Spring School on Multi-Time Wave Functions
Lecture 1: Introduction and Overview

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1 Relativistic space-time

$\mathscr{M} = \mathbb{R}^4$ Minkowski space-time [with points] $x = x^\mu = (x^0, \boldsymbol{x}) = (ct, \boldsymbol{x})$, $c = 1$, metric

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} . \tag{1}$$

[picture: light cone, sets of timelike/spacelike/lightlike = null vectors]
Lorentz group $\mathscr{L} = O(1,3)$ [= set of all Lorentz transformations including rotations in space], rotation group = $SO(3)$.
Restricted Lorentz group $\mathscr{L}^+ \subset \mathscr{L}$ [contains those that don't reverse either time or space]
Poincaré group

$$\mathscr{P} = \Bigl\{ x \mapsto a + \Lambda x \ : \ a \in \mathscr{M}, \Lambda \in \mathscr{L} \Bigr\} \tag{2}$$

[includes space-time translations, correspondingly $\mathscr{P}^+$]

2 Dirac equation

Wave function $\psi(t, \boldsymbol{x}) \in \mathbb{C}^4$ spin space, Dirac eq ($\hbar = 1$)

$$i\frac{\partial \psi}{\partial t} = -i\boldsymbol{\alpha} \cdot \nabla \psi + \beta m \psi =: H_{\text{Dirac}} \psi \tag{3}$$

or, [equivalently,]

$$i\gamma^\mu \partial_\mu \psi = m\psi \quad \text{or} \quad i\partial\!\!\!/ \psi = m\psi \, . \tag{4}$$

Here, $\beta = \gamma^0$, $\alpha^i = \gamma^0 \gamma^i$ ($i = 1, 2, 3$), $v\!\!\!/ = v_\mu \gamma^\mu$.
Clifford relation $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} I$, in particular $\gamma^0 \gamma^0 = I$.

Hamiltonian formulation: Hilbert space $\mathscr{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$ with [inner product]

$$\langle\phi|\psi\rangle = \int_{\mathbb{R}^3} d^3\boldsymbol{x}\, \phi^\dagger(\boldsymbol{x})\, \psi(\boldsymbol{x}) \tag{5}$$

with [inner product in spin space]

$$\phi^\dagger\psi = \sum_{s=1}^{4} \phi_s^*\, \psi_s\,. \tag{6}$$

$\exists$ self-adjoint version of $H_{\text{Dirac}} \Rightarrow U_t := e^{-iHt} : \mathscr{H} \to \mathscr{H}$ is unitary.

Space-time formulation: $\psi : \mathscr{M} = \mathbb{R}^4 \to \mathbb{C}^4$. Spinors transform under $\Lambda \in \mathscr{L}^+$ [according to]

$$\pm\, S(\Lambda) : \mathbb{C}^4 \to \mathbb{C}^4\,, \tag{7}$$

[i.e., S is a (projective) *representation* of $\mathscr{L}^+$. Thus,]

$$\psi'(x) = S(\Lambda)\psi(\Lambda^{-1}x)\,. \tag{8}$$

[If Λ = rotation through angle φ, then $S(\Lambda)$ is a rotation through angle $\varphi/2$; that's why it's called spin-$\frac{1}{2}$.]

Theorem. Every $\Lambda \in \mathscr{L}^+$ leaves γ^μ invariant, $\gamma^\mu = (\gamma')^\mu = S(\Lambda)\, \Lambda^\mu_\nu \gamma^\nu S(\Lambda)^{-1}$.

Corollary. The Dirac eq is Lorentz invariant.

Remark. [The inner product] (6) is *not* Lorentz invariant. However, the following product is:

$$\overline{\phi}\,\psi := \phi^\dagger\gamma^0\psi\,. \tag{9}$$

Definition. For every $\psi : \mathscr{M} \to \mathbb{C}^4$, the *probability current 4-vector field* is

$$j^\mu(x) = \overline{\psi(x)}\gamma^\mu\psi(x)\,. \tag{10}$$

- $j(x)$ is defined in a covariant way,
- causal = timelike or lightlike, $j^\mu(x)\, j_\mu(x) \geq 0$.
- future-pointing, $j^0(x) \geq 0$.
- $j^0 = \overline{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi = \sum_{s=1}^4 |\psi_s|^2 = \rho$ = probability density according to Born's rule
- $\partial_\mu j^\mu = 0$ (continuity eq $\partial_t\rho = -\text{div}_3\boldsymbol{j}$) [exercise]

- By the Gauss integral theorem, for spacelike hypersurfaces Σ, Σ',

$$\int_\Sigma d^3\sigma(x)\, j^\mu(x)\, n_\mu(x) = \int_{\Sigma'} d^3\sigma(x)\, j^\mu(x)\, n_\mu(x) \tag{11}$$

  with $d^3\sigma(x) = d^3x\sqrt{\det{}^3g(x)}$ = 3-volume defined by metric on Σ, provided $j^\mu \to 0$ fast enough as "$x \to \infty$ spacelike." [exercise]

- propagation locality: If $\psi(t=0)$ is concentrated in $A \subset \mathbb{R}^3$ (i.e., $\psi(0, \boldsymbol{x}) = 0$ for all $\boldsymbol{x} \notin A$), then ψ is concentrated in future$(\{0\} \times A) \cup$ past$(\{0\} \times A)$. [picture] [no propagation faster than light]

3 What is a multi-time wave function?

Ordinary wf of QM of N particles

$$\varphi(t, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \tag{12}$$

[evolves according to] Schrödinger eq

$$i\frac{\partial\varphi}{\partial t} = H\varphi \quad \Leftrightarrow \quad \varphi(t) = e^{-iHt}\varphi(0) \tag{13}$$

with H = Hamiltonian operator.
[Uniquely determined by] initial data $\varphi(0)$ on $\mathbb{R}^{3N}$.
Not covariant: [refers to space-time points] $(t, \boldsymbol{x}_1), \ldots, (t, \boldsymbol{x}_N)$ simultaneous [w.r.t. chosen Lorentz frame; picture]

Multi-time wf [Dirac 1932]

$$\psi(t_1, \boldsymbol{x}_1, \ldots, t_N, \boldsymbol{x}_N) = \psi(x_1, \ldots, x_N) \tag{14}$$

Example 1 (non-interacting). $\psi_{s_1 s_2}(x_1, x_2)$, $\psi : \mathscr{M}^2 \to \mathbb{C}^4 \otimes \mathbb{C}^4$,

$$\psi(t_1, \cdot, t_2, \cdot) = e^{-iH_1t_1 - iH_2t_2} \tag{15}$$

with $H_j = H_{\text{Dirac}}$ acting on $\boldsymbol{x}_j, s_j$.
[Uniquely determined by] initial data $\psi(0,0) : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{C}^4 \otimes \mathbb{C}^4$.
Obeys multi-time Schrödinger eqs

$$i\frac{\partial\psi}{\partial t_1} = H_1\psi \tag{16}$$
$$i\frac{\partial\psi}{\partial t_2} = H_2\psi \tag{17}$$

Note:

$$\varphi(t,\boldsymbol{x}_1,\boldsymbol{x}_2) := \psi(t,\boldsymbol{x}_1,t,\boldsymbol{x}_2) \tag{18}$$

obeys [by the chain rule]

$$i\frac{\partial\varphi}{\partial t} = i\frac{\partial\psi}{\partial t_1}\Big|_{(t,t)} + i\frac{\partial\psi}{\partial t_2}\Big|_{(t,t)} = (H_1+H_2)\varphi, \tag{19}$$

so $H = H_1 + H_2$ (non-interacting).
Remark: works also for non-relativistic Schrödinger eq. $H_j = -\frac{1}{2m_j}\Delta_j$
Likewise for N particles.
Challenge: include interaction. (Lectures 2–7)

Example 2 (second-order equations). [Also possible] $\psi : \mathscr{M}^2 \to \mathbb{C}$,

$$\Box_1\psi = m_1^2\psi \tag{20}$$
$$\Box_2\psi = m_2^2\psi \tag{21}$$

with $\Box = \partial^\mu\partial_\mu = \partial_t^2 - \Delta$ d'Alembertian.

Example 3 (if you know QFT). Field operators $\Phi(t,\boldsymbol{x}) = e^{iHt}\Phi(0,\boldsymbol{x})e^{-iHt}$ (Heisenberg picture) on Fock space $\mathscr{H}$,

$$\psi(x_1...x_N) := \frac{1}{\sqrt{N!}}\langle\emptyset|\Phi(x_1)\cdots\Phi(x_N)|\Psi\rangle \tag{22}$$

with $|\emptyset\rangle$ = Fock vacuum, $|\Psi\rangle \in \mathscr{H}$ = state vector.
[Conversely, we will use ψ to construct QFTs in Lecture 4.]

Example 4 (detectors). N non-interacting particles, ideal hard detectors along timelike hypersurface $\Sigma = \partial\Omega$ [picture].
What is the probability of detection at $(y_1...y_N) \in \Sigma^N$?
Answer [Werner 1987, Tumulka 2016]: Solve (17) for $1...N$ on Ω^N with boundary condition (BC)

$$\not{n}_j(x_j)\,\psi(x_1...x_N) = \not{u}_j(x_j)\,\psi(x_1...x_N) \tag{23}$$

for all $j \in \{1...N\}$, $x_j \in \Sigma$, $x_1...x_N \in \overline{\Omega}$. Here $n_\mu(x)$ = unit normal to Σ, $u_\mu(x)$ = unit timelike vector of detector frame (tangent to Σ), and $\not{v}_j$ means $v_\mu\gamma^\mu$ acting on s_j.

$$\text{Prob}\Big(Y_1 \in d^3y_1,...,Y_N \in d^3y_N\Big) = \overline{\psi}(y_1...y_N)\,\not{n}_1(y_1)\cdots\not{n}_N(y_N)\,\psi(y_1...y_N)\,d^3\sigma(y_1)\cdots d^3\sigma(y_N)\,. \tag{24}$$

Remark: also for non-relativistic with BC $\boldsymbol{n}(x_j)\cdot\nabla_j\psi(x_1...x_N) = i\kappa(x_j)\psi(x_1...x_N)$ with detector-dependent $\kappa > 0$.

Example 5 (scattering cross section). [Soft detectors along distant sphere,] $\Omega = \mathbb{R} \times B_R(\mathbf{0})$ in the limit $R \to \infty$; still, use (17) and (24), no BC necessary in the limit; no interaction after initial period [because particles are far from each other].
[Leads to] prob distr on $\big((\text{time axis}) \times \mathbb{S}^2\big)^N$ with $\mathbb{S}^2 = \partial B_1(\mathbf{0})$ given in the non-rel. case by [Dürr and Teufel 2004]

$$\lim_{R\to\infty} \mathrm{Prob}\Big(Y_1 \in R dt_1 R d^2\boldsymbol{\omega}_1, ..., Y_N \in R dt_N R d^2\boldsymbol{\omega}_N\Big) = \Big|\mathscr{F}\varphi_0\big(\tfrac{m\boldsymbol{\omega}_1}{t_1}, \ldots, \tfrac{m\boldsymbol{\omega}_N}{t_N}\big)\Big|^2 dt_1 d^2\boldsymbol{\omega}_1 \cdots dt_N d^2\boldsymbol{\omega}_N \,. \quad (25)$$

with $\mathscr{F}$ = Fourier transformation, φ_0 = initial wf after interaction period

Example 6 (curved Born rule). N non-interacting Dirac particles, detectors along spacelike hypersurface Σ [picture], detection at $Y_1, ..., Y_N$, prob again given by (24). In short, [Bloch 1934]

$$\text{“ } \rho_\Sigma = |\psi_\Sigma|^2 \text{ ”} \quad (26)$$

[in the appropriate basis in spin space] with

$$\psi_\Sigma(x_1...x_N) = \psi(x_1...x_N). \quad (27)$$

In fact, $\psi_\Sigma \in \mathscr{H}_\Sigma$, which contains functions $\Sigma^N \to (\mathbb{C}^4)^{\otimes N}$ with inner product

$$\langle\chi|\varphi\rangle = \int_{\Sigma^N} d^3\sigma(x_1) \cdots d^3\sigma(x_N)\, \overline{\chi}(x_1...x_N)\, \not{n}_1(x_1) \cdots \not{n}_N(x_N)\, \varphi(x_1...x_N)\,. \quad (28)$$

[More detail and interacting case in Lecture 6 on Friday morning.]

4 Multi-time Schrödinger eqs

Non-interacting: Want $\psi(x_1...x_N)$ determined by initial data for $t_1 = ... = t_N = 0$; this suggests [Dirac 1932] (alternative: integral eqs $\to$ Lecture 7)

$$i\frac{\partial\psi}{\partial t_1} = H_1\psi \quad (29)$$

$$\vdots \quad (30)$$

$$i\frac{\partial\psi}{\partial t_N} = H_N\psi, \quad (31)$$

again $\varphi(t, \boldsymbol{x}_1..., \boldsymbol{x}_N) = \psi(t, \boldsymbol{x}_1, ..., t, \boldsymbol{x}_N) \quad \Rightarrow \quad H = \sum_{j=1}^N H_j\Big|_{(t,t...t)}$.
H_j = “partial Hamiltonian,” now not the free H.

Consistency question: Suppose first that each $H_j : L^2(\mathbb{R}^{3N},\mathbb{C}) \to L^2(\mathbb{R}^{3N},\mathbb{C})$ is time independent. Then [picture]

$$e^{-iH_2t_2}e^{-iH_1t_1}\psi(0,0) = \psi(t_1,t_2) = e^{-iH_1t_1}e^{-iH_2t_2}\psi(0,0) \tag{32}$$

If $\psi(0,0)$ can be arbitrary, this requires that

$$\Big[e^{-iH_1t_1}, e^{-iH_2t_2}\Big] = 0 \;\; \forall t_1,t_2 \qquad \Leftrightarrow \qquad \big[H_1,H_2\big] = 0, \tag{33}$$

the consistency condition [Bloch 1934]. [More in Lecture 2.]

Example 7 (quantum control). $t_1 =$ time, $t_2 ... t_N =$ parameters that experimenters can control (external fields). We vary $t_2(t) ... t_N(t)$ for $t \in [0,T]$ from $t_j(0)$ to $t_j(T)$. If the eqs satisfy the consistency condition, then $\varphi(T)$ depends only on the final parameters $t_j(T)$ but not on the path $t_j(T)$ in parameter space.

Definition. set of spacelike configurations of N particles

$$\mathscr{S}_N = \Big\{(x_1,...,x_N) \in \mathscr{M}^N \;:\; \forall j,k : (x_j - x_k)^\mu (x_j - x_k)_\mu < 0 \text{ or } x_j = x_k\Big\} \tag{34}$$

[picture]
Often, $\psi : \mathscr{S}_N \to \mathbb{C}^K$ instead of $\psi : \mathscr{M}^N \to \mathbb{C}^K$.

Tomonaga–Schwinger approach: [Closely related to multi-time wf.] Suppose we have

- $\mathscr{H}_\Sigma$ for every spacelike hypersurface Σ,
- $U_\Sigma^{\Sigma'} : \mathscr{H}_\Sigma \to \mathscr{H}_{\Sigma'}$ unitary time evolution,
- $\psi_{\Sigma'} = U_\Sigma^{\Sigma'} \psi_\Sigma$
- $F_\Sigma^{\Sigma'} : \mathscr{H}_\Sigma \to \mathscr{H}_{\Sigma'}$ free unitary time evolution.

Fix Σ_0. Define interaction picture

$$\Psi_\Sigma := F_\Sigma^{\Sigma_0}\, U_{\Sigma_0}^{\Sigma} \psi_{\Sigma_0}\,. \tag{35}$$

Tomonaga–Schwinger eq: For Σ' infinitesimally close to Σ [picture]

$$i(\Psi_{\Sigma'} - \Psi_\Sigma) = \int_\Sigma^{\Sigma'} d^4x\, \mathcal{H}_I(x)\, \Psi_\Sigma\,. \tag{36}$$

$\mathcal{H}_I(x)$ is called the interaction Hamiltonian density. Consistency condition

$$\big[\mathcal{H}_I(x), \mathcal{H}_I(y)\big] = 0 \quad \text{for spacelike separated } x,y. \tag{37}$$

[More in Lectures 4–6.]