Tropical Elliptic Curves and their j-Invariant

(joint work with Eric Katz and Hannah Markwig)

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15th February, 2008

Main Result

"The tropical j-invariant is the tropicalisation of the j-invariant."

Elliptic Curves
Tropical Curves
Main Result

1. Plane Cubics

Main Object of Interest

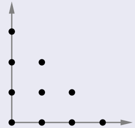
A plane curve of degree 3 with equation

$$F = a_{30} \cdot x^3 + a_{21} \cdot x^2 y + a_{12} \cdot x y^2 + a_{03} \cdot y^3$$
$$+ a_{20} \cdot x^2 + a_{11} \cdot xy + a_{02} \cdot y^2 + a_{10} \cdot x + a_{01} \cdot y + a_{00}$$

where the coefficients a_{ij} belong to some field K.

Notation

$$\mathcal{A} = \{(i,j) \mid a_{ij} \neq 0\} = \mathrm{supp}(F)$$
$$\underline{a} = (a_{ij} \mid (i,j) \in \mathcal{A})$$
$$C_F = \{(X,Y) \in K^2 \mid F(X,Y) = 0\}$$



Tropicalisation

Definition

On $(\mathbb{K}^*)^2$ we have the tropicalisation

$$\mathrm{Trop} : (\mathbb{K}^*)^2 \longrightarrow \mathbb{R}^2 : (X, Y) \mapsto \big(\mathrm{ord}(X), \mathrm{ord}(Y)\big)$$

and thus for $F \in \mathbb{K}[x, y]$ we have

$$\mathcal{T}_F = \mathrm{Trop}\left(C_F \cap (\mathbb{K}^*)^2\right) \subset \mathbb{R}^2.$$

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Problem

- Trop forgets an awful lot of information!
- The definition is not too helpful to compute $\mathcal{T}_F$.

Example (continued)

Find the vertices of the tropical curve $\mathcal{T}_F$ with

$$\text{trop}(F) = \max\{3x, 3y, x + y + 1, 0\}!$$

- $3x = 3y = x + y + 1 \geq 0 \quad \rightsquigarrow \quad (x, y) = (1, 1).$

Example (continued)

Find the vertices of the tropical curve $\mathcal{T}_F$ with

$$\mathrm{trop}(F) = \max\{3x, 3y, x+y+1, 0\}!$$

- $3x = 3y = x + y + 1 \geq 0 \rightsquigarrow (x, y) = (1, 1).$
- $3x = x + y + 1 = 0 \geq 3y \rightsquigarrow (x, y) = (0, -1).$
- $3y = x + y + 1 = 0 \geq 3x \rightsquigarrow (x, y) = (-1, 0).$
- $3x = 3y = 0 \geq x + y + 1 \rightsquigarrow \emptyset.$

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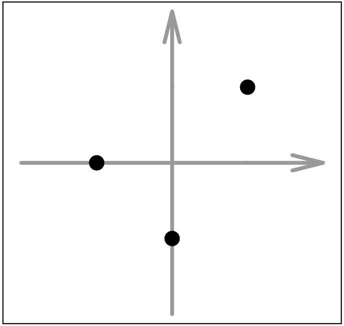

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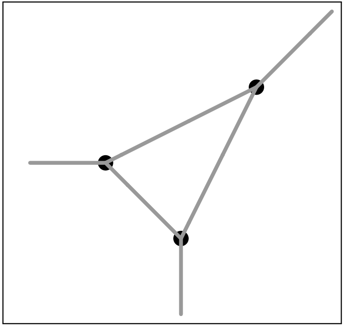

Main Result

Theorem

When $\deg(F) = 3$, $g(\mathcal{T}_F) = 1$ and $u_{ij} = \operatorname{ord}(a_{ij})$, then

$$j(\mathcal{T}_F) = -ord_u(j) \leq -\operatorname{ord}\left(j(C_F)\right).$$

If u lies in a full dimensional cone of the secondary fan of $\mathcal{A}$, then

$$j(\mathcal{T}_F) = -\operatorname{ord}\left(j(C_F)\right).$$

Corollary

If $\deg(F) = 3$ and $\operatorname{ord}\left(j(C_F)\right) \geq 0$, then $\mathcal{T}_F$ has no cycle.