Eigenvalues of Simplicial Rook Graphs

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Graphs, Laplacians, and Spectra

$G = (V, E)$: connected simple graph (no loops or parallel edges)
Adjacency matrix: $A = A(G) =$ matrix with rows and columns indexed by $V(G)$ with 1s for edges, 0s for non-edges

Laplacian matrix: $L = D - A$, where $D =$ diagonal matrix of vertex degrees

- A, L real symmetric $\therefore$ diagonalizable, real eigenvalues
- If G is δ-regular ($D = \delta\mathbb{1}$): A, L have same eigenvectors

Spectral graph theory: study of spectra (multisets of eigenvalues) of $A(G)$, $L(G)$

- A: isoperimetric problems, clustering, expanders...
- L: algebraic combinatorics, Matrix-Tree Theorem, integrality

Graphs, Laplacians, and Spectra

Many classes of graphs have nice (i.e., integral) Laplacian eigenvalues:

- complete graphs
- complete bipartite graphs
- hypercubes
- threshold graphs (Merris)
- Kneser graphs (Godsil/Royle? Haemers?)

This talk is about a mostly-unstudied (as far as we know) class of graphs that appear to be Laplacian integral and have nice combinatorics.

$d, n =$ positive integers

$$n\Delta^{d-1} = \text{dilated simplex } \{\mathbf{v} = (v_1, \dots, v_d) \in \mathbb{R}^d \colon \sum_{i=1}^{d} v_i = n\}$$
$$= \text{conv}\{n\mathbf{e}_1, \dots, n\mathbf{e}_d\} \subseteq \mathbb{R}^d$$

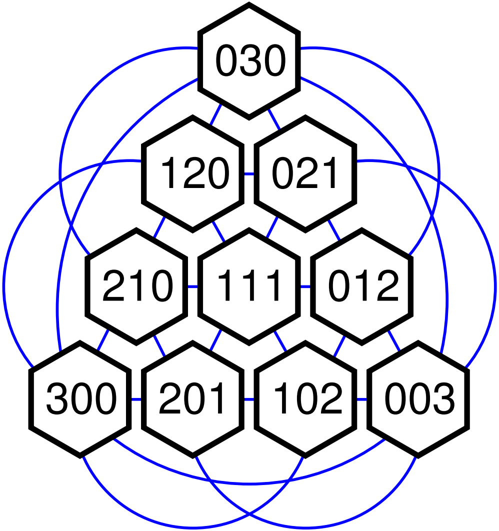
Definition

The simplicial rook graph $SR(d, n)$ is the graph with vertices

$$V(d, n) = n\Delta^{d-1} \cap \mathbb{N}^d$$

with two vertices adjacent iff they differ in exactly two coordinates (i.e., they lie on a common "lattice line").

Simplicial Rook Graphs



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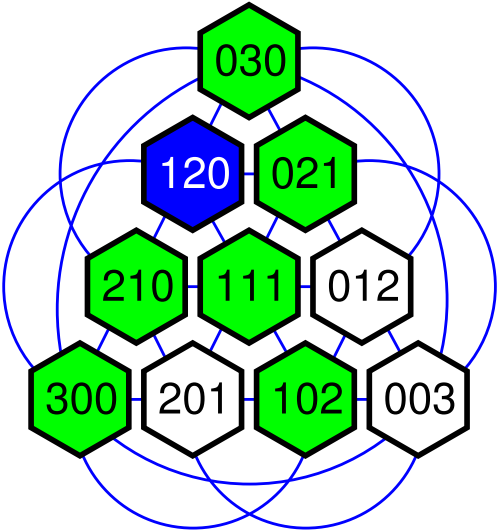

Simplicial Rook Graphs

- $G = SR(d,n)$ has $\binom{n+d-1}{d-1}$ vertices.
- G is regular of degree $\delta = (d-1)n$.
  Thus spectra of $A(G)$ and $L(G)$ contain same information.
- Independence number $\alpha(SR(d,n))$ = maximum number of nonattacking rooks on a "simplicial chessboard".
- $\alpha(SR(3,n)) = \lfloor(2n+3)/3\rfloor$.
  [Nivasch–Lev 2005; Blackburn–Paterson–Stinson 2011]

The Spectrum of $A(3, n)$

Theorem (JLM/JDW, 2012)

The eigenvalues of $A(3,n) = A(SR(3,n))$ *are as follows:*

$\mathbf{n = 2m+1}$ **odd**		$\mathbf{n = 2m}$ **even**	
Eigenvalue	**Multiplicity**	**Eigenvalue**	**Multiplicity**
-3	$\binom{2m}{2}$	-3	$\binom{2m-1}{2}$
$-2, \ldots, m-3$	3	$-2, \ldots, m-4$	3
$m-1$	2	$m-3$	2
$m, \ldots, n-2$	3	$m-1, \ldots, n-2$	3
$2n$	1	$2n$	1

Corollary

The number of spanning trees of $SR(3,n)$ is

$$\begin{cases} \dfrac{32(2n+3)^{\binom{n-1}{2}} \displaystyle\prod_{a=n+2}^{2n+2} a^3}{3(n+1)^2(n+2)(3n+5)^3} & \text{if } n \text{ is odd,} \\[2em] \dfrac{32(2n+3)^{\binom{n-1}{2}} \displaystyle\prod_{a=n+2}^{2n+2} a^3}{3(n+1)(n+2)^2(3n+4)^3} & \text{if } n \text{ is even.} \end{cases}$$

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Conjecture

The graph $SR(d, n)$ *is integral for all* d *and* n.

(Strong evidence — we know "most" of the eigenvalues — but no proof yet.)

Hexagon Vectors

For each interior vertex $\mathbf{v} \in V(3, n)$ (i.e., $v_i > 0$ for all i), the signed characteristic vector of the hexagon centered at $\mathbf{v}$ is an eigenvector with eigenvalue -3.

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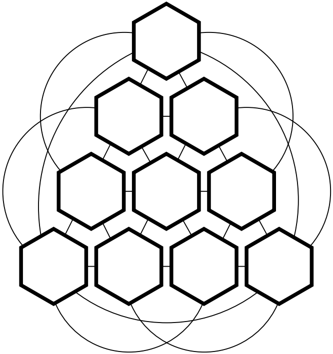

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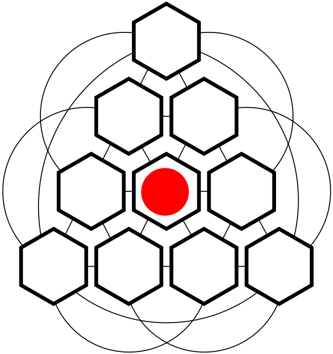

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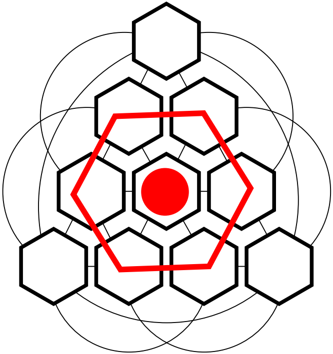

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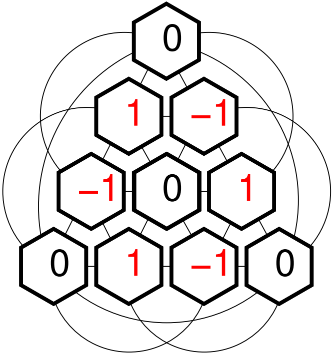

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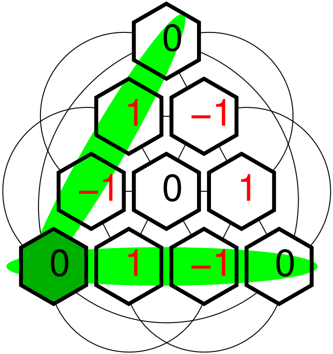

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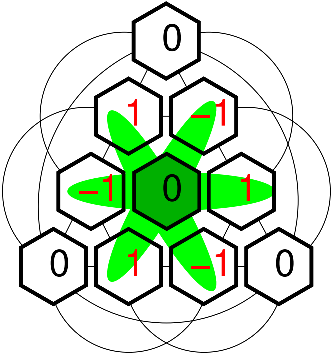

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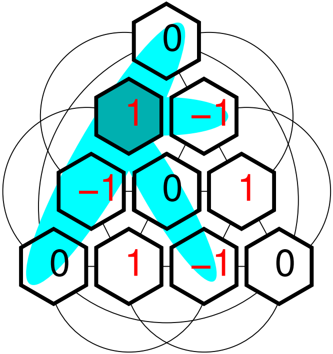

Hexagon Vectors

- Number of possible centers for a hexagon vector = number of interior vertices of $n\Delta^{d-1}$ =

$$\binom{n-1}{2}.$$

- The hexagon vectors are all linearly independent.

- The other $\binom{n+2}{2} - \binom{n-2}{2} = 3v$ eigenvectors have explicit formulas in terms of characteristic vectors of lattice lines (the part that required the most staring at data).

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The good news: Hexagon vectors generalize to permutohedron vectors: linearly independent eigenvectors with eigenvalue $-\binom{d}{2}$. These account for "most" eigenvalues.

The bad news: The remaining eigenvalues and their multiplicities are much more obscure. Someone in the audience should figure out the pattern!

Permutohedron Vectors in $G(d, n)$

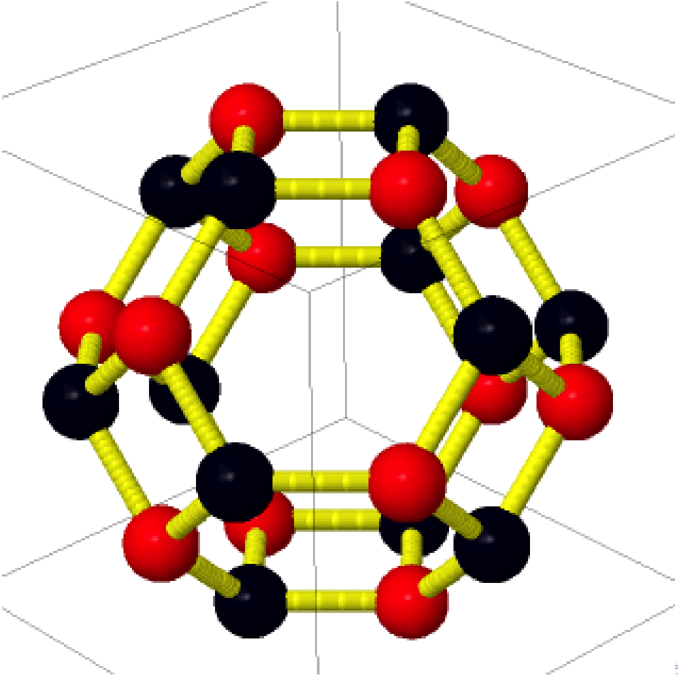
Definition

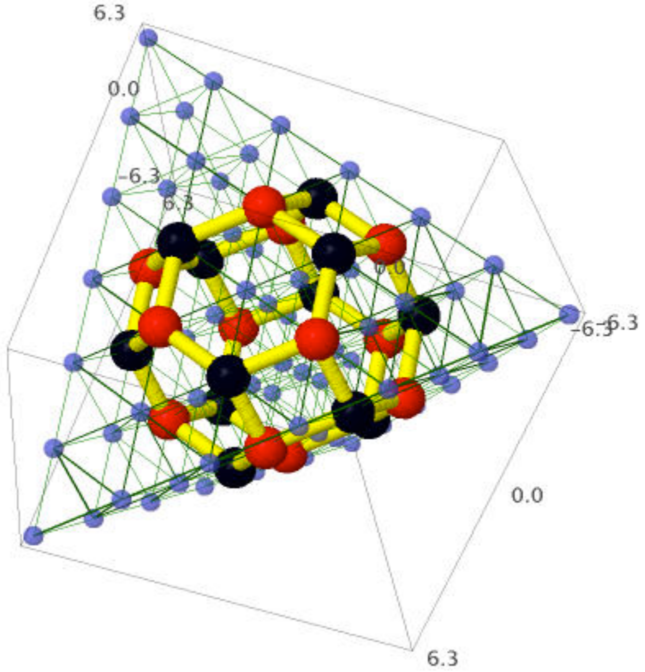
A lattice permutohedron is a polytope in $\mathbb{R}^d$ with vertices

$$\{\mathbf{p} + \sigma \colon \sigma \in \mathfrak{S}_d\}$$

where $\mathbf{p} \in \mathbb{Z}^d$ and $\mathfrak{S}_d$ is the symmetric group (with elements regarded as vectors of length d).

"Most" eigenvectors of $SR(d, n)$ are signed characteristic vectors $\mathcal{H}_{\mathbf{p}}$ of lattice permutohedra inscribed in the simplex $n\Delta^{d-1}$.




6.3
0.0
-6.3
6.3
0.0
-6.3
-6.3
0.0
6.3

Permutohedron Eigenvectors

For each permutohedron P with vertices in $SR(d,n)$, let H_P be its signed characteristic vector:

$$H_P = \sum_{\sigma \in \mathfrak{S}_d} \epsilon(\sigma) \mathbf{e}_{\mathbf{p}+\sigma}$$

- Each H_P is an eigenvalue of $A(d,n)$ with eigenvalue $-\binom{d}{2}$
- The H_P are linearly independent
- Permutohedron vectors account for "most" eigenvectors:

$$\lim_{n\to\infty} \frac{\#\{\mathbf{p} \colon \operatorname{Per}(\mathbf{p}) \subset V(SR(d,n))\}}{|V(SR(d,n))|} = \lim_{n\to\infty} \frac{\binom{n-\binom{d-1}{2}}{d-1}}{\binom{n+d-1}{d-1}} = 1.$$

The Case $n < \binom{d}{2}$

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Multiplicity of smallest eigenvalue = **Mahonian number** $M(d,n)$
= number of permutations in $\mathfrak{S}_d$ with n inversions
= coefficient of q^n in $(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{d-1})$

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Construction uses (ordinary, non-simplicial) rook theory.

The Case $n < \binom{d}{2}$

Permutation $\pi \in \mathfrak{S}_d$ with n inversions

$\rightsquigarrow$ **inversion word** $\mathbf{a} = (a_1, \ldots, a_d)$, where
$a_i = \#\{j \in [d] : \pi_i > \pi_j\}$ (note: $\sum a_i = n$)

$\rightsquigarrow$ **skyline Ferrers board** with column heights given by $\mathbf{a} + \pi$

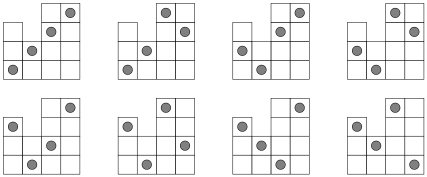
$\rightsquigarrow$ **eigenvector** $X_\pi = \sum_{\sigma \in R(\pi)} \varepsilon(\sigma)\, \mathbf{e}_{\pi + \mathbf{a} - \sigma}$

where $R(\pi)$ = set of maximal rook placements on $\mathbf{a} + \pi$

- Proof that X_π is an eigenvector: sign-reversing involution moving rooks around

The Case $n < \binom{d}{2}$

Example: $d = 4$, $\pi = 3142$, $\mathbf{a} = (2, 0, 1, 0)$, $\mathbf{a} + \pi = (3, 2, 4, 4)$



$$X_\pi = \mathbf{e}_{2010} - \mathbf{e}_{2001} - \mathbf{e}_{1110} + \mathbf{e}_{1101} - \mathbf{e}_{0120} + \mathbf{e}_{0102} + \mathbf{e}_{0030} - \mathbf{e}_{0003}$$

Open Problems

- (The big one.) Prove that $SR(d, n)$ has integral spectrum for all d, n. (Verified for lots of d, n.)

- The induced subgraphs

$$SR(d, n)|_{V(d,n)\cap P},$$

where P is a lattice permutohedron, also appear to be Laplacian integral for all $d, n, \mathbf{p}$. (Verified for $d \leq 6$.)

- Is $A(d, n)$ determined up to isomorphism by its spectrum? (We don't know.)

Thank you!

Preprint: `arxiv:1209.3493`
Slides + Sage: `http://www.math.ku.edu/~jmartin`