

Commutative Algebra

Due date: Friday, 18/12/2009, 14h00

Exercise 28: Let M be an R -module and $\varphi : M \rightarrow M$ an R -linear map. Show:

- If M is noetherian and φ is surjective, then φ is an isomorphism.
- If M is artinian and φ is injective, then φ is an isomorphism.

Hint, consider the kernel respectively cokernel of φ^n for $n \in \mathbb{N}$.

Exercise 29: Which of the following rings R_i is noetherian?

- $R_1 = \left\{ \frac{g}{h} \in \text{Quot}(\mathbb{C}[x]) \mid h(z) \neq 0 \text{ for } |z| = 1 \right\}$.
- $R_2 = \left\{ f \in \mathbb{C}\{x\} \mid f \text{ has infinite radius of convergence} \right\}$.
- $R_3 = \left\{ f \in \mathbb{C}[x] \mid \frac{\partial^i f}{\partial x^i}(0) = 0 \text{ for } i = 1, \dots, k \right\}$, k fixed.

Exercise 30: Let $\mathbb{Q} \subseteq K$ be a field extension such that $\dim_{\mathbb{Q}}(K) < \infty$, and suppose R is a subring of K containing \mathbb{Z} such that $I \cap \mathbb{Z} \neq \{0\}$ for each ideal $0 \neq I \trianglelefteq R$. Show that R is noetherian.

Hint, show first that $\dim_{\mathbb{Z}/p\mathbb{Z}}(R/pR) \leq \dim_{\mathbb{Q}}(K)$ for any prime number p . Then conclude that for $0 \neq m \in I \cap \mathbb{Z}$ the set R/mR (and hence I/mR) is finite by induction on the number of prime factors of $m = p_1 \cdots p_k$, p_i prime number. – Remark: using a bit field theory one can show that the assumption $I \cap \mathbb{Z} \neq \{0\}$ is always fulfilled.

Exercise 31: Let $R \subseteq R' \subseteq R''$ be rings, $R'' = R[a_1, \dots, a_n]$ a finitely generated R -algebra and R'' finitely generated as an R' -module. Show, if R noetherian, then R' is finitely generated as an R -algebra and noetherian.

Recall, $R[a_1, \dots, a_n] = \{f(a_1, \dots, a_n) \mid f \in R[x_1, \dots, x_n]\}$ is the set of all polynomial expressions in a_1, \dots, a_n with coefficients in R . Hint, if $R'' = \langle b_1, \dots, b_m \rangle_{R'}$, then write a_i and $b_i \cdot b_j$ as linear combinations of the b_v and consider the R -algebra generated by the coefficients.

In-Class Exercise 17: Show that $\mathbb{C}[x, y]/I$ with $I = \langle x^3 - x^2, x^2y + 2x^2, xy - y, y^2 + 2y \rangle$ is an artinian ring and decompose it as a direct sum of two local artinian rings.

In-Class Exercise 18: Is $K(x) = \text{Quot}(K[x])$ a noetherian $K[x]_{\langle x \rangle}$ -module?