Spring School on Multi-Time Wave Functions

Exercise Session 1

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Exercise 1. *No-go theorem for potentials in multi-time equations with Laplacians*

Consider the multi-time system

$$\begin{aligned} i\partial_{t_1}\psi &= (-\Delta_1 + V_1(\mathbf{x}_1,\mathbf{x}_2))\psi, \\ i\partial_{t_2}\psi &= (-\Delta_2 + V_2(\mathbf{x}_1,\mathbf{x}_2))\psi \end{aligned} \tag{1}$$

for a multi-time wave function $\psi : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{C}$. Here, Δ_i denotes the Laplacian with respect to $\mathbf{x}_i$, $i = 1,2$ and $V_1, V_2 : \mathbb{R}^6 \to \mathbb{R}$ are smooth functions.

(a) State the appropriate consistency condition.

(b) Show that this consistency condition is only satisfied if V_1 does not depend on $\mathbf{x}_2$ and V_2 does not depend on $\mathbf{x}_1$.

Exercise 2. *Space-like configurations*

Consider the case of $N = 2$ particles. We denote the set of space-like configurations (including collision configurations) by

$$\mathscr{S} = \{(x_1,x_2) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x_1^0 - x_2^0| < |\mathbf{x}_1 - \mathbf{x}_2| \text{ or } x_1^0 = x_2^0, \mathbf{x}_1 = \mathbf{x}_2\}. \tag{2}$$

Show that $\mathscr{S}$ is the smallest Poincaré invariant set which contains the equal-time configurations

$$\mathscr{E} = \{(x_1,x_2) \in \mathbb{R}^4 \times \mathbb{R}^4 : x_1^0 = x_2^0\}. \tag{3}$$

Exercise 3. *Multi-time equations for ϕ^4 theory*

ϕ^4 theory is a quantum field theory model in which the Heisenberg field operators $\phi(x)$ obey the evolution equation

$$(\Box + m^2)\phi(x) = \phi^3(x). \tag{4}$$

Use this equation and the expression of multi-time wave functions via field operators,

$$\psi^{(n)}(x_1,...,x_n) = \frac{1}{\sqrt{n!}}\langle 0|\phi(x_1)\cdots\phi(x_n)|\psi_H\rangle, \tag{5}$$

to derive multi-time equations for $\psi^{(n)}$. (These equations should only contain $\psi^{(m)}$ for different values for m, not any field operators.)

Exercise 4. *Continuity equation from Dirac equation*

Derive the continuity equation $\partial_\mu j^\mu = 0$ from the Dirac equation $i\gamma^\mu \partial_\mu \psi = m\psi$ and the definition $j^\mu = \overline{\psi}\gamma^\mu\psi$.

Hint: Use that the adjoint of γ^μ is $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$, as can be verified in (e.g.) the standard representation

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

with $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Generalization: Suppose that $\psi : \mathbb{R}^{4N} \to (\mathbb{C}^4)^{\otimes N}$ satisfies the free multi-time Dirac equations $i\gamma_j^\mu \partial_{x_j^\mu}\psi = m\psi$, where γ_j^μ is γ^μ acting on s_j. Let $\overline{\psi} = \psi^\dagger \gamma_1^0 \cdots \gamma_N^0$ and

$$j^{\mu_1 \dots \mu_N}(x_1 \dots x_N) = \overline{\psi}(x_1 \dots x_N)\, \gamma_1^{\mu_1} \cdots \gamma_N^{\mu_N}\, \psi(x_1 \dots x_N).$$

Show that $\partial_{x_j^{\mu_j}} j^{\mu_1 \dots \mu_N}(x_1 \dots x_N) = 0$ for all $j = 1 \dots N$.

Exercise 5. *Creation and Annihilation Operators*

Let us consider the scalar bosonic creation and annihilation operators defined by

$$\begin{aligned} (a(\mathbf{x})\varphi)(\mathbf{x}_1, \dots, \mathbf{x}_N) &= \sqrt{N+1}\, \varphi(\mathbf{x}_1, \dots, \mathbf{x}_N, \mathbf{x}), \\ (a^\dagger(\mathbf{x})\varphi)(\mathbf{x}_1, \dots, \mathbf{x}_N) &= \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \delta^{(3)}(\mathbf{x}_j - \mathbf{x})\varphi(\mathbf{x}_1, \dots, \widehat{\mathbf{x}}_j, \dots, \mathbf{x}_N), \end{aligned} \tag{6}$$

where $\widehat{(\cdot)}$ denotes omission.
Show that for any operator $H : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ we have that

$$\int_{\mathbb{R}^3} d^3\mathbf{x}\, a^\dagger(\mathbf{x}) H_{\mathbf{x}} a(\mathbf{x}) \varphi(\mathbf{x}_1, \dots, \mathbf{x}_N) = \sum_{j=1}^{N} H_{\mathbf{x}_j} \varphi(\mathbf{x}_1, \dots, \mathbf{x}_N). \tag{7}$$

Exercise 6. *Poincaré invariant interaction potential in multi-time Dirac equations*

Consider the Poincaré invariant multi-time equations

$$\left(i\gamma_k^\mu \partial_{x_k^\mu} - m_k - \frac{e^2}{2\sqrt{|(x_1 - x_2)^2|}} \right) \psi(x_1, x_2) = 0, \quad k = 1, 2, \tag{8}$$

where $(x_1 - x_2)^2 = (x_1^0 - x_2^0)^2 - |\mathbf{x}_1 - \mathbf{x}_2|^2$.

(a) Demonstrate that the single-time wave function $\varphi(t, \mathbf{x}_1, \mathbf{x}_2) = \psi(t, \mathbf{x}_1, t, \mathbf{x}_2)$ satisfies a Schrödinger-like equation with a potential $\propto \frac{e^2}{|\mathbf{x}_1 - \mathbf{x}_2|}$.

(b) Write down the appropriate consistency condition for (8).

(c) Show through an explicit calculation that the consistency condition is violated.

Exercise 7. *Probability conservation on space-like hypersurfaces*

Let $N \in \mathbb{N}$ and $\psi \in C^1(\mathbb{R}^{4N}, \mathbb{C}^{4^N})$ be a solution of the free multi-time Dirac equations $(i\gamma_k^\mu \partial_{x_k^\mu} - m_k)\psi = 0,\ k = 1, ..., N$ which is compactly supported in space for all fixed time variables. For every smooth space-like hypersurface Σ with future-pointing unit normal vector field n, we define

$$P(\Sigma) = \int_\Sigma d\sigma(x_1) \cdots \int_\Sigma d\sigma(x_N)\ \overline{\psi}(x_1, ..., x_N)\, \not{n}_1(x_1) \cdots \not{n}_N(x_N)\, \psi(x_1, ..., x_N).$$

(a) Show that $P(\Sigma) = P(\Sigma')$ for all pairs of smooth space-like hypersurfaces Σ, Σ'. *Hint:* Apply the Gauss integral theorem to the volume between Σ and Σ', with a limit of mantle surfaces moving to spacelike infinity.

(b) Let ψ, ϕ be two solutions of the same initial value problem $\psi|_{\Sigma_0^N} = \phi|_{\Sigma_0^N} = \psi_0$ for some given function $\psi_0 \in C_c^\infty(\Sigma_0^N, \mathbb{C}^{4^N})$. Show that (a) implies $\psi|_{\Sigma^N} = \phi|_{\Sigma^N}$ for all smooth spacelike hypersurfaces Σ.
Hint: You can use that $\overline{\psi}(x_1, ..., x_N)\, \not{n}_1(x_1) \cdots \not{n}_N(x_N)\psi(x_1, ..., x_N) \geq 0$ for all future-pointing time-like or light-like vector fields n.

Exercise 8. *Finite Propagation Speed (Domain of Dependence)*

(a) Consider the 4-volume C depicted in a 2-dimensional way in Figure 1. C is the volume enclosed by Σ_0, Σ_t, and Σ^s. Let $j : \mathbb{R}^4 \to \mathbb{R}^4$ be a continuously differentiable vector field. Taking $\mathbb{R}^4$ as a coordinate space with Euclidean metric, what are the outward unit normal vectors for Σ_0, Σ_t, and Σ^s? Then, write out explicitly the 4-dimensional Gauss integral theorem for $\int_C d^4x \, \mathrm{div}_4(j)$.

(b) Consider the one-particle Dirac equation $i\gamma^\mu \partial_\mu \psi = \big(m + V(\mathbf{x})\big)\psi$ with smooth self-adjoint external potential $V \in C^\infty(\mathbb{R}^3, \mathbb{C}^{4\times 4})$. For smooth initial data $\psi_0 \in C^\infty(\mathbb{R}^3, \mathbb{C}^4)$ it is known that there is a unique smooth solution $\psi \in C^\infty(\mathbb{R}^4, \mathbb{C}^4)$. We denote the open ball with radius r around $\mathbf{y}$ by $B_r(\mathbf{y}) := \big\{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x} - \mathbf{y}| < r\big\}$. Prove that $\psi(t, \mathbf{x})$ for $\mathbf{x} \in B_{T-t}(\mathbf{y})$ is uniquely determined by specifying the initial conditions on $B_T(\mathbf{y})$.
Hint: Because of linearity, it suffices to consider $\psi(0, \mathbf{x}) = 0$. Use $\partial_\mu j^\mu = 0$ and part (a).

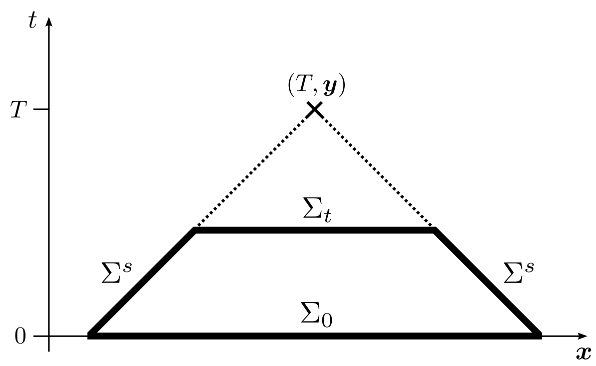

Figure 1: Σ_0 and Σ_t are parts of equal time hypersurfaces, Σ^s is part of the past light cone of $(T, \mathbf{y})$. Σ_0, Σ_t and Σ^s enclose a volume in $\mathbb{R}^4$, a truncated cone.