1 Relativistic space-time

$\mathcal{M} = \mathbb{R}^4$ Minkowski space-time [with points] $x = x^\mu = (x^0, \mathbf{x}) = (ct, \mathbf{x})$, $c = 1$, metric

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}. \tag{1}$$

[picture: light cone, sets of timelike/spacelike/lightlike = null vectors]

Lorentz group $\mathcal{L} = O(1, 3) \ [\text{set of all Lorentz transformations including rotations in space}]$, rotation group $= SO(3)$.

Restricted Lorentz group $\mathcal{L}^+ \subset \mathcal{L} \ [\text{contains those that don’t reverse either time or space}]$

Poincaré group

$$\mathcal{P} = \{ x \mapsto a + \Lambda x : a \in \mathcal{M}, \Lambda \in \mathcal{L} \} \tag{2}$$

[includes space-time translations, correspondingly $\mathcal{P}^+$]

2 Dirac equation

Wave function $\psi(t, \mathbf{x}) \in \mathbb{C}^4$ spin space, Dirac eq $(\hbar = 1)$

$$i \frac{\partial \psi}{\partial t} = -i \mathbf{\alpha} \cdot \nabla \psi + \beta m \psi =: H_{\text{Dirac}} \psi \tag{3}$$

or, [equivalently,]

$$i \gamma^\mu \partial_\mu \psi = m \psi \quad \text{or} \quad i \not{\partial} \psi = m \psi. \tag{4}$$

Here, $\beta = \gamma^0$, $\mathbf{\alpha}^i = \gamma^0 \gamma^i$ $(i = 1, 2, 3)$, $\not{\partial} = \gamma^\mu \partial_\mu$.

Clifford relation $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} I$, in particular $\gamma^0 \gamma^0 = I$. 

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Hamiltonian formulation: Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$ with [inner product]
\[
\langle \phi | \psi \rangle = \int_{\mathbb{R}^3} d^3x \, \bar{\phi}^\dagger(x) \psi(x)
\]
with [inner product in spin space]
\[
\bar{\phi}^\dagger \psi = \sum_{s=1}^4 \phi_s^* \psi_s.
\]

There exists a self-adjoint version of $H_{\text{Dirac}} \Rightarrow U_t := e^{-iHt} : \mathcal{H} \rightarrow \mathcal{H}$ is unitary.

Space-time formulation: $\psi : \mathcal{M} = \mathbb{R}^4 \rightarrow \mathbb{C}^4$. Spinors transform under $\Lambda \in \mathcal{L}^+$ [according to]
\[
\pm S(\Lambda) : \mathbb{C}^4 \rightarrow \mathbb{C}^4,
\]
[i.e., $S$ is a (projective) representation of $\mathcal{L}^+$. Thus,]
\[
\psi'(x) = S(\Lambda) \psi(\Lambda^{-1}x).
\]

[If $\Lambda$ is a rotation through angle $\varphi$, then $S(\Lambda)$ is a rotation through angle $\varphi/2$; that’s why it’s called spin-$\frac{1}{2}$.]

**Theorem.** Every $\Lambda \in \mathcal{L}^+$ leaves $\gamma^\mu$ invariant, $\gamma^\mu = (\gamma')^\mu = S(\Lambda) \Lambda_\nu^\mu \gamma^\nu S(\Lambda)^{-1}$. 

**Corollary.** The Dirac eq is Lorentz invariant.

**Remark.** [The inner product] (6) is not Lorentz invariant. However, the following product is:
\[
\bar{\phi} \psi := \bar{\phi}^\dagger \gamma^0 \psi.
\]

**Definition.** For every $\psi : \mathcal{M} \rightarrow \mathbb{C}^4$, the probability current 4-vector field is
\[
j^\mu(x) = \bar{\psi}(x) \gamma^\mu \psi(x).
\]

- $j(x)$ is defined in a covariant way,
- causal = timelike or lightlike, $j^\mu(x) \, j_\mu(x) \geq 0$.
- future-pointing, $j^0(x) \geq 0$.
- $j^0 = \bar{\psi} \gamma^0 \psi = \psi^\dagger \gamma^0 \gamma^0 \psi = \psi^\dagger \psi = \sum_{s=1}^4 |\psi_s|^2 = \rho$ = probability density according to Born’s rule
- $\partial_\mu j^\mu = 0$ (continuity eq $\partial_t \rho = -\text{div}_A j$) [exercise]
• By the Gauss integral theorem, for spacelike hypersurfaces $\Sigma, \Sigma'$,

$$\int_{\Sigma} d^3 \sigma(x) j^\mu(x) n_\mu(x) = \int_{\Sigma'} d^3 \sigma(x) j^\mu(x) n_\mu(x)$$

with $d^3 \sigma(x) = d^3 x \sqrt{\text{det} g(x)}$ is 3-volume defined by metric on $\Sigma$, provided $j^\mu \to 0$ fast enough as $x \to \infty$ spacelike.” [exercise]

• propagation locality: If $\psi(t = 0)$ is concentrated in $A \subset \mathbb{R}^3$ (i.e., $\psi(0, x) = 0$ for all $x \notin A$), then $\psi$ is concentrated in future($\{0\} \times A$) $\cup$ past($\{0\} \times A$). [picture]

[no propagation faster than light]

3 What is a multi-time wave function?

Ordinary wf of QM of $N$ particles

$$\varphi(t, x_1, \ldots, x_N)$$

[evolves according to] Schrödinger eq

$$i \frac{\partial \varphi}{\partial t} = H \varphi \quad \Rightarrow \quad \varphi(t) = e^{-iH t} \varphi(0)$$

with $H = $ Hamiltonian operator.

[Uniquely determined by] initial data $\varphi(0)$ on $\mathbb{R}^{3N}$.

Not covariant: [refers to space-time points] $(t, x_1), \ldots, (t, x_N)$ simultaneous [w.r.t. chosen Lorentz frame; picture]

Multi-time wf [Dirac 1932]

$$\psi(t_1, x_1, \ldots, t_N, x_N) = \psi(x_1, \ldots, x_N)$$

Example 1 (non-interacting). $\psi_{s_1 s_2}(x_1, x_2)$, $\psi : \mathcal{M}^2 \to \mathbb{C}^4 \otimes \mathbb{C}^4$,

$$\psi(t_1, \cdot, t_2, \cdot) = e^{-iH_{t_1} - iH_{t_2}}$$

with $H_j = H_{\text{Dirac}}$ acting on $x_j, s_j$.

[Uniquely determined by] initial data $\psi(0, 0): \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C}^4 \otimes \mathbb{C}^4$.

Obeys multi-time Schrödinger eqs

$$i \frac{\partial \psi}{\partial t_1} = H_1 \psi$$

$$i \frac{\partial \psi}{\partial t_2} = H_2 \psi$$

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Note:
\[ \varphi(t, \mathbf{x}_1, \mathbf{x}_2) := \psi(t, \mathbf{x}_1, t, \mathbf{x}_2) \]  
(18)
obeys [by the chain rule]
\[ i \frac{\partial \varphi}{\partial t} = i \frac{\partial \psi}{\partial t_1} \bigg|_{(t, t)} + i \frac{\partial \psi}{\partial t_2} \bigg|_{(t, t)} = (H_1 + H_2) \varphi, \]  
(19)
so \( H = H_1 + H_2 \) (non-interacting).
Remark: works also for non-relativistic Schrödinger eq. \( H_j = -\frac{1}{2m_j} \Delta_j \)
Likewise for \( N \) particles.
Challenge: include interaction. (Lectures 2–7)

**Example 2 (second-order equations).** [Also possible] \( \psi : \mathcal{M}^2 \to \mathbb{C} \),
\[ \Box_1 \psi = m_1^2 \psi \]  
(20)
\[ \Box_2 \psi = m_2^2 \psi \]  
(21)
with \( \Box = \partial^\mu \partial_\mu = \partial_t^2 - \Delta \) d’Alembertian.

**Example 3 (if you know QFT).** Field operators \( \Phi(t, \mathbf{x}) = e^{iHt} \Phi(0, \mathbf{x}) e^{-iHt} \) (Heisenberg picture) on Fock space \( \mathcal{H} \),
\[ \psi(x_1...x_N) := \frac{1}{\sqrt{N!}} \langle \emptyset | \Phi(x_1) \cdots \Phi(x_N) | \Psi \rangle \]  
(22)
with \( |\emptyset\rangle = \) Fock vacuum, \( |\Psi\rangle \in \mathcal{H} = \) state vector.
[Conversely, we will use \( \psi \) to construct QFTs in Lecture 4.]

**Example 4 (detectors).** \( N \) non-interacting particles, ideal hard detectors along timelike hypersurface \( \Sigma = \partial \Omega \) [picture].
What is the probability of detection at \((y_1...y_N)\) in \( \Sigma^N \)?
Answer [Werner 1987, Tumulka 2016]: Solve (17) for \( 1...N \) on \( \Omega^N \) with boundary condition (BC)
\[ \psi_j(x_j) \psi(x_1...x_N) = \psi_j(x_j) \psi(x_1...x_N) \]  
(23)
for all \( j \in \{1...N\} \), \( x_j \in \Sigma \), \( x_1...x_N \in \overline{\Omega} \). Here \( n_\mu(x) = \) unit normal to \( \Sigma \), \( u_\mu(x) = \) unit timelike vector of detector frame (tangent to \( \Sigma \)), and \( \psi_j \) means \( u_\mu \gamma^\mu \) acting on \( s_j \).

\[ \text{Prob} \left( Y_1 \in d^3y_1, ..., Y_N \in d^3y_N \right) = \right. \]
\[ \overline{\psi}(y_1...y_N) \psi_1(y_1) \cdots \psi_N(y_N) \psi(y_1...y_N) d^3\sigma(y_1) \cdots d^3\sigma(y_N). \]  
(24)
Remark: also for non-relativistic with BC \( \mathbf{n}(x_j) \cdot \nabla_j \psi(x_1...x_N) = i\kappa(x_j)\psi(x_1...x_N) \) with detector-dependent \( \kappa > 0 \).
Example 5 (scattering cross section). [Soft detectors along distant sphere,] \( \Omega = \mathbb{R} \times B_R(0) \) in the limit \( R \to \infty \); still, use (17) and (24), no BC necessary in the limit; no interaction after initial period [because particles are far from each other].

[Leads to] prob distr on \( ((\text{time axis}) \times S^2)^N \) with \( S^2 = \partial B_1(0) \) given in the non-rel. case by [Dürr and Teufel 2004]

\[
\lim_{R \to \infty} \text{Prob}\left( Y_1 \in R dt_1 R d^2 \mathbf{\omega}_1, \ldots, Y_N \in R dt_N R d^2 \mathbf{\omega}_N \right) = \left| \mathcal{F} \varphi_0 \left( \frac{m \omega_1}{t_1}, \ldots, \frac{m \omega_N}{t_N} \right) \right|^2 dt_1 d^2 \mathbf{\omega}_1 \cdots dt_N d^2 \mathbf{\omega}_N. \tag{25}
\]

with \( \mathcal{F} = \text{Fourier transformation} \), \( \varphi_0 = \text{initial wf after interaction period} \).

Example 6 (curved Born rule). \( N \) non-interacting Dirac particles, detectors along spacelike hypersurface \( \Sigma \) [picture], detection at \( Y_1, \ldots, Y_N \), prob again given by (24). In short, [Bloch 1934]

\[
\rho_\Sigma = |\psi_\Sigma|^2 \tag{26}
\]

[in the appropriate basis in spin space] with

\[
\psi_\Sigma(x_1 \ldots x_N) = \psi(x_1 \ldots x_N). \tag{27}
\]

In fact, \( \psi_\Sigma \in \mathcal{H}_\Sigma \), which contains functions \( \Sigma^N \to (\mathbb{C}^4)^\otimes N \) with inner product

\[
\langle \chi | \varphi \rangle = \int_{\Sigma^N} d^3 \sigma (x_1) \cdots d^3 \sigma (x_N) \overline{\chi}(x_1 \ldots x_N) \psi_1(x_1) \cdots \psi_N(x_N) \varphi(x_1 \ldots x_N). \tag{28}
\]

[More detail and interacting case in Lecture 6 on Friday morning.]

4 Multi-time Schrödinger eqs

Non-interacting: Want \( \psi(x_1 \ldots x_N) \) determined by initial data for \( t_1 = \ldots = t_N = 0 \); this suggests [Dirac 1932] (alternative: integral eqs \( \to \) Lecture 7)

\[
i \frac{\partial \psi}{\partial t_1} = H_1 \psi \tag{29}
\]

\[
\vdots \tag{30}
\]

\[
i \frac{\partial \psi}{\partial t_N} = H_N \psi, \tag{31}
\]

again \( \varphi(t, x_1, \ldots, x_N) = \psi(t, x_1, \ldots, t, x_N) \Rightarrow H = \sum_{j=1}^N H_j \bigg|_{(t, t \ldots t)} \).

\( H_j = \text{“partial Hamiltonian,” now not the free } H. \)
Consistency question: Suppose first that each $H_j : L^2(\mathbb{R}^{3N}, \mathbb{C}) \to L^2(\mathbb{R}^{3N}, \mathbb{C})$ is time independent. Then [picture]

$$
e^{-iH_2 t_2}e^{-iH_1 t_1} \psi(0,0) = \psi(t_1, t_2) = e^{-iH_1 t_1}e^{-iH_2 t_2} \psi(0,0) \quad (32)$$

If $\psi(0,0)$ can be arbitrary, this requires that

$$\left[ e^{-iH_1 t_1}, e^{-iH_2 t_2} \right] = 0 \quad \forall t_1, t_2 \iff [H_1, H_2] = 0, \quad (33)$$

the consistency condition [Bloch 1934]. [More in Lecture 2.]

**Example 7 (quantum control).** $t_1 =$ time, $t_2, \ldots t_N =$ parameters that experimenters can control (external fields). We vary $t_2(t), \ldots t_N(t)$ for $t \in [0, T]$ from $t_j(0)$ to $t_j(T)$. If the eqs satisfy the consistency condition, then $\varphi(T)$ depends only on the final parameters $t_j(T)$ but not on the path $t_j(T)$ in parameter space.

**Definition.** set of spacelike configurations of $N$ particles

$$\mathcal{J}_N = \left\{ (x_1, \ldots, x_N) \in \mathcal{M}^N : \forall j, k : (x_j - x_k)^\mu (x_j - x_k)_\mu < 0 \text{ or } x_j = x_k \right\} \quad (34)$$

[picture] Often, $\psi : \mathcal{J}_N \to \mathbb{C}^K$ instead of $\psi : \mathcal{M}^N \to \mathbb{C}^K$.

**Tomonaga–Schwinger approach:** [Closely related to multi-time wf.] Suppose we have

- $\mathcal{H}_\Sigma$ for every spacelike hypersurface $\Sigma$,
- $U_{\Sigma'}^\Sigma : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$ unitary time evolution,
- $\psi_{\Sigma'} = U_{\Sigma'}^\Sigma \psi_{\Sigma}$
- $F_{\Sigma'}^\Sigma : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$ free unitary time evolution.

Fix $\Sigma_0$. Define interaction picture

$$\Psi_{\Sigma} := F_{\Sigma}^{\Sigma_0} U_{\Sigma_0}^{\Sigma'} \psi_{\Sigma_0} \quad (35)$$

Tomonaga–Schwinger eq: For $\Sigma'$ infinitesimally close to $\Sigma$ [picture]

$$i(\Psi_{\Sigma'} - \Psi_{\Sigma}) = \int_{\Sigma} d^4x \mathcal{H}_I(x) \Psi_{\Sigma} \quad (36)$$

$\mathcal{H}_I(x)$ is called the interaction Hamiltonian density. Consistency condition

$$[\mathcal{H}_I(x), \mathcal{H}_I(y)] = 0 \quad \text{for spacelike separated } x, y. \quad (37)$$

[More in Lectures 4–6.]

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