Updown Numbers and the Initial Monomials of the Slope Variety

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AMS Central Sectional Meeting
University of Notre Dame
November 6, 2010

The Slope Variety

$P_1 = (x_1, y_1), \ldots, P_n = (x_n, y_n)$: labeled points in the plane $\mathbb{C}^2$, with all x_i distinct

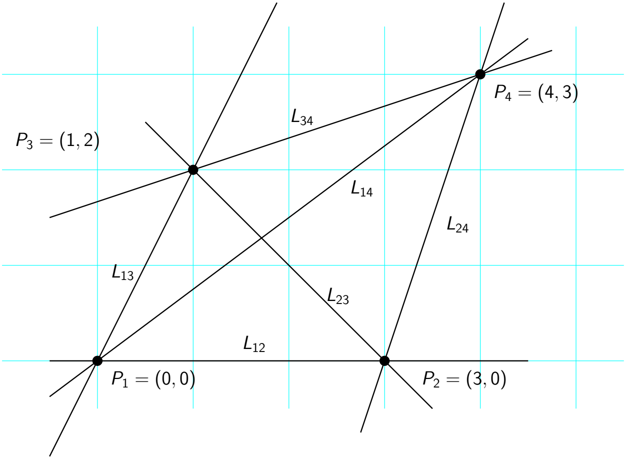
L_{ij} = line determined by P_i and P_j

$m_{ij} = \frac{y_i - y_j}{x_i - x_j}$ = slope of L_{ij}

Definition The **slope variety** S_n is the set of slope vectors

$$\mathbf{m} = (m_{12}, m_{13}, \ldots, m_{n-1,n}) \in \mathbb{C}^{\binom{n}{2}}$$

arising from some labeled point set $(P_1, \ldots, P_n)$.



$m_{12} = 0$

$m_{13} = 2$

$m_{14} = 3/4$

$m_{23} = -1$

$m_{24} = 3$

$m_{34} = 1/3$

The Slope Variety

$R_n = \mathbb{C}[m_{12}, \ldots, m_{n-1,n}]$

$$\begin{aligned} I_n &= \text{ideal of all polynomials that vanish on } S_n \\ &= \text{constraints on slope vectors} \end{aligned}$$

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- Translation and scaling don't change the slopes
- $\dim S_n = \dim R_n/I_n = 2n - 3$.
- For $n \geq 4$, $S_n \neq \mathbb{C}^{\binom{n}{2}}$ and $I_n \neq 0$.

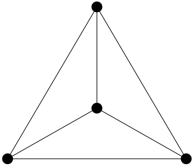
Example For $n = 4$, S_4 is the hypersurface defined by $\tau(K_4) = 0$, where

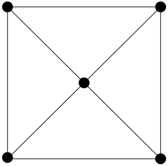
$$\begin{aligned}\tau(K_4) \;=\; & m_{12}m_{14}m_{23} - m_{13}m_{14}m_{23} - m_{12}m_{13}m_{24} \\ & + m_{13}m_{14}m_{24} + m_{13}m_{23}m_{24} - m_{14}m_{23}m_{24} \\ & + m_{12}m_{13}m_{34} - m_{12}m_{14}m_{34} - m_{12}m_{23}m_{34} \\ & + m_{14}m_{23}m_{34} + m_{12}m_{24}m_{34} - m_{13}m_{24}m_{34} \;.\end{aligned}$$

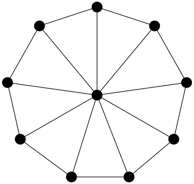
For general $n \geq 4$, the ideal I_n has at least $\binom{n}{4}$ cubic generators like this (and possibly others).

Wheels

A *k-wheel* is a graph consisting of a cycle of length $k \geq 3$, and a center vertex adjacent to all vertices of the cycle.







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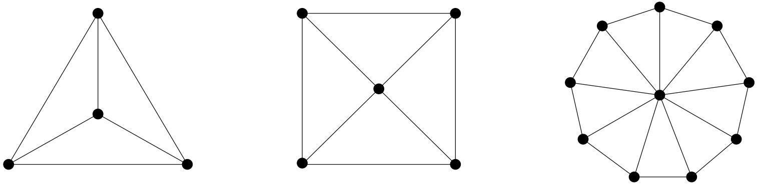

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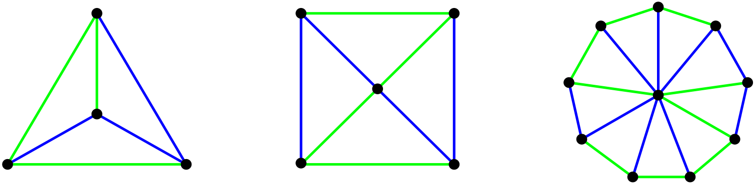

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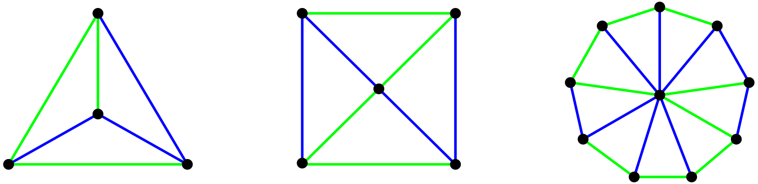

- Every wheel can be decomposed into two disjoint spanning trees ("**coupled** trees") in $2^k - 2$ ways
- For $k = 3$: coupled trees = spanning paths = permutations of $\{1, 2, 3, 4\}$ modulo reversal

Tree Polynomials

The **tree polynomial** of W is

$$\tau(W) = \sum_{T \in \mathscr{T}(W)} \varepsilon(T) \prod_{e \in T} m_e$$

where

$$\begin{aligned} \mathscr{T}(W) &= \{\text{coupled spanning trees of } W\}, \\ \varepsilon(T) &\in \{+1, -1\}. \end{aligned}$$

Example $\tau(K_4) = \frac{1}{2} \sum_{\sigma \in \mathfrak{S}_4} \varepsilon(\sigma) m_{\sigma(1)\sigma(2)} m_{\sigma(2)\sigma(3)} m_{\sigma(3)\sigma(4)}.$

The Ideal of Tree Polynomials

Theorem [JLM '06] Order the variables in R_n by

$$m_{12} < m_{13} < \cdots < m_{1n} < m_{23} < \cdots$$

and order monomials either by glex or rlex order. Then:

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is a Gröbner basis for the ideal I_n.

- R_n/I_n is reduced and Cohen-Macaulay, and its Hilbert series has a combinatorial interpretation using perfect matchings.

G-Words and R-Words

Definition Let $n \geq 4$. A sequence $w = (w_1, \ldots, w_n)$ of distinct positive integers is a **G-word** if:

1. $w_1 = \max(w_1, \ldots, w_n)$;
2. $w_n = \max(w_2, \ldots, w_n)$;
3. $w_2 > w_{n-1}$.

A G-word is *primitive* if no proper subword is a G-word and no reversal of a proper subword is a G-word.

An **R-word** is defined similarly by reversing the inequality in (3).

G-Words and R-Words

G-words with digits $\{1, 2, 3, 4, 5\}$:

52314	(primitive)
53214	(primitive)
53124	(not primitive: 3124 is the reverse of a G-word)

R-words with digits $\{1, 2, 3, 4, 5\}$:

51324	(primitive)
51234	(not primitive: 5123 is an R-word)
52134	(primitive)

Two Initial Ideals

Theorem [JLM '06] The glex (resp. rlex) initial ideal of I_n is generated by the squarefree monomials

$$m_w \;:=\; m_{w_1 w_2} m_{w_2 w_3} \cdots m_{w_{d-1} w_d}$$

for all G-words (resp. R-words) $w = (w_1, \ldots, w_d)$ with $\{w_1, \ldots, w_d\} \subseteq [n]$.

Example For $n = 5$:

$$\mathrm{in}_{\mathrm{glex}}(I_5) \;=\; \langle m_{4213},\ m_{5213},\ m_{5214},\ m_{5314},\ m_{5324},\ m_{52314},\ m_{53214} \rangle$$
$$\mathrm{in}_{\mathrm{rlex}}(I_5) \;=\; \langle m_{4123},\ m_{5123},\ m_{5124},\ m_{5134},\ m_{5234},\ m_{51324},\ m_{52134} \rangle$$

Updown Permutations

Definition A permutation $w = w_1 \cdots w_n$ is called **updown** if

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Fun Fact $\displaystyle\sum_{n\geq 0} \frac{u_n}{n!} x^n = \tan x + \sec x.$

Decreasing 012-Trees

Definition A **decreasing 012-tree** is a rooted tree with vertices labeled by distinct positive integers, such that

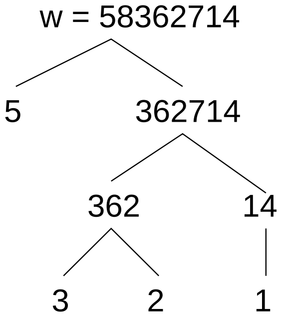
- the labels decrease as you move down the tree; and
- every vertex has 0, 1 or 2 children.

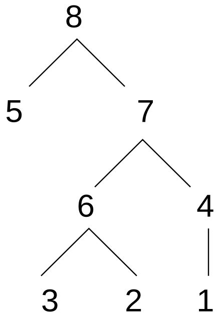
Theorem [Donaghey, 1975] There is a bijection

$$\left\{ \begin{array}{c} \text{updown permutations} \\ w \in \mathfrak{S}_n \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{decreasing 012-trees} \\ \text{on vertex set } [n] \end{array} \right\}.$$

Donaghey's Bijection

- Largest digit in updown permutation → label of vertex
- Left and right subwords → left and right subtrees





The Main Result

Theorem [JLM and Wagner, 2009] There are bijections between

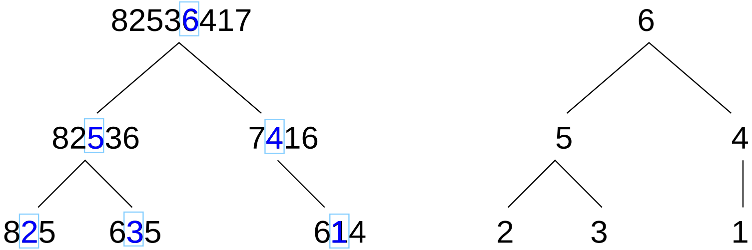
- Primitive G-words of length n
- Primitive R-words of length n
- Decreasing 012-trees of length $n-2$

Corollary For all $3 \leq d \leq n-1$, the number of degree-d generators of $\text{in}(I_n)$ (where $\text{in} = \text{in}_{\text{glex}}$ or $\text{in} = \text{in}_{\text{rlex}}$) is

$$\binom{n}{d+1} u_{d-1}.$$

Example

- Start with a primitive G-word of length n (e.g., 82536417).
- Construct a rooted tree by splitting w at $n-2$ and labeling children with subwords.
- In right-hand branches, swap $n-2$ and $n-1$.



Open Problems

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- T. Enkosky–JLM: Let J_n be the ideal generated by the $\tau(K_4)$s. Then $\operatorname{Spec} R_n/J_n$ is either S_n, or (at worst) has an embedded component of dimension 1.

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- No known combinatorial interpretation (yet) over other finite fields
- Unclear whether the ideal I_n has additional structure in positive characteristic

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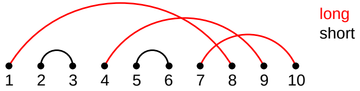
- Some progress on understanding the higher-dimensional analogues of tree polynomials, but no algebraic results yet

The Hilbert Series of R_n/I_n

A *matching* M is a partition of $[2N] = \{1, 2, \ldots, 2N\}$ into N pairs. A pair $\{x, y\} \in M$ is *long* if $|x - y| \geq 2$.



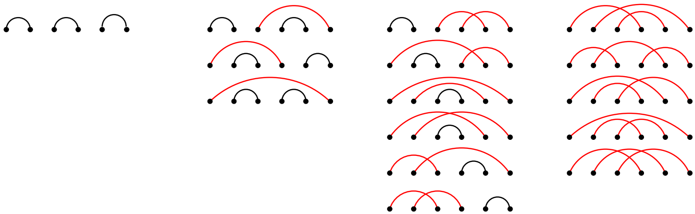
Ξ_N = set of matchings on $[N]$

Theorem [JLM '06] For $n \geq 4$, the Hilbert series of $T = R_n/I_n$ is

$$\sum_{k \geq 0} q^k \cdot \dim_{\mathbb{C}}(T_k) \quad = \quad \frac{\displaystyle\sum_{M \in \Xi_{2n-4}} q^{\#\text{ long pairs of } M}}{(1-q)^{2n-3}}.$$

The Hilbert Series of R_5/I_5

Example For $n = 5$, the matchings on $[2n-4] = [6]$ are:



The Hilbert series of R_5/I_5 is

$$\frac{1 + 3q + 6q^2 + 5q^3}{(1-q)^7}.$$