Exercise 1. No-go theorem for potentials in multi-time equations with Laplacians

Consider the multi-time system

\[ i \partial_{t_1} \psi = (-\Delta_1 + V_1(x_1, x_2)) \psi, \]
\[ i \partial_{t_2} \psi = (-\Delta_2 + V_2(x_1, x_2)) \psi \]  

for a multi-time wave function \( \psi : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{C} \). Here, \( \Delta_i \) denotes the Laplacian with respect to \( x_i \), \( i = 1, 2 \) and \( V_1, V_2 : \mathbb{R}^6 \to \mathbb{R} \) are smooth functions.

(a) State the appropriate consistency condition.

(b) Show that this consistency condition is only satisfied if \( V_1 \) does not depend on \( x_2 \) and \( V_2 \) does not depend on \( x_1 \).

Exercise 2. Space-like configurations

Consider the case of \( N = 2 \) particles. We denote the set of space-like configurations (including collision configurations) by

\[ \mathcal{S} = \{(x_1, x_2) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x_1^0 - x_2^0| < |x_1 - x_2| \text{ or } x_1^0 = x_2^0, x_1 = x_2\}. \]

Show that \( \mathcal{S} \) is the smallest Poincaré invariant set which contains the equal-time configurations

\[ \mathcal{E} = \{(x_1, x_2) \in \mathbb{R}^4 \times \mathbb{R}^4 : x_1^0 = x_2^0\}. \]

Exercise 3. Multi-time equations for \( \phi^4 \) theory

\( \phi^4 \) theory is a quantum field theory model in which the Heisenberg field operators \( \phi(x) \) obey the evolution equation

\[ (\Box + m^2)\phi(x) = \phi^3(x). \]

Use this equation and the expression of multi-time wave functions via field operators,

\[ \psi^{(n)}(x_1, ..., x_n) = \frac{1}{\sqrt{n!}} \langle 0|\phi(x_1) \cdots \phi(x_n)|\psi_H \rangle, \]

to derive multi-time equations for \( \psi^{(n)} \). (These equations should only contain \( \psi^{(m)} \) for different values for \( m \), not any field operators.)
Exercise 4. Continuity equation from Dirac equation

Derive the continuity equation \( \partial_\mu j^\mu = 0 \) from the Dirac equation \( i\gamma_\mu \partial_\mu \psi = m\psi \) and the definition \( j^\mu = \bar{\psi} \gamma^\mu \psi \).

Hint: Use that the adjoint of \( \gamma^\mu \) is \( \gamma^\mu = \gamma^0 \gamma_\mu \gamma^0 \), as can be verified in (e.g.) the standard representation

\[
\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}
\]

with \( I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and Pauli matrices

\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Generalization: Suppose that \( \psi : \mathbb{R}^{4N} \to (\mathbb{C}^4)^\otimes N \) satisfies the free multi-time Dirac equations \( i\gamma_j^\mu \partial_\mu \psi = m\psi \), where \( \gamma_j^\mu \) is \( \gamma^\mu \) acting on \( s_j \). Let \( \psi = \psi^\dagger \gamma^0_1 \cdots \gamma^0_N \) and

\[
j^{\mu_1 \cdots \mu_N}(x_1 \ldots x_N) = \overline{\psi}(x_1 \ldots x_N) \gamma^{\mu_1} \cdots \gamma^{\mu_N} \psi(x_1 \ldots x_N).
\]

Show that \( \partial_{x^\mu_j} j^{\mu_1 \cdots \mu_N}(x_1 \ldots x_N) = 0 \) for all \( j = 1 \ldots N \).

Exercise 5. Creation and Annihilation Operators

Let us consider the scalar bosonic creation and annihilation operators defined by

\[
(a(x)\varphi)(x_1, \ldots, x_N) = \sqrt{N+1} \varphi(x_1, \ldots, x_N, x),
\]

\[
(a^\dagger(x)\varphi)(x_1, \ldots, x_N) = \frac{1}{\sqrt{N}} \sum_{j=1}^N \delta^{(3)}(x_j - x) \varphi(x_1, \ldots, \hat{x}_j, \ldots, x_N), \tag{6}
\]

where \( \hat{\cdot} \) denotes omission.

Show that for any operator \( H : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \) we have that

\[
\int_{\mathbb{R}^3} d^3x a^\dagger(x) H_x a(x) \varphi(x_1, \ldots, x_N) = \sum_{j=1}^N H_{x_j} \varphi(x_1, \ldots, x_N). \tag{7}
\]

Exercise 6. Poincaré invariant interaction potential in multi-time Dirac equations

Consider the Poincaré invariant multi-time equations

\[
\left( i\gamma_k^\mu \partial_\mu - m_k - \frac{e^2}{2\sqrt{|(x_1 - x_2)^2|}} \right) \psi(x_1, x_2) = 0, \quad k = 1, 2, \tag{8}
\]

where \((x_1 - x_2)^2 = (x_1^0 - x_2^0)^2 - |x_1 - x_2|^2\).

(a) Demonstrate that the single-time wave function \( \varphi(t, x_1, x_2) = \psi(t, x_1, t, x_2) \) satisfies a Schrödinger-like equation with a potential \( \propto \frac{e^2}{|x_1 - x_2|} \).

(b) Write down the appropriate consistency condition for (8).

(c) Show through an explicit calculation that the consistency condition is violated.
Exercise 7. Probability conservation on space-like hypersurfaces

Let $N \in \mathbb{N}$ and $\psi \in C^1(\mathbb{R}^{4N}, \mathbb{C}^{4N})$ be a solution of the free multi-time Dirac equations $(i\gamma^\mu \partial_{x^\mu} - m_k)\psi = 0$, $k = 1, \ldots, N$ which is compactly supported in space for all fixed time variables. For every smooth space-like hypersurface $\Sigma$ with future-pointing unit normal vector field $n$, we define

$$P(\Sigma) = \int_{\Sigma} \sigma(x_1) \cdots \int_{\Sigma} \sigma(x_N) \bar{\psi}(x_1, \ldots, x_N) \psi_1(x_1) \cdots \psi_N(x_N) \psi(x_1, \ldots, x_N).$$

(a) Show that $P(\Sigma) = P(\Sigma')$ for all pairs of smooth space-like hypersurfaces $\Sigma, \Sigma'$. Hint: Apply the Gauss integral theorem to the volume between $\Sigma$ and $\Sigma'$, with a limit of mantle surfaces moving to spacelike infinity.

(b) Let $\psi, \phi$ be two solutions of the same initial value problem $\psi|_{\Sigma_0} = \phi|_{\Sigma_0} = \psi_0$ for some given function $\psi_0 \in C^\infty_c(\Sigma_0^N, \mathbb{C}^{4N})$. Show that (a) implies $\psi|_{\Sigma_N} = \phi|_{\Sigma_N}$ for all smooth spacelike hypersurfaces $\Sigma$. Hint: You can use that $\bar{\psi}(x_1, \ldots, x_N) \psi_1(x_1) \cdots \psi_N(x_N) \psi(x_1, \ldots, x_N) \geq 0$ for all future-pointing time-like or light-like vector fields $n$. 

Exercise 8. Finite Propagation Speed (Domain of Dependence)

(a) Consider the 4-volume $C$ depicted in a 2-dimensional way in Figure 1. $C$ is the volume enclosed by $\Sigma_0$, $\Sigma_t$, and $\Sigma^s$. Let $j : \mathbb{R}^4 \to \mathbb{R}^4$ be a continuously differentiable vector field. Taking $\mathbb{R}^4$ as a coordinate space with Euclidean metric, what are the outward unit normal vectors for $\Sigma_0$, $\Sigma_t$, and $\Sigma^s$? Then, write out explicitly the 4-dimensional Gauss integral theorem for $\int_C d^4x \text{div}_4(j)$.

(b) Consider the one-particle Dirac equation $i\gamma^\mu \partial_\mu \psi = (m + V(x)) \psi$ with smooth self-adjoint external potential $V \in C^\infty(\mathbb{R}^3, \mathbb{C}^{4 \times 4})$. For smooth initial data $\psi_0 \in C^\infty(\mathbb{R}^3, \mathbb{C}^4)$ it is known that there is a unique smooth solution $\psi \in C^\infty(\mathbb{R}^4, \mathbb{C}^4)$. We denote the open ball with radius $r$ around $y$ by $B_r(y) := \{ x \in \mathbb{R}^3 : |x - y| < r \}$. Prove that $\psi(t, x)$ for $x \in B_{T-t}(y)$ is uniquely determined by specifying the initial conditions on $B_T(y)$. 

Hint: Because of linearity, it suffices to consider $\psi(0, x) = 0$. Use $\partial_\mu j^\mu = 0$ and part (a).

Figure 1: $\Sigma_0$ and $\Sigma_t$ are parts of equal time hypersurfaces, $\Sigma^s$ is part of the past light cone of $(T, y)$. $\Sigma_0$, $\Sigma_t$ and $\Sigma^s$ enclose a volume in $\mathbb{R}^4$, a truncated cone.