *in particular to no exceptional curve.*

Proof:

(i) By definition $|B|_a$ is a connected component of Hilb_Σ, whose number is at most countable. If in addition $\dim |B|_a = 0$, then $|B|_a = \{B\}$ which proves the claim.

(ii) Curves of negative self-intersection are not algebraically equivalent to any other curve (cf. [IS96] p. 153).

(iii) Follows from (i).

□

A.4 Example (Kodaira)
Let $z_1, \ldots, z_9 \in \mathbb{P}^2_{\mathbb{C}}$ be in very general position[19] and let $\Sigma = \mathrm{Bl}_{\underline{z}}\left(\mathbb{P}^2_{\mathbb{C}}\right)$ be the blow up of $\mathbb{P}^2_{\mathbb{C}}$ in $\underline{z} = (z_1, \ldots, z_9)$. Then Σ contains infinitely many irreducible smooth rational -1-curves, i. e. exceptional curves of the first kind.

Proof: It suffices to find an infinite number of irreducible curves C in $\mathbb{P}^2_{\mathbb{C}}$ such that

$$d^2 - \sum_{i=1}^{9} m_i^2 = -1, \tag{A.1}$$

and

$$p_a(C) - \sum_{i=1}^{9} \frac{m_i(m_i-1)}{2} = 0, \tag{A.2}$$

where $m_i = \mathrm{mult}_{z_i}(C)$ and $d = \deg(C)$, since the expression in (A.1) is the self intersection of the strict transform $\widetilde{C} = \mathrm{Bl}_{\underline{z}}(C)$ of C and (A.2) gives its arithmetical genus. In particular $\widetilde{C}$ cannot contain any singularities, since they would contribute to the arithmetical genus, and, being irreducible anyway, $\widetilde{C}$ is an exceptional curve of the first kind.

We are going to deduce the existence of these curves with the aid of quadratic Cremona transformations.

Claim: If for some $d > 0$ and $m_1, \ldots, m_9 \geq 0$ with $3d - \sum_{i=1}^{9} m_i = 1$ there is an irreducible curve $C \in \left|\mathcal{J}_{X(\underline{m};\underline{z})}(d)\right|_l$, then $T(C) \in \left|\mathcal{J}_{X(\underline{m}';\underline{z}')}(d+a)\right|_l$ is an irreducible curve, where

[19] To be precise, no three of the nine points should be collinear, and after any finite number of quadratic Cremona transformations centred at the z_i (respectively the newly obtained centres) still no three should be collinear. Thus the admissible tuples in $(\mathbb{P}^2_{\mathbb{C}})^9$ form a very general set, cf. [Har77] Ex. V.4.15.

- $\{i,j,k\} \subset \{1,\ldots,9\}$ are such that $m_i+m_j+m_k<d$,
- $T : \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ is the quadratic Cremona transformation at z_i, z_j, z_k,
- $z'_\nu = \begin{cases} z_\nu, & \text{if } \nu \neq i,j,k, \\ T(\overline{z_\lambda z_\mu}), & \text{if } \{\nu,\lambda,\mu\} = \{i,j,k\}, \end{cases}$
- $m'_\nu = \begin{cases} m_\nu, & \text{if } \nu \neq i,j,k, \\ m_\nu + a, & \text{else, and} \end{cases}$
- $a = d-(m_i+m_j+m_k)$.

Note that, $3(d+a) - \sum_{i=1}^9 m'_i = 1$, i. e. we may iterate the process since the hypothesis of the claim will be preserved.

Since $3d > \sum_{i=1}^9 m_i$, there must be a triple (i,j,k) such that $d > m_i+m_j+m_k$. Let us now consider the following diagram

$$\begin{tikzcd}[column sep=small] & \Sigma = \mathrm{Bl}_{z_i,z_j,z_k}(\mathbb{P}^2_{\mathbb{C}}) = \mathrm{Bl}_{z'_i,z'_j,z'_k}(\mathbb{P}^2_{\mathbb{C}}) \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ \mathbb{P}^2_{\mathbb{C}} \arrow[rr, dashed, "T"] & & \mathbb{P}^2_{\mathbb{C}}, \end{tikzcd}$$

and let us denote the exceptional divisors of π by E_i and those of π' by E'_i. Moreover, let $\widetilde{C} = \mathrm{Bl}_{z_i,z_j,z_k}(C)$ be the strict transform of C under π and let $\widetilde{T(C)} = \mathrm{Bl}_{z'_i,z'_j,z'_k}\big(T(C)\big)$ be the strict transform of $T(C)$ under π'. Then of course $\widetilde{C} = \widetilde{T(C)}$, and $T(C)$, being the projection $\pi'\big(\widetilde{C}\big)$ of the strict transform $\widetilde{C}$ of the irreducible curve C, is of course an irreducible curve. Note that the condition $d > m_i+m_j+m_k$ ensures that $\widetilde{C}$ is not one of the curves which are contracted. It thus suffices to verify

$$\deg\big(T(C)\big) = d+a,$$

and

$$m'_i = \mathrm{mult}_{z'_i}\big(T(C)\big) = \begin{cases} m_\nu, & \text{if } \nu \neq i,j,k, \\ m_\nu + a, & \text{else.} \end{cases}$$

Since outside the lines $\overline{z_iz_j}$, $\overline{z_iz_k}$, and $\overline{z_kz_j}$ the transformation T is an isomorphism and since by hypothesis none of the remaining z_ν belongs to one of these lines we clearly have $m'_\nu = m_\nu$ for $\nu \neq i,j,k$. Moreover, we have

$$\begin{aligned} m'_i &= \widetilde{T(C)}.E'_i = \widetilde{C}.\,\mathrm{Bl}_{z_i,z_j,z_k}(\overline{z_jz_k}) \\ &= \big(\pi^*C - \textstyle\sum m_lE_l\big).\big(\pi^*\overline{z_jz_k} - E_j - E_k\big) \\ &= C.\overline{z_jz_k} - m_j - m_k = d - m_j - m_k = m_i + a. \end{aligned}$$

Analogously for m'_j and m'_k.

Finally we find

$$\begin{aligned} \deg\big(T(C)\big) &= T(C).\overline{z'_iz'_j} = \pi'^*T(C).\pi'^*\overline{z'_iz'_j} \\ &= \big(\widetilde{T(C)} + \textstyle\sum m'_\nu E'_\nu\big).\big(E_k + E'_i + E'_j\big) \\ &= \widetilde{C}.E_k + \textstyle\sum m'_\nu E'_\nu.E_k \\ &= m_k + m'_i + m'_j = d+a. \end{aligned}$$

This proves the claim.

Let us now show by induction that for any $d > 0$ there is an irreducible curve C of degree $d' \geq d$ satisfying (A.1) and (A.2). For $d = 1$ the line $C = \overline{z_1 z_2}$ through z_1 and z_2 gives the induction start. Given some suitable curve of degree $d' \geq d$ the above claim then ensures that through points in very general position there is an irreducible curve of higher degree satisfying (A.1) and (A.2), since $a = d - (m_1 + m_2 + m_3) > 0$. Thus the induction step is done. $\square$

The example shows that a smooth projective surface Σ may indeed carry an infinite number of exceptional curves - even of the first kind. According to Nagata ([Nag60] Theorem 4a, p. 283) the example is due to Kodaira. For further references on the example see [Har77] Ex. V.4.15, [BS95] Example 4.2.7, or [Fra41]. [IS96] p. 198 Example 3 shows that also $\mathbb{P}^2_{\mathbb{C}}$ blown up in the nine intersection points of two plane cubics carries infinitely many exceptional curves of the first kind.

Appendix B. Condition (2.5)

B.1 Proposition
Suppose that $B \subset \Sigma$ is an irreducible curve with $B^2 = 0$ and $\dim |B|_a \geq 1$, then

(B.1) *$|B|_a$ is an irreducible projective curve, and*

(B.2) *there is a fibration $f : \Sigma \to H$ whose fibres are just the elements of $|B|_a$, and H is the normalisation of $|B|_a$.*

We are proving the proposition in several steps.

B.2 Proposition
Let $f : Y' \to Y$ be a finite flat morphism of noetherian schemes with Y irreducible such that for some point $y_0 \in Y$ the fibre $Y'_{y_0} = f^{-1}(y_0) = Y' \times_Y \operatorname{Spec}\big(k(y_0)\big)$ is a single reduced point.

Then the structure map $f^\# : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_{Y'}$ is an isomorphism, and hence so is f.

Proof: Since there is at least one connected reduced fibre Y'_{y_0}, semicontinuity of flat, proper morphisms in the version [GD67] IV.12.2.4 (vi) implies that there is an open dense subset $U \subseteq Y$ such that Y'_y is connected and reduced, hence a single reduced point, $\forall\, y \in U$. (U dense in Y is due to the fact that Y is irreducible.)

Thus the assumptions are stable under restriction to open subschemes of Y, and since the claim that we have to show is local on Y, we may assume that $Y = \operatorname{Spec}(A)$ is affine. Moreover, f being finite, thus affine, we have $Y' = \operatorname{Spec}(B)$ is also affine.

Since f is flat it is open and hence dominates the irreducible affine variety Y and, therefore, induces an inclusion of rings $A \hookrightarrow B$. It now suffices to show:

Claim: $A \hookrightarrow B$ is an isomorphism.

By assumption there exists a $y = P \in \operatorname{Spec}(A) = Y$ such that $Y'_y = f^{-1}(y) = \operatorname{Spec}(B_P/PB_P)$ is a single point with reduced structure. In particular we have for the multiplicity of $Y'_y = \mathrm{Spec}(B_P/PB_P)$ over $\{y\} = \mathrm{Spec}(A_P/PA_P)$

$$1 = \mu(Y'_y) = \mathrm{length}_{A_P/PA_P}(B_P/PB_P),$$

which implies that

$$A_P/PA_P \hookrightarrow B_P/PB_P$$

is an isomorphism. Hence by Nakayama's Lemma also

$$A_P \hookrightarrow B_P$$

is an isomorphism, that is, B_P is free of rank 1 over A_P. B being locally free over A, with $A/\sqrt{0}$ an integral domain, thus fulfils

$$A_Q \hookrightarrow B_Q$$

is an isomorphism for all $Q \in \mathrm{Spec}(A)$, and hence the claim follows. □

B.3 Proposition (Principle of Connectedness)
Let X and Y be noetherian schemes, Y connected, and let $\pi : X \to Y$ be a flat projective morphism such that for some $y_0 \in Y$ the fibre $X_{y_0} = \pi^{-1}(y_0)$ is reduced and connected.

Then for all $y \in Y$ the fibre $X_y = \pi^{-1}(y)$ is connected.

Proof: Considering points in the intersections of the finite number of irreducible components of Y we can reduce to the case Y irreducible.

Stein Factorisation (cf. [GD67] III.4.3.3) gives a factorisation of π of the form

$$\pi : X \xrightarrow{\pi'} Y' = \mathrm{Spec}(\pi_*\mathcal{O}_X) \xrightarrow{f} Y,$$

with

(1) π' connected (i. e. its fibres are connected),

(2) f finite,

(3) $f_*\mathcal{O}_{Y'} = \pi_*\mathcal{O}_X$ locally free over $\mathcal{O}_Y$, since π is flat, and

(4) $Y'_{y_0} = f^{-1}(y_0)$ is connected and reduced, i. e. a single reduced point.

Because of (1) it suffices to show that f is connected, and we claim that they are reduced as well. Since f is finite (3) is equivalent to saying that f is flat. Hence f fulfils the assumptions of Proposition B.2, and we conclude that $\mathcal{O}_Y = f_*\mathcal{O}_{Y'}$ and the proposition follows from [Har77] III.11.3.

Alternatively, from [GD67] IV.15.5.9 (ii) it follows that there is an open dense subset $U \subseteq Y$ such that X_y is connected for all $y \in U$. Since, moreover, by the same theorem the number of connected components of the fibres is a lower semi-continuous function on Y the special fibres cannot have more connected components than the general ones, that is, all fibres are connected. □

B.4 Lemma
Under the hypotheses of Proposition B.1 let $C \in |B|_a$ then C is connected.

Proof: Consider the universal family

$$\begin{tikzcd} |B|_a \times \Sigma \arrow[dr, "\mathrm{pr}_{|B|_a}"'] & \bigcup_{C\in|B|_a} \{C\} \times C =: S \arrow[l, hook'] \arrow[d, "\pi \text{ flat}"] \\ & |B|_a \end{tikzcd} \tag{B.3}$$

over the connected projective scheme $|B|_a \subseteq \mathrm{Hilb}_\Sigma$.

Since the projection π is a flat projective morphism, and since the fibre $\pi^{-1}(B) = \{B\} \times B$ is connected and reduced, the result follows from Proposition B.3. □

B.5 Lemma
Under the hypotheses of Proposition B.1 let $C \in |B|_a$ with $B \subseteq C$, then $C = B$.

Proof: Suppose $B \subsetneqq C$, then the Hilbert polynomials of B and C are different in contradiction to $B \sim_a C$. □

B.6 Lemma
Under the hypotheses of Proposition B.1 let $C \in |B|_a$ with $C \neq B$, then $C \cap B = \emptyset$.

Proof: Since B is irreducible by Lemma B.5 B and C do not have a common component. Suppose $B \cap C = \{p_1, \ldots, p_r\}$, then $B^2 = B.C \geq r > 0$ in contradiction to $B^2 = 0$. □

B.7 Proposition (Zariski's Lemma)
Under the hypotheses of Proposition B.1 let $C = \sum_{i=1}^r n_i C_i \in |B|_a$, where the C_i are pairwise different irreducible curves, $n_i > 0$ for $i = 1, \ldots, r$.

Then the intersection matrix $Q = (C_i.C_j)_{i,j=1,\ldots,r}$ is negative semi-definite, and, moreover, C, considered as an element of the vectorspace $\bigoplus_{i=1}^r \mathbb{Q} \cdot C_i$, generates the annihilator of Q.

In particular, $D^2 \leq 0$ for all curves $D \subseteq C$, and, moreover, $D^2 = 0$ if and only if $D = C$.

Proof: By Lemma B.4 C is connected. We are going to apply [BPV84] I.2.10, and thus we have to verify three conditions.

(i') $C.C_i = B.C_i = 0$ for all $i = 1, \ldots, r$ by Lemma B.6. Thus C is an element of the annihilator of Q with $n_i > 0$ for all $i = 1, \ldots, r$.

(ii) $C_i.C_j \geq 0$ for all $i \neq j$.

(iii) Since C is connected there is no non-trivial partition $I \cup J$ of $\{1, \ldots r\}$ such that $C_i.C_j = 0$ for all $i \in I$ and $j \in J$.

Thus [BPV84] I.2.10 implies that $-Q$ is positive semi-definite. □

B.8 Lemma
Under the hypotheses of Proposition B.1 let $C, C' \in |B|_a$ be two distinct curves, then $C \cap C' = \emptyset$.

Proof: Suppose $C = A + D$ and $C' = A + D'$ such that D and D' have no common component.

We have
$$0 = B^2 = (A + D)^2 = (A + D')^2 = (A + D).(A + D'),$$
and thus
$$(A + D)^2 + (A + D')^2 = 2(A + D).(A + D'),$$
which implies that
$$D^2 + D'^2 = 2D.D',$$
where each summand on the left hand side is less than or equal to zero by Proposition B.7, and the right hand side is greater than or equal to zero, since the curves D and D' have no common component. We thus conclude
$$D^2 = D'^2 = D.D' = 0.$$
But then again Proposition B.7 implies that $D = C$ and $D' = C'$, that is, C and C' have no common component.

Suppose $C \cap C' = \{p_1, \ldots, p_r\}$, then $B^2 = C.C' \geq r > 0$ would be a contradiction to $B^2 = 0$. Hence, $C \cap C' = \emptyset$. □

B.9 Lemma

Under the hypotheses of Proposition B.1 consider once more the universal family (B.3) together with its projection onto Σ,

$$\begin{tikzcd}
|B|_a \times \Sigma \arrow[drr, bend left, "\mathrm{pr}_\Sigma"] \arrow[ddr, bend right, "\mathrm{pr}_{|B|_a}"'] & & \\
& S \arrow[ul, hook'] \arrow[r, "\pi'"] \arrow[d, "\pi"] & \Sigma \\
& |B|_a. &
\end{tikzcd} \tag{B.4}$$

Then S is an irreducible projective surface, $|B|_a$ is an irreducible curve, and π' is surjective.

Proof:

Step 1: S is an irreducible projective surface and π' is surjective.

The universal property of $|B|_a$ implies that S is an effective Cartier divisor of $|B|_a \times \Sigma$, and thus in particular projective of dimension at least $2 \leq \dim |B|_a + \dim(\Sigma) - 1$. Since π' is projective, its image is closed in Σ and of dimension 2, hence it is the whole of Σ, since Σ is irreducible.

By Lemma B.8 the fibres of π' are all single points, and thus, by [Har92] Theorem 11.14, S is irreducible.

Moreover,
$$\dim(S) = \dim(\Sigma) + \dim(\text{fibre}) = 2.$$

Step 2: $\dim|B|_a = \dim(|B|_a \times \Sigma) - \dim(\Sigma) = \dim(S) + 1 - 2 = 1.$

Step 3: $|B|_a$ is irreducible.

Let V be any irreducible component of $|B|_a$ of dimension one, then we have a universal family over V and the analogue of Step 1 for V shows that the curves in V cover Σ. But then by Lemma B.8 there can be no further curve in $|B|_a$, since any further curve would necessarily have a non-empty intersection with one of the curves in V. □

B.10 Lemma
Let's consider the following commutative diagram of projective morphisms

$$\begin{tikzcd} S \arrow[r, "\pi'"] \arrow[d, "\pi"'] & \Sigma \\ |B|_a & S_{red} \arrow[l, "\varphi"] \arrow[u, "\varphi'"'] \arrow[ul, hook] \end{tikzcd} \tag{B.5}$$

The map $\varphi' : S_{red} \longrightarrow \Sigma$ is birational.

Proof: Since S_{red} and Σ are irreducible and reduced, and since φ' is surjective, we may apply [Har77] III.10.5, and thus there is an open dense subset $U \subseteq S_{red}$ such that $\varphi'_{|} : U \to \Sigma$ is smooth. Hence, in particular $\varphi'_{|}$ is flat and the fibres are single reduced points. Since $\varphi'_{|} : U \to \varphi'(U)$ is projective and quasi-finite, it is finite (cf. [Har77] Ex. III.11.2), and it follows from Proposition B.2 that $\varphi'_{|}$ is an isomorphism onto its image, i. e. φ' is birational. □

B.11 Lemma
If $\psi : \Sigma \dashrightarrow S_{red}$ denotes the rational inverse of the map φ' in (B.5), then ψ is indeed a morphism, i. e. φ' is an isomorphism.

Proof: By Lemma B.8 the fibres of φ' over the possible points of indeterminacy of φ' are just points, and thus the result follows from [Bea83] Lemma II.9. □

B.12 Lemma
The map $g : \Sigma \to |B|_a$ assigning to each point $p \in \Sigma$ the unique curve $C \in |B|_a$ with $p \in C$ is a morphism, and is thus a fibration whose fibres are the curves in $|B|_a$.

Proof: We just have $g = \varphi \circ \psi$. □

Proof of Proposition B.1: Let $\nu : H \to |B|_a$ be the normalisation of the irreducible curve $|B|_a$. Then H is a smooth irreducible curve.

Moreover, since Σ is irreducible and smooth, and since $g : \Sigma \to |B|_a$ is surjective, g factorises over H, i. e. we have the following commutative diagram

$$\begin{tikzcd} \Sigma \arrow[r, "g"] \arrow[d, "\exists f"'] & |B|_a \\ H. \arrow[ur, "\nu"] \end{tikzcd}$$

Then f is the desired fibration. □

Appendix C. Some Facts used in the Proofs of Section 3

In this section we are, in particular, writing down some identifications of certain sheaves respectively of their global sections. Doing this we try to be very formal. However, in a situation of the kind $X \overset{i}{\hookrightarrow} Y \xrightarrow{\pi} Z$ we usually do not distinguish between $\mathcal{O}_X$ and $i_*\mathcal{O}_X$, or between π and any restriction of π to X.

C.1 Lemma
Let $\varphi(x,y,t) = \sum_{i=0}^{\infty} \varphi_i(x,y)\cdot t^i \in \mathbb{C}\{x,y,t\}$ *with* $\varphi(x,y,t) \in (x,y)^m$ *for every fixed* t *in some small disc* Δ *around* 0. *Then* $\varphi_i(x,y) \in (x,y)^m$ *for every* $i \in \mathbb{N}_0$.

Proof: We write the power series as $\varphi = \sum_{\alpha+\beta=0}^{\infty}(\sum_{i=0}^{\infty} c_{\alpha,\beta,i}\cdot t^i)x^\alpha y^\beta$.
$\varphi(x,y,t) \in (x,y)^m$ for every $t \in \Delta$ implies

$$\sum_{i=0}^{\infty} c_{\alpha,\beta,i}\cdot t^i = 0 \;\; \forall\, \alpha+\beta < m \text{ and } t \in \Delta.$$

The identity theorem for power series in $\mathbb{C}$ then implies that

$$c_{\alpha,\beta,i} = 0 \;\; \forall\, \alpha+\beta < m \text{ and } i \geq 0.$$

□

C.2 Lemma
Let X *be a noetherian scheme,* $i : C \hookrightarrow X$ *a closed subscheme,* $\mathcal{F}$ *a sheaf of modules on* C, *and* $\mathcal{G}$ *a sheaf of modules on* X. *Then*

$$i_*\mathcal{F} \cong i_*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_C, \tag{C.1}$$

$$H^0(C,\mathcal{F}) = H^0(X, i_*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_C), \tag{C.2}$$

$$\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{O}_C \cong i_*i^*(\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{O}_C), \text{ and} \tag{C.3}$$

$$H^0\big(C, i^*(\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{O}_C)\big) = H^0(X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{O}_C). \tag{C.4}$$

Proof:(C.1) For $U \subseteq X$ open, we define

$$\begin{array}{ccccc} \Gamma(U, i_*\mathcal{F}) & \to & \Gamma(U, i_*\mathcal{F}) \otimes_{\Gamma(U,\mathcal{O}_X)} \Gamma(U,\mathcal{O}_C) & \subseteq & \Gamma(U, i_*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_C) \\ s & \mapsto & s \otimes 1. & & \end{array}$$

This morphism induces on the stalks the isomorphism

$$i_*\mathcal{F}_x = \left\{ \begin{array}{ccc} \mathcal{F}_x, \text{ (if } x \in C) & = & \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/I_{C,x} \\ 0, \text{ (else)} & = & 0 \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/I_{C,x} \end{array} \right\} \cong i_*\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{C,x},$$

where $I_{C,x}$ is the ideal defining C in X locally at x.
(C.2) The identification (C.1) together with [Har77] III.2.10 gives:

$$H^0(C,\mathcal{F}) = H^0(X, i_*\mathcal{F}) = H^0(X, i_*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_C).$$

(C.3) The adjoint property of i_* and i^* together with $i^*i_* \cong \mathrm{id}$ gives rise to the following isomorphisms:

$$\mathrm{End}\big(i_*i^*(\mathcal{G}\otimes\mathcal{O}_C)\big) \cong \mathrm{Hom}\big(i^*i_*i^*(\mathcal{G}\otimes\mathcal{O}_C), i^*(\mathcal{G}\otimes\mathcal{O}_C)\big)$$
$$\cong \mathrm{End}\big(i^*(\mathcal{G}\otimes\mathcal{O}_C)\big) \cong \mathrm{Hom}\big(\mathcal{G}\otimes\mathcal{O}_C, i_*i^*(\mathcal{G}\otimes\mathcal{O}_C)\big).$$

That means, that the identity morphism on $i_*i^*(\mathcal{G}\otimes\mathcal{O}_C)$ must correspond to an isomorphism from $\mathcal{G}\otimes\mathcal{O}_C$ to $i_*i^*(\mathcal{G}\otimes\mathcal{O}_C)$ via these identifications.

(C.4) follows from (C.3) and once more [Har77] III.2.10.

C.3 Corollary

In the situation of Lemma 3.4 we have:

$$H^0\big(C,\pi_*\mathcal{O}_{\widetilde{C}}(E)\otimes_{\mathcal{O}_C}\mathcal{O}_C(C)\big) = H^0\big(\Sigma,\pi_*\mathcal{O}_{\widetilde{C}}(E)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C(C)\big), \text{ and} \tag{C.5}$$

$$H^0\big(C,\pi_*\mathcal{O}_{\widetilde{\Sigma}}(E)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C(C)\big) = H^0\big(\Sigma,\pi_*\mathcal{O}_{\widetilde{\Sigma}}(E)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C(C)\big). \tag{C.6}$$

Proof: We denote by $j:\widetilde{C}\hookrightarrow\widetilde{\Sigma}$ and $i:C\hookrightarrow\Sigma$ respectively the given embeddings.

(C.5) By (C.2) in Lemma C.2 we have:

$$H^0\big(C,\pi_*\mathcal{O}_{\widetilde{C}}(E)\otimes_{\mathcal{O}_C}\mathcal{O}_C(C)\big) = H^0\Big(\Sigma, i_*\big(\pi_*\mathcal{O}_{\widetilde{C}}(E)\otimes_{\mathcal{O}_C}\mathcal{O}_C(C)\big)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C\Big).$$

By the projection formula this is just equal to:

$$H^0\Big(\Sigma,\big(i_*\pi_*\mathcal{O}_{\widetilde{C}}(E)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_\Sigma(C)\big)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C\Big) = H^0\big(\Sigma,\pi_*j_*\mathcal{O}_{\widetilde{C}}(E)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C(C)\big)$$
$$=_{\mathrm{def}} H^0\big(\Sigma,\pi_*\mathcal{O}_{\widetilde{C}}(E)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C(C)\big).$$

(C.6) Using (C.4) in Lemma C.2 we get:

$$H^0\big(C,\pi_*\mathcal{O}_{\widetilde{\Sigma}}(E)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C(C)\big) =_{\mathrm{def}} H^0\Big(C, i^*\big(\pi_*\mathcal{O}_{\widetilde{\Sigma}}(E)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C(C)\big)\Big) =$$
$$H^0\big(\Sigma,\pi_*\mathcal{O}_{\widetilde{\Sigma}}(E)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C(C)\big).$$

□

C.4 Lemma

With the notation of Lemma 3.4 we show that $\mathrm{supp}\big(\mathrm{Ker}(\gamma)\big)\subseteq\{z_1,\ldots,z_r\}$.

Proof: Since $\pi:\widetilde{\Sigma}\setminus\big(\bigcup_{i=1}^r E_i\big)\longrightarrow\Sigma\setminus\{z_1,\ldots,z_r\}$ is an isomorphism, we have for any sheaf $\mathcal{F}$ of $\mathcal{O}_{\widetilde{\Sigma}}$-modules and $y\in\widetilde{\Sigma}\setminus\big(\bigcup_{i=1}^r E_i\big)$:

$$(\pi_*\mathcal{F})_{\pi(y)} = \lim_{\pi(y)\in V}\mathcal{F}(\pi^{-1}(V)) = \lim_{y\in U}\mathcal{F}(U) = \mathcal{F}_y.$$

In particular,

$$\big(\pi_*\mathcal{O}_{\widetilde{\Sigma}}(E)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C(C)\big)_{\pi(y)} \cong \mathcal{O}_{\widetilde{\Sigma},y}\otimes_{\mathcal{O}_{\Sigma,\pi(y)}}\mathcal{O}_{C,\pi(y)} \cong \mathcal{O}_{\Sigma,\pi(y)}\otimes_{\mathcal{O}_{\Sigma,\pi(y)}}\mathcal{O}_{C,\pi(y)},$$

and

$$\big(\pi_*\mathcal{O}_{\widetilde{C}}(E)\otimes_{\mathcal{O}_\Sigma}\mathcal{O}_C(C)\big)_{\pi(y)} \cong \mathcal{O}_{\widetilde{C},y}\otimes_{\mathcal{O}_{\Sigma,\pi(y)}}\mathcal{O}_{C,\pi(y)} \cong \mathcal{O}_{C,\pi(y)}\otimes_{\mathcal{O}_{\Sigma,\pi(y)}}\mathcal{O}_{C,\pi(y)}.$$

Moreover, the morphism $\gamma_{\pi(y)}$ becomes under these identifications just the morphism given by $a\otimes\overline{b}=1\otimes\overline{ab}\mapsto\overline{a}\otimes\overline{b}=1\otimes\overline{ab}$, which is injective. Thus, $0=\mathrm{Ker}(\gamma_{\pi(y)})=\mathrm{Ker}(\gamma)_{\pi(y)}$, and $\pi(y)\not\in\mathrm{supp}(\mathrm{Ker}(\gamma))$. $\square$

C.5 Lemma

Let X be an irreducible noetherian scheme, $\mathcal{F}$ a coherent sheaf on X, and $s\in H^0(X,\mathcal{F})$ such that $\dim\big(\mathrm{supp}(s)\big)<\dim(X)$. Then $s\in H^0\big(X,\mathrm{Tor}(\mathcal{F})\big)$.

Proof: The multiplication by s gives rise to the following exact sequence:

$$0\longrightarrow\mathrm{Ker}(\cdot s)\longrightarrow\mathcal{O}_X\stackrel{\cdot s}{\longrightarrow}\mathcal{F}.$$

Since $\mathcal{O}_X$ and $\mathcal{F}$ are coherent, so is $\mathrm{Ker}(\cdot s)$, and hence $\mathrm{supp}\,\big(\mathrm{Ker}(\cdot s)\big)$ is closed in X. Now,

$$\mathrm{supp}(\mathrm{Ker}(\cdot s))=\{z\in X\mid\exists\,0\neq r_z\in\mathcal{O}_{X,z}:r_z\cdot s_z=0\}$$

$$=\{z\in X\mid s_z\in\mathrm{Tor}(\mathcal{F}_z)\}.$$

But then the complement $\{z\in X|s_z\not\in\mathrm{Tor}(\mathcal{F}_z)\}$ is open and is contained in $\mathrm{supp}(s)$ (since $s_z=0$ implies that $s_z\in\mathrm{Tor}(\mathcal{F}_z)$), and is thus empty since X is irreducible and $\mathrm{supp}(s)$ of lower dimension. $\square$

Appendix D. The Degree of a Line Bundle on a Curve

D.1 Remark

Let $C=C_1\cup\ldots\cup C_k$ be a reduced curve on a smooth projective surface Σ over $\mathbb{C}$, where the C_i are irreducible, and let $\mathcal{L}$ be a line bundle on C. Then we define the *degree* of $\mathcal{L}$ with the aid of the normalisation $\nu:C'\to C$. We have $H^2(C,\mathbb{Z})\cong\bigoplus_{i=1}^k H^2(C_i',\mathbb{Z})=\mathbb{Z}^k$, and thus the image of $\mathcal{L}$ in $H^2(C,\mathbb{Z})$, which is the first Chern class of $\mathcal{L}$, can be viewed as a vector $(l_1,\ldots,l_k)$ of integers, and we may define the degree of $\mathcal{L}$ by

$$\deg(\mathcal{L}):=l_1+\cdots+l_k.$$

In particular, if C is irreducible, we get:

$$\deg(\mathcal{L})=\deg(\nu^*\mathcal{L})=c_1(\nu^*\mathcal{L}).$$

Since $H^0(C,\mathcal{L})\neq0$ implies that $H^0(C',\nu^*\mathcal{L})\neq0$, and since the existence of a non-vanishing global section of $\nu^*\mathcal{L}$ on the smooth curve C' implies that the corresponding divisor is effective, we get the following lemma. (cf. [BPV84] Section II.2)

D.2 Lemma

Let C be an irreducible reduced curve on a smooth projective surface Σ, and let $\mathcal{L}$ be a line bundle on C. If $H^0(C,\mathcal{L})\neq0$, then $\deg(\mathcal{L})\geq0$.

Appendix E. Two Results used in the Proof of Theorem 4.1

E.1 Remark
Let $\Sigma \subset \mathbb{P}^n_{\mathbb{C}}$ be a (not necessarily) smooth projective surface and let $z \in \Sigma$ be fixed. We consider the secant variety

$$\widehat{S}_z := \big\{(z', r) \in \Sigma \times \mathbb{P}^n_{\mathbb{C}} \;\big|\; z' \neq z, r \in \overline{z, z'}\big\},$$

which is locally closed in $\Sigma \times \mathbb{P}^n_{\mathbb{C}}$, together with the to projections $\widehat{S}_z \to \Sigma \setminus \{z\}$ and $\widehat{S}_z \to S_z := \bigcup_{z \neq z' \in \Sigma} \overline{zz'} \subset \mathbb{P}^n_{\mathbb{C}}$. The isomorphism

$$\begin{aligned} \big(\Sigma \setminus \{z\}\big) \times \mathbb{P}^1_{\mathbb{C}} &\longrightarrow \widehat{S}_z \\ \big(z', (a : b)\big) &\longmapsto (z', a \cdot z + b \cdot z') \end{aligned}$$

shows that $\widehat{S}_z$ is an irreducible $\mathbb{P}^1_{\mathbb{C}}$-bundle. But thus also S_z is irreducible. Moreover, since $\widehat{S}_z$ has dimension 3, the dimension of S_z is at most 3, and since $\Sigma \cap S_z = \Sigma$ is closed in the irreducible variety S_z we have either $\dim(S_z) = 3$ or $S_z = \Sigma$. The latter happens if Σ is linear in $\mathbb{P}^n_{\mathbb{C}}$, and might happen when z is a singular point, e. g. $\Sigma = \{x_0x_1 - x_2 = 0\} \subset \mathbb{P}^3_{\mathbb{C}}$ and $z = (0 : 0 : 0 : 1)$.

E.2 Lemma
Let $\Sigma \subseteq \mathbb{P}^n_{\mathbb{C}}$ a non-linear projective surface.

(i) *If $z \in \Sigma$ is not singular, then a generic secant line through z is not contained in Σ. In particular,* $\dim(S_z) = 3$.

(ii) *If Σ is smooth, there is a very general subset $U \subset \Sigma \times \Sigma$ such that for $(z, z') \in U$ the secant line $\overline{zz'} \not\subset \Sigma$.*

Proof: Part b. is an immediate consequence of Part a. Since $z \in \Sigma$ is regular, $\mathcal{O}_{\Sigma,z} \cong \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}/(f_1, \ldots, f_{n-2})$, where $(f_1, \ldots, f_{n-2})$ is a regular sequence in $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}$. For generic linear forms $l, l' \in \mathbb{C}[x_0, \ldots, x_n]_1$ through the point z, the sequence $(f_1, \ldots, f_{n-2}, l, l')$ will then be regular. This in particular means that the linear $n-2$-space $H = V(l, l')$ intersects Σ in z transversally, i. e. with intersection multiplicity 1 and z is an isolated point of $H \cap \Sigma$. Since Σ is not linear H intersects Σ in at least one more point z' by the Theorem of Bézout, and then $\overline{zz'} \not\subset \Sigma$. This proves the first assertion, and by Remark E.1 we also know that $\dim(S_z) = 3$. □

E.3 Lemma
Let L be very ample over $\mathbb{C}$ on the smooth projective surface Σ.

There is a very general subset $U \subset \Sigma \times \Sigma$ such that for $\big(z, z'\big) \in U$, there is a smooth connected curve through z and z' in $|L|_l$. Indeed, a generic curve in $|L|_l$ through z and z' will be so.

Proof: Considering the embedding $\Sigma \subseteq \mathbb{P}^n_{\mathbb{C}}$ defined by L the curves in $|L|_l$ are in one-to-one correspondence with the hyperplane sections.

By the Theorems of Bertini (cf. [Har77] II.8.18 and III.9.9.1) we know that there is an open dense subset of the linear system of all hyperplane sections of Σ which are irreducible and smooth. Moreover, the linear systems $\mathcal{L}_z = \big\{C \in |L|_l \;\big|\; z \in C\big\}$ forms a subsystem of codimension one for any $z \in \Sigma$. Thus for all but possibly a finite number of points $z_1, \ldots, z_s$ the linear system $\mathcal{L}_z$ has an open dense subset which consists of irreducible and smooth curves. Note that the linear system $\mathcal{L}_{z,z'} = \big\{C \in |L|_l \;\big|\; z, z' \in C\big\}$ forms a linear subsystem of $\mathcal{L}_z$ of codimension one, and that $\mathcal{L}_{z,z'} = \mathcal{L}_{z,z''}$ only if z, z' and z'' are collinear. Thus fixing a point $z \in \Sigma \setminus \{z_1, \ldots, z_s\}$ we find that for all z' but those in a finite number of lines $l_{1,z}, \ldots, l_{s_z,z}$ the system $\mathcal{L}_{z,z'}$ contains an irreducible and smooth curve. This proves the claim.

$\square$

E.4 Remark

A slightly closer investigation shows that for two points $z, z' \in \Sigma$ the linear system $|L|_l$ in Lemma E.3 does not contain a smooth and irreducible curve C trough z and z' if and only if the linear system has a fix component through z and z'. Having embedded the surface Σ into $\mathbb{P}^n_{\mathbb{C}}$ via L this means that the secant line through z and z' lies in Σ. Therefore, Lemma E.2 also implies the result of Lemma E.3, Part a.

If we consider e. g. $\Sigma = \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ with $L = \mathcal{O}_\Sigma(1,1)$, then for two points on the lines $p \times \mathbb{P}^1_{\mathbb{C}}$ respectively on $\mathbb{P}^1_{\mathbb{C}} \times \{p\}$ there is no smooth curve in $|L|_l$ through the two points.

E.5 Lemma

Let $L \subset \Sigma$ be a smooth curve and $X \subset \Sigma$ a zero-dimensional scheme. If $D \in \mathrm{Div}(\Sigma)$ such that

$$h^1\big(\Sigma, \mathcal{J}_{X:L/\Sigma}(D-L)\big) = 0, \text{ and} \tag{E.1}$$

$$\deg(X \cap L) \leq D.L + 1 - 2g(L), \tag{E.2}$$

then

$$h^1\big(\Sigma, \mathcal{J}_{X/\Sigma}(D)\big) = 0.$$

Proof: Condition (E.2)) implies

$$\begin{aligned} 2g(L) - 2 < D.L - \deg(X \cap L) &= \deg\big(\mathcal{O}_L(D)\big) + \deg\big(\mathcal{J}_{X\cap L/L}\big) \\ &= \deg\big(\mathcal{J}_{X\cap L/L}(D)\big), \end{aligned}$$

and thus by Riemann-Roch (cf. [Har77] IV.1.3.4)

$$h^1\big(\mathcal{J}_{X\cap L/L}(D)\big) = 0.$$

Consider now the exact sequence

$$0 \longrightarrow \mathcal{J}_{X:L/\Sigma}(D-L) \xrightarrow{\cdot L} \mathcal{J}_{X/\Sigma}(D) \longrightarrow \mathcal{J}_{X\cap L/L}(D) \longrightarrow 0.$$

The result then follows from the corresponding long exact cohomology sequence

$$0 = H^1\big(\mathcal{J}_{X:L/\Sigma}(D-L)\big) \longrightarrow H^1\big(\mathcal{J}_{X/\Sigma}(D)\big) \longrightarrow H^1\big(\mathcal{J}_{X\cap L/L}(D)\big) = 0.$$

□

Appendix F. Product Surfaces

Througout this section we stick to the notation of Section 6.c and 6.d. Let C_1 and C_2 be two smooth projective curves of genus $g_1 \geq 0$ and $g_2 \geq 0$ respectively, and let $\Sigma = C_1 \times C_2$.

Supposed that one of the curves is rational, the surface is geometrically ruled and the Picard number of Σ is two. Whereas, if both C_1 and C_2 are of strictly positive genus, this need no longer be the case as we have seen in Remark 6.9. Thus the following proposition is the best we may expect.

F.1 Proposition

For a generic choice of smooth projective curves C_1 and C_2 the Neron-Severi group of $\Sigma = C_1 \times C_2$ is $\mathrm{NS}(\Sigma) \cong C_1\mathbb{Z} \oplus C_2\mathbb{Z}$.

More precisely, fixing g_1 and g_2 there is a very general subset $U \subseteq M_{g_1} \times M_{g_2}$ such that for any $(C_1, C_2) \in U$ the Picard number of $C_1 \times C_2$ is two, where M_{g_i} denotes the moduli space of smooth projective curves of genus g_i, $i = 1, 2$.

Proof: As already mentioned, if either g_1 or g_2 is zero, then we may take $U = M_{g_1} \times M_{g_2}$.

Suppose that $g_1 = g_2 = 1$. Given an elliptic curve C_1 there is a countable union V of proper subvarieties of M_1 such that for any $C_2 \in M_1 \setminus V$ the Picard number of $C_1 \times C_2$ is two - namely, if τ_1 and τ_2 denote the periods as in Section 6.d, then we have to require that there exists no invertible integer matrix $\left(\begin{smallmatrix} z_1 & z_2 \\ z_3 & z_4 \end{smallmatrix}\right)$ such that $\tau_2 = \frac{z_4 - z_3\tau_1}{z_2 - z_1\tau_1}$. (Compare also [GH94] p. 286.)

We, therefore, may assume that $g_1 \geq 2$ and $g_2 \geq 1$. The claim then follows from Lemma F.2, which is due to Denis Gaitsgory. □

F.2 Lemma (Denis Gaitsgory)

Let C_2 be any smooth projective curve of genus $g_2 \geq 1$. Then for any $g_1 \geq 2$ there is a very general subset U of the moduli space M_{g_1} of smooth projective curves of genus g_1 such that the Picard number of $C_1 \times C_2$ is two for any $C_1 \in U$.

Proof: We note that a curve $B \subset \Sigma = C_1 \times C_2$ with $C_1 \not\sim_a B \not\sim_a C_2$ induces a non-trivial morphism $\mu_B : C_1 \to \mathrm{Pic}(C_2) : p \mapsto \mathrm{pr}_{2*}\big(\mathrm{pr}_1^*(p)\big)$, where $\mathrm{pr}_i : \Sigma \to C_i$, $i = 1, 2$, denote the canonical projections. It thus makes sense to study the moduli problem of (non-trivial) maps from curves of genus g_1 into $\mathrm{Pic}(C_2)$.

More precisely, let $k \in \mathbb{N}$ and let $0 \neq \beta \in H_2\big(\mathrm{Pic}_k(C_2), \mathbb{Z}\big) = \mathbb{Z}^{2g_2}$ be given, where $\mathrm{Pic}_k(C_2)$ is the Picard variety of divisors of degree k on C_2. Following the notation of [FP97] we denote by $M_{g_1,0}\big(\mathrm{Pic}_k(C_2), \beta\big)$ the moduli space of pairs (C_1, μ), where C_1 is a smooth projective curve of genus g_1 and $\mu : C_1 \to \mathrm{Pic}_k(C_2)$ a morphism with $\mu_*\big([C_1]\big) = \beta$. We then have the canonical morphism

$$F_{k,\beta} : M_{g_1,0}\big(\mathrm{Pic}_k(C_2), \beta\big) \to M_{g_1} : (C_1, \mu) \mapsto C_1,$$

just forgetting the map μ, and the proposition reduces to the following claim:

Claim: For no choice of $k \in \mathbb{N}$ and $0 \neq \beta \in H_2\big(\operatorname{Pic}_k(C_2), \mathbb{Z}\big)$ the morphism $F_{k,\beta}$ is dominant.

Let $\mu : C_1 \to \operatorname{Pic}_k(C_2)$ be any morphism with $\mu_*\big([C_1]\big) = \beta$. Then μ is not a contraction and the image of C_1 is a projective curve in the abelian variety $\operatorname{Pic}_k(C_2)$. Moreover, we have the following exact sequence of sheaves

$$0 \longrightarrow \mathcal{T}_{C_1} \xrightarrow{d\mu} \mu^*\mathcal{T}_{\operatorname{Pic}_k(C_2)} = \mathcal{O}_{C_1}^{g_2} \longrightarrow \mathcal{N}_\mu := \operatorname{coker}(d\mu) \longrightarrow 0. \tag{F.1}$$

Since $d\mu$ is a non-zero inclusion, its dual $d\mu^\vee : (\mu^*\mathcal{T}_{\operatorname{Pic}_k(C_2)})^\vee = \mathcal{O}_{C_1}^{g_2} \to \Omega_{C_1} = \omega_{C_1}$ is not zero on global sections, that is

$$H^0\big(d\mu^\vee\big) : H^0\big(C_1, \mathcal{O}_{C_1}^{g_2}\big) = \operatorname{Hom}_{\mathcal{O}_{C_1}}\big(\mathcal{O}_{C_1}^{g_2}, \mathcal{O}_{C_1}\big) \to H^0(C_1, \omega_{C_1}) = \operatorname{Hom}_{\mathcal{O}_{C_1}}\big(\mathcal{T}_{C_1}, \mathcal{O}_{C_1}\big)$$

is not the zero map. Since $g_1 \geq 2$ we have $h^0(C_1, \omega_{C_1}) = 2g_1 - 2 > 0$, and thus ω_{C_1} has global sections. Therefore, the induced map $H^0\big(C_1, \omega_{C_1} \otimes \mathcal{O}_{C_1}^{g_2}\big) \to H^0(C_1, \omega_{C_1} \otimes \omega_{C_1})$ is not the zero map, which by Serre duality gives that the map

$$H^1(d\mu) : H^1(C_1, \mathcal{T}_{C_1}) \to H^1\big(C_1, \mu^*\mathcal{T}_{\operatorname{Pic}_k(C_2)}\big)$$

from the long exact cohomology sequence of (F.1) is not zero. Hence the coboundary map

$$\delta : H^0(C_1, \mathcal{N}_\mu) \to H^1(C_1, \mathcal{T}_{C_1})$$

cannot be surjective. According to [Har98] p. 96 we have

$$\delta = dF_{k,\beta} : \mathcal{T}_{M_{g_1,0}(\operatorname{Pic}_k(C_2),\beta)} = H^0(C_1, \mathcal{N}_\mu) \longrightarrow \mathcal{T}_{M_{g_1}} = H^1(C_1, \mathcal{T}_{C_1}).$$

But if the differential of $F_{k,\beta}$ is not surjective, then $F_{k,\beta}$ itself cannot be dominant. $\square$

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