Spring School on Multi-Time Wave Functions

Exercise Session 2

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Exercise 1. *Gauge transformations*

Consider the multi-time equations

$$i\partial_{t_k}\psi = (H_k^0 + V_k)\psi \tag{1}$$

for $j = 1, \ldots, N$, where H_k^0 is the free Dirac Hamiltonian $H_k^0 = -i\sum_{a=1}^3 \gamma_k^0\gamma_k^a\partial_{x_k^a} + \gamma_k^0 m_k$, and $V_k \in C^\infty(\mathbb{R}^{4N}, \mathbb{R})$. For some $\theta \in C^\infty(\mathbb{R}^{4N}, \mathbb{R})$ we define the gauge transformation

$$\widetilde{\psi}(x_1, \ldots, x_N) = e^{i\theta(x_1,\ldots,x_N)}\psi(x_1, \ldots, x_N). \tag{2}$$

Compute which equations $\widetilde{\psi}$ satisfies.

Note: The result will show that some potentials in the Dirac equation do not really lead to interaction but are only due to gauge effects.

Exercise 2. *Derivation of a potential from a multi-time integral equation*

Consider the following multi-time integral equation in 1+1 dimensions:

$$\psi(t_1, z_1, t_2, z_2) = \psi^{\text{free}}(t_1, z_1, t_2, z_2) + \int_{\mathbb{R}^4} dt_1'\, dz_1'\, dt_2'\, dz_2'\; G_1(t_1 - t_1', z_1 - z_1') \\ \times G_2(t_2 - t_2', z_2 - z_2') H(-(t_1' - t_2')^2 + |z_1' - z_2'|^2)\psi(t_1', z_1', t_2', z_2'), \tag{3}$$

where H denotes the Heaviside function. Show that if one treats the product $G_1(t_1 - t_1', z_1 - z_1')G_2(t_2 - t_2', z_2 - z_2')\psi(t_1', z_1', t_2', z_2')$ as constant in the time interval $t_2' \in [t_1' - |z_1' - z_2'|, t_1' + |z_1' - z_2'|]$, the single-time wave function $\varphi(t, z_1, z_2) = \psi(t, z_1, t, z_2)$ satisfies an integral version of the Schrödinger equation with potential $V(t, z_1, z_2) \propto |z_1 - z_2|$.

Exercise 3. *(PL) $\Rightarrow$ (NCFV)*

Explain why for any hypersurface evolution $(\mathscr{H}_\Sigma, P_\Sigma, U_\Sigma^{\Sigma'})_{\Sigma,\Sigma'}$, propagation locality (PL) implies there is no creation from the vacuum (NCFV). These properties are defined as follows.

(PL) Whenever ψ_Σ is concentrated in $A \subseteq \Sigma$, then $\psi_{\Sigma'} = U_\Sigma^{\Sigma'}\psi_\Sigma$ is concentrated in $\mathrm{Gr}(A, \Sigma')$.

(Here, ψ is *concentrated* in A if it vanishes for all configurations with any particle outside of A, and $\mathrm{Gr}(A, \Sigma') = [\text{future}(A) \cup \text{past}(A)] \cap \Sigma'$.)

(NCFV) $U_\Sigma^{\Sigma'}\, P_\Sigma(\{\emptyset\})\, U_{\Sigma'}^{\Sigma} = P_{\Sigma'}(\{\emptyset\})$.

(Here, $P_\Sigma(\{\emptyset\})$ is the projection to the vacuum state in $\mathscr{H}_\Sigma$.)

Exercise 4. *Probability current balance for an IBC model*

Here is an interior-boundary condition (IBC) for particle creation in a non-relativistic model. Suppose an x-particle is fixed at the origin in $\mathbb{R}^3$, and the number N of y-particles can be 0 or 1, so the y-configuration space is $\mathcal{Q} = \{\emptyset\} \cup \mathbb{R}^3$, and the Hilbert space is $\mathscr{H} = \mathbb{C} \oplus L^2(\mathbb{R}^3)$ (a truncated Fock space). On wave functions $\varphi = (\varphi^{(0)}, \varphi^{(1)}) \in \mathscr{H}$ satisfying the IBC

$$\lim_{\mathbf{y}\to\mathbf{0}} \Big(|\mathbf{y}|\, \varphi^{(1)}(\mathbf{y}) \Big) = -\frac{mg}{2\pi}\, \varphi^{(0)}\,, \tag{4}$$

where g is a coupling constant, the Hamiltonian is defined by

$$(H\varphi)^{(0)} = \frac{g}{4\pi} \int\limits_{\mathbb{S}^2} d^2\boldsymbol{\omega}\, \lim_{r\searrow 0} \partial_r \Big(r\varphi^{(1)}(r\boldsymbol{\omega}) \Big) \tag{5}$$

$$(H\varphi)^{(1)}(\mathbf{y}) = -\frac{\hbar^2}{2m} \Delta\varphi^{(1)}(\mathbf{y})\,. \tag{6}$$

Show that then the following balance condition between the loss of probability in the 1-particle sector at $\mathbf{0} \in \mathbb{R}^3$ and the gain in the 0-particle sector due to the source term on the right-hand side of (5) holds:

$$-\lim_{r\searrow 0} \Big(r^2 \int_{\mathbb{S}^2} d^2\boldsymbol{\omega}\, \boldsymbol{\omega} \cdot \mathbf{j}(r\boldsymbol{\omega}) \Big) = \frac{\partial |\varphi^{(0)}|^2}{\partial t}\,. \tag{7}$$

Here, the probability current is defined (as usual) by $\mathbf{j}(\mathbf{y}) = \frac{1}{m} \mathrm{Im}\, \varphi^{(1)}(\mathbf{y})^* \nabla_{\mathbf{y}} \varphi^{(1)}(\mathbf{y})$.
Hint: write $\mathbf{y} = r\boldsymbol{\omega}$ and note that $\boldsymbol{\omega} \cdot \nabla = \partial_r$.

Exercise 5. *Solution of a Volterra integral equation*

Let $T > 0$. Show that for $K \in C([0,T]^2)$ the Volterra integral equation

$$f(t) = f_0(t) + \int_0^t dt'\; K(t,t') f(t') \tag{8}$$

has a unique solution $f(t) \in C([0,T])$ for every $f_0 \in C([0,T])$.

Hint: Let $\widehat{K}$ be the integral operator defined by $\widehat{K} f = \int_0^t dt'\, K(t,t') f(t')$. Show through a direct computation that $f = \sum_{n=0}^{\infty} \widehat{K}^n f_0$ converges in the Banach space $\mathscr{B} = C([0,T])$ equipped with the supremum norm and solves the equation.

Exercise 6. *Consistency condition for the emission-absorption model*

Recall the emission-absorption model (now with a single x-particle),

$$\begin{aligned} i\partial_{x^0}\psi &= \left(H_x^0 + H_x^{\mathrm{int}}\right)\psi, \\ i\partial_{y_k^0}\psi &= H_{y_k}^0\psi, \end{aligned} \tag{9}$$

where $H^0 = -i\sum_{a=1}^3 \gamma^0\gamma^a\partial_{x^a} + \gamma^0 m$ is the free Dirac Hamiltonian, and

$$\begin{aligned} \left(H_x^{\mathrm{int}}\psi\right)(x, y^{4N}) = \sqrt{N+1} &\sum_{s_{N+1}=1}^{4} g_{s_{N+1}}^* \psi_{s_{N+1}}\big(x, (y^{4N}, x)\big) \\ &+ \frac{1}{\sqrt{N}}\sum_{k=1}^{N} G_{s_k}(y_k - x)\,\psi_{\widehat{s}_k}\big(x, y^{4N}\setminus y_k\big), \end{aligned} \tag{10}$$

with some function $G : \mathbb{R}^4 \to \mathbb{C}^4$ and $g \in \mathbb{C}^4$. Here, we have assumed that there is only one x-particle. Carefully compute the consistency condition

$$\left[i\partial_{x^0} - H_x, i\partial_{y_k^0} - H_{y_k}\right] = 0. \tag{11}$$

What is a good choice for G to fulfill the condition?

Exercise 7. *Consistent interaction potentials*

Consider the multi-time equations

$$\begin{aligned} i\partial_{t_1}\psi &= (H_1^0 + V_1)\psi, \\ i\partial_{t_2}\psi &= (H_2^0 + V_2)\psi, \end{aligned} \tag{12}$$

where H_k^0 is the free Dirac Hamiltonian $H_k^0 = -i\sum_{a=1}^3 \gamma_k^0\gamma_k^a\partial_{x_k^a} + \gamma_k^0 m_k$. We now consider the matrix valued potentials

$$\begin{aligned} V_1(x_1, x_2) &= \gamma_1^\mu C_\mu \exp\left(2i\gamma_1^5 c_\lambda(x_2^\lambda - x_1^\lambda)\right) - m_1\gamma_1^0, \\ V_2 &= \gamma_1^5\gamma_2^0\gamma_2^\nu c_\nu, \end{aligned} \tag{13}$$

for arbitrary non-zero $c, C \in \mathbb{C}^4$ and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Then $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ for $\mu = 0, 1, 2, 3$. Show that the multi-time equations (12) are consistent.

Note: It can be shown that these multi-time equations are indeed interacting, i.e., there is no gauge transformation which makes them non-interacting. However, they are not Poincaré invariant.