Tropical Resultants for Curves

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Why Resultants?

- Resultants are classical powerful elimination tools
- Related to tropical elimination
- Needed in the context of geometric construction to codify intersection of curves

Notation

- $\mathbb{K}$ is an algebraically closed field of arbitrary characteristic provided by a non trivial valuation $v : \mathbb{K}^* \to \mathbb{T} \subseteq \mathbb{R}$.
- A valuation v is an homomorphism

$$v : \mathbb{K}^* \longrightarrow \mathbb{T}$$

  from the multiplicative group $\mathbb{K}^*$ onto an ordered abelian group such that, if $a + b \neq 0$

$$v(a+b) \geq \min\{v(a), v(b)\}$$

- R is the valuation ring of $\mathbb{K}$ with maximal ideal m.

$$R = \{x \in \mathbb{K}^* | v(x) \geq 0\} \cup \{0\}$$

$$m = \{x \in \mathbb{K}^* | v(x) > 0\} \cup \{0\}$$

Notation

- k denotes the residual field with respect to the valuation; $k = R/m$.
- Depending on the characteristics of $\mathbb{K}$ and k we will distinguish three cases.
  - char($\mathbb{K}$)=char(k)=0, the equicharacteristic zero case. This is the most common to develop tropical geometry.
    Example: The Puiseux series $\mathbb{C}\{t\}$
    Residual field $\mathbb{C} \subseteq \mathbb{C}\{t\}$
  - char($\mathbb{K}$)=char(k)=$p > 0$, the equicharacteristic p case.
    Example: The algebraic closure of $\mathbb{F}_p(t)$ of rational functions in one variable.
    $$v(t) = 1$$
    Residual field $\overline{\mathbb{F}}_p \subseteq \overline{\mathbb{F}_p[t]}$

Notation

- char($\mathbb{K}$)=0, char(k)=$p > 0$. The p-adic case.
  Example $\overline{\mathbb{Q}}_5$ the algebraic closure of the 5-adics.
  If $x = 5^{k}\frac{p}{q} \in \mathbb{Q}$, $k \in \mathbb{Z}$, 5 does not divide p and q, then

  $$v(x) = k.$$

  This valuation can be extended to the algebraic numbers.
  Residual field $\overline{\mathbb{F}}_5 \not\subseteq \overline{\mathbb{Q}}_5$.

Notation

- $\pi : R \to k = R/m$ denotes the residual class map.
- The tropicalization $T : \mathbb{K}^* \to \mathbb{T}$ is either v (if working with min or $-v$ if working with max)
- We suppose that we have a multiplicative subgroup $\{t^\gamma \mid \gamma \in \mathbb{T}\} \subseteq \mathbb{K}^*$ isomorphic to $\mathbb{T}$ by the homomorphism v
- Let $x \in \mathbb{K}^*$, we denote the principal coefficient $Pc(x)$ as the element $\pi(xt^{-v(x)}) \in k$ (this is also called angular component.)
- $x \in \mathbb{K}^*$ is residually generic if $Pc(x)$ is generic in k.

Two Examples of Principal Coefficient

- $\mathbb{K} = \mathbb{C}\{t\}$ the Puiseux series field with the standard valuation. $k = \mathbb{C}$. The group $\{t^\gamma \mid \gamma \in \mathbb{T}\}$ is just the set of elements $t^{m/n}$.

$$x = \frac{2}{t} + 1 + t + t^2,$$

$$v(x) = -1, Pc(x) = \pi(tx) = 2 \in \mathbb{C}.$$

Two Examples of Principal Coefficient

- $\mathbb{K} = \overline{\mathbb{Q}_5}$ the algebraic closure of the rationals with a $5-$adic valuation. $k = \overline{\mathbb{F}_5}$. We may take the group t^γ as an appropriate set of the form $5^{n/m}$.

If x is a root of valuation -1 of

$$\frac{2}{5^5} + 3z + z^5 + 75z^6,$$

then $5x$ is a root of $2 + 3 \cdot 5^4 z + z^5 + 3 \cdot 5z^6$,

$$Pc(x) = \pi(5x) \text{ is a root of } 2 + z^5 \in \overline{\mathbb{F}_5},$$

$$Pc(x) = 2$$

Tropicalization... (again)

- If $V \subseteq (\mathbb{K}^*)^n$ is an algebraic variety, the tropicalization $T(V)$ is the image of V under the map:

$$\begin{array}{cccc} T: & (\mathbb{K}^*)^n & \to & \mathbb{T}^n \\ & (x_1,\ldots,x_n) & \mapsto & (-v(x_1),\ldots,-v(x_n)) \end{array}$$

If $f = \text{``}\sum_{i\in I} a_i x^i\text{''} = \max\{a_i + <i,x>\}$ is a tropical polynomial, its zero set $\mathcal{T}(f)$ is the set of points is attained for at least two different indices i.

Theorem (Kapranov)

Let $\widetilde{f} = \sum_{i\in I} a_i x^i \in \mathbb{K}[x_1,\ldots,x_n]$. *Let*

$$f = \text{``}\sum_{i\in I} T(a_i)x^i\text{''} \in \mathbb{T}[x_1,\ldots,x_n]$$

Then

$$T(V(\widetilde{f})) = \mathcal{T}(f)$$

Classical Univariate Resultant

I, $J \subseteq \mathbb{N}$ finite and of cardinality at least 2 such that $0 \in I \cap J$.

$\widetilde{f} = \sum_{i \in I} a_i x^i$, $\widetilde{g} = \sum_{j \in J}^{m} b_j x^j \in \mathbb{K}[x]$, of support I and J.

Let p be the characteristic of $\mathbb{K}$.

There is a unique polynomial in $\mathbb{Z}/(p\mathbb{Z})[a_i, b_j]$, up to a constant factor, called the resultant, such that it vanishes if and only if $\widetilde{f}$ and $\widetilde{g}$ have a common root.

Tropical Univariate Resultant Polynomial

Definition

We denote by $R(I, J, \mathbb{K}) \in \mathbb{Z}/(p\mathbb{Z})[a_i, b_j]$ *this resultant, denote by* $R_t(I, J, \mathbb{K})$ *the tropicalization of the resultant. This tropical polynomial is called the tropical resultant of supports* I *and* J *over* $\mathbb{K}$*. The tropical variety is denoted by* $\mathcal{T}(R_t(I, J, \mathbb{K}))$*.*

- The tropical resultant polynomial depends on the field $\mathbb{K}$ and the valuation!

Example of Different Tropicalizations of the Resultant

Let $f = a + bx + cx^2$, $g = p + qx + rx^2$.
If char($\mathbb{K}$)$\neq 2$ then
$R(\{0,1,2\},\{0,1,2\},\mathbb{K}) = r^2a^2 - 2racp + c^2p^2 - qrba - qbcp + cq^2a + prb^2$.
If char($\mathbb{K}$)$=2$ then
$R(\{0,1,2\},\{0,1,2\},\mathbb{K}) = r^2a^2 + c^2p^2 - qrba - qbcp + cq^2a + prb^2$.

So the tropical resultant polynomial is: Puiseux series:
$P_1 =$ "$0r^2a^2 + 0 \quad racp + 0c^2p^2 + 0qrba + 0qbcp + 0cq^2a + 0prb^2$".
2-adics:
$P_2 =$ "$0r^2a^2 + (-1)racp + 0c^2p^2 + 0qrba + 0qbcp + 0cq^2a + 0prb^2$".
char 2:
$P_3 =$ "$0r^2a^2 + \qquad + 0c^2p^2 + 0qrba + 0qbcp + 0cq^2a + 0prb^2$".

Geometric Meaning of the Resultant

Theorem

The tropical variety defined by $R_t(I, J, \mathbb{K})$ does not depend on the field $\mathbb{K}$.
$f =$ "$\sum_{i \in I} a_i x^i$", $g =$ "$\sum_{j \in J}^{m} b_j x^j$" have a common tropical root if and only if the point (a_i, b_j) belongs to the variety defined by $R_t(I, J, \mathbb{K})$.

Puiseux series:
$P_1 =$ "$0r^2a^2 + 0 \quad racp + 0c^2p^2 + 0qrba + 0qbcp + 0cq^2a + 0prb^2$".
2-adics:
$P_2 =$ "$0r^2a^2 + (-1)racp + 0c^2p^2 + 0qrba + 0qbcp + 0cq^2a + 0prb^2$".
char 2:
$P_3 =$ "$0r^2a^2 + \qquad 0c^2p^2 + 0qrba + 0qbcp + 0cq^2a + 0prb^2$".
The tropical variety defined by P_1, P_2 and P_3 is the same.

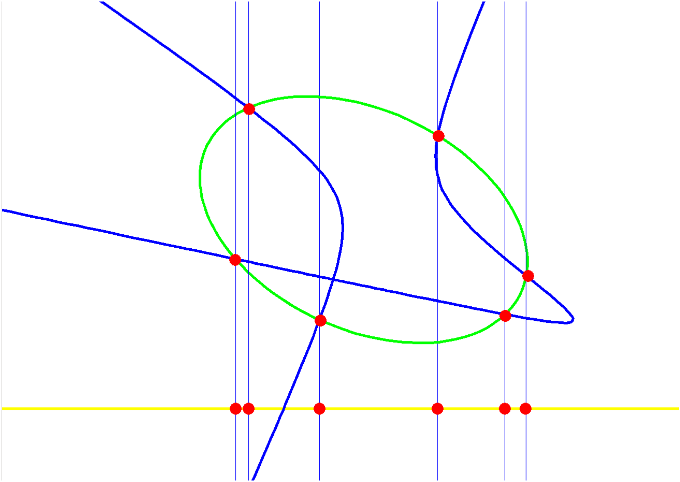
Idea of the Proof

- Direct: In the resultant, the coefficients of monomials that are vertices of the Newton polytope are always ± 1 [Sturmfels, 1994]

- Lifting Proof: Two tropical polynomials with a common tropical root can always be lifted to two algebraic polynomials with a common algebraic root. This happens because they form an acyclic incidence configuration and Kapranov's theorem.

Bivariate Resultants

Let $\widetilde{f} = \sum \widetilde{a}_{i,j} x^i y^j$, $\widetilde{g} = \sum \widetilde{b}_{k,l} x^k y^l \in \mathbb{K}[x, y]$.
$\text{Res}(\widetilde{f}, \widetilde{g}, y)$ is a polynomial in $\mathbb{K}[x]$ such that its roots are the x-th coordinates of the finite set $\{\widetilde{f} = \widetilde{g} = 0\}$.



Tropical Bivariate Resultants

$f = \text{“}\sum_{i,j} a_{i,j}x^iy^j\text{”}$, $g = \text{“}\sum_{k,l} b_{k,l}x^ky^l\text{”} \in \mathbb{T}[x,y]$
We define the resultant of f and g with respect to y as an specialization of the corresponding univariate resultant: ej:
$f = \text{“}0 + 2x + 3y\text{”}$, $g = \text{“}2 + 3x + 3y + 3xy + 2x^2 + 0y^2\text{”}$

- Rewrite them as polynomials in $\mathbb{T}[y][x]$,

$$f = (0 + 3y) + (2)x, \;\; g = (2 + 3y + 0y^2) + (3 + 3y)x + (2)x^2$$

- Compute the univariate resultant corresponding to the supports of the polynomials w.r.t. x: $I = \{0,1\}$, $J = \{0,1,2\}$,

$$R(I, J, \mathbb{C}\{t\}) = Res_x(A_0 + A_1x, B_0 + B_1x + B_2x^2) =$$

$$= A_1^2B_0 - A_0A_1B_1 + B_2A_0^2$$

$$R_t(I, J, \mathbb{C}\{t\}) = \text{“}0A_1^2B_0 + 0A_0A_1B_1 + 0B_2A_0^2\text{”}$$

Tropical Bivariate Resultants

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$$R(I, J, \mathbb{C}\{t\}, x) = Res_x(A_0 + A_1 x, B_0 + B_1 x + B_2 x^2) =$$

$$= A_1^2 B_0 - A_0 A_1 B_1 + B_2 A_0^2$$

$$R_t(I, J, \mathbb{C}\{t\}, x) = \text{“}0A_1^2 B_0 + 0A_0 A_1 B_1 + 0B_2 A_0^2\text{”}$$

- Evaluate this tropical resultant in the coefficients of f and g

$$R_t(f, g, x) = \text{“}2^2(2{+}3y{+}0y^2){+}(0{+}3y)(2)(3{+}3y){+}(2)(0{+}3y)^2\text{”} =$$

$$= \text{“}6 + 8y + 8y^2\text{”}$$

Tropical Bivariate Resultant

- This resultant polynomial may vary depending essentially on the characteristics of $\mathbb{K}$ and k.
- However, the roots of the resultant polynomial do not depend on $\mathbb{K}$ and v, only of f and g.
- Practical hints to decrease the number of monomials in a resultant polynomial:
  - In the univariate algebraic resultant, we can get rid off every monomial whose coefficient is not ± 1.
  - In the specialization of the coefficients, we can use the equality "$(a+b)^n$" = "$a^n + b^n$".

Valuation Meaning of the Roots of the Resultant

Theorem

Let $\widetilde{f} = \sum_{i,j} \widetilde{a}_{i,j} x^i y^j$, $\widetilde{g} = \sum_{k,l} \widetilde{b}_{k,l} x^k y^l \in \mathbb{K}[x, y]$.
Let $f = T(\widetilde{f}) =$ "$\sum_{i,j} T(\widetilde{a}_{i,j}) x^i y^j$", $g = T(\widetilde{g}) =$ "$\sum_{k,l} T(\widetilde{b}_{k,l}) x^k y^l$"
Let $h(y) = Res_x(\widetilde{f}, \widetilde{g})$.
If the coefficients of $\widetilde{f}, \widetilde{g}$ *are residually generic then*

$$T(\{h(y) = 0\}) = \mathcal{T}(Res_x(f, g, \mathbb{K}))$$

That is, the tropical resultant encodes the tropicalization of the algebraic resultant, whenever the coefficients are generic enough.

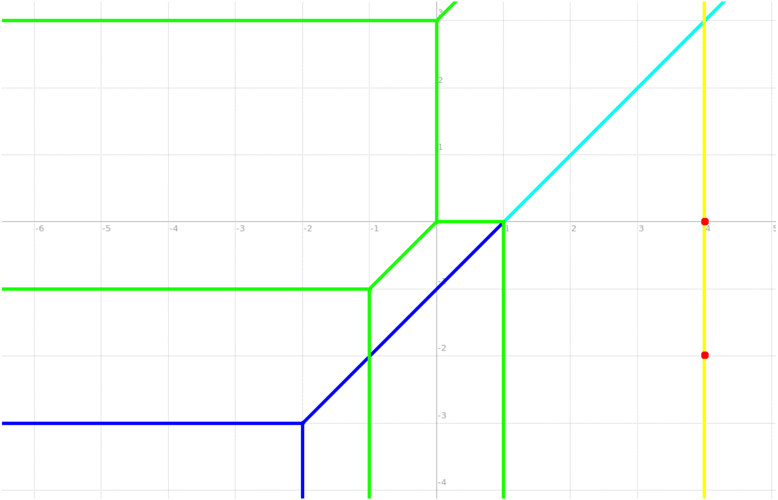
Geometric Meaning of the Roots of the Resultant

- Recall the notion of stable intersection. Given two tropical curves, there is a well defined finite set of intersection points that varies continuously as the curves are perturbed and that verifies Bernstein-Koushnirenko theorem.

Geometric Meaning of the Roots of the Resultant

$$f = \text{“}0 + 2x + 3y\text{”}, g = \text{“}2 + 3x + 3y + 3xy + 2x^2 + 0y^2\text{”}$$

$$R_t(f, g, x) = \text{“}6 + 8y + 8y^2\text{”} \text{ roots: -2, 0}$$



Geometric Meaning of the Roots of the Resultant

Lemma

Let f and g be two tropical polynomials in two variables. Then, for any two lifts $\widetilde{f}$, $\widetilde{g}$ such that their coefficients are residually generic, the intersection of the algebraic curves projects into the stable intersection.

$$T(\widetilde{f} \cap \widetilde{g}) \subseteq \mathcal{T}(f) \cap_{st} \mathcal{T}(g)$$

Idea: If q is a non stable intersection point, it belongs to the interior of two parallel edges of $\mathcal{T}(f)$ and $\mathcal{T}(g)$. The residual polynomials $\widetilde{f}_q$ and $\widetilde{g}_q$ over q can be written as

$$\widetilde{f}_q = \sum_{i=0}^{n} \alpha_i (x^r y^s)^i, \widetilde{g}_q = \sum_{j=0}^{m} \beta_i (x^r y^s)^j.$$

The resultant of the polynomials $\sum_{i=0}^{n} \alpha_i z^i$, $\sum_{j=0}^{m} \beta_i z^j$ with respect to z must vanish. So there is an algebraic dependence among these residual coefficients.

Geometric Meaning of the Roots of the Resultant

Theorem

Let $\widetilde{f}, \widetilde{g} \in \mathbb{K}[x, y]$. Then, it can be computed a finite set of polynomials in the principal coefficients of $\widetilde{f}, \widetilde{g}$ such that, if no one of them vanish, the tropicalization of the intersection of $\widetilde{f}, \widetilde{g}$ is the stable intersection of f and g. Moreover, the multiplicities are conserved.

$$\sum_{\substack{\widetilde{q} \in \widetilde{f} \cap \widetilde{g} \\ T(\widetilde{q}) = q}} mult(\widetilde{q}) = mult_t(q)$$

This theorem is proved using the correspondence between the several algebraic and tropical resultants and the previous lemma.

Geometric Meaning of the Roots of the Resultant

Corollary

Let f, $g \in \mathbb{T}[x, y]$ be two tropical polynomials. Let

$$h(y) \in \mathbb{T}[y] = Res_x(f, g, \mathbb{K})$$

be a tropical resultant of f and g with respect to x. Then, the tropical roots of h are exactly the y-th coordinates of the stable intersection of f and g.

- This is an indirect prove that all the polynomial resultants define the same points.

How to Compute the Stable Intersection and the Compatibility with the Algebraic Case

- In the tropical setting

$$f \cap_{st} g \subseteq f \cap g \cap Res_x(f,g) \cap Res_y(f,g)$$

the right-hand set is finite but may be greater than the stable intersection.
Solution: Let a be such that $x - ay$ is injective in the right-hand set.
Let $z = xy^{-a}$.
The resultant

$$\mathrm{Res}_y(\widetilde{f}(zy^a, y), \widetilde{g}(zy^a, y)) = \widetilde{R}(z) = \widetilde{R}(xy^{-a})$$

has as roots the values that the function $x - ay$ takes on the stable intersection.

Applications

- Transfer a proof of Berstein-Koushnirenko theorem in the plane to the positive characteristic case. (See Rojas 1999 for an alternative proof of this theorem in positive characteristic in the general context.)
- If $\widetilde{f}$, $\widetilde{g} \in \mathbb{K}[x, y]$, $R(x) = Res_y(\widetilde{f}, \widetilde{g})$, $R(y) = Res_x(\widetilde{f}, \widetilde{g})$, Let a such that $x - ay$ is injective in $T(\widetilde{f}) \cap T(\widetilde{g}) \cap T(R(x)) \cap T(R(y))$, then

$$\widetilde{f}, \widetilde{g}, R(x), R(y), Res_y(\widetilde{f}(zy^a, y), \widetilde{f}(zy^a, y))(xy^{-a})$$

is a tropical basis of the ideal $(\widetilde{f}, \widetilde{g})$ (This result has been generalized independently by Hept and Theobald, 2007)
- Resultants are a tool used in the construction method to prove classical theorems in tropical geometry, ej: converse Pascal theorem or Cayley-Bacharach.

Open Problem

- How to compute the tropical resultant directly?

The tropical determinant of the Sylvester matrix should work, this is a problem of deciding if the Newton polytope of the determinant and the permanent of the Sylvester matrix is equal or not.