Lecture 2: Consistency Conditions and Interaction Potentials

Consider multi-time wave function \( \Psi(x_1, ..., x_N) \), where \( x_i = (x_i^0, x_i^1, x_i^2, x_i^3) = (x_i^0, x_i^1) \in \mathbb{R}^4 \) 

\( N \) multi-time equations

\[ i \frac{\partial \Psi}{\partial x_j^0} = H_j \Psi, \quad j = 1, ..., N \]

Recall example from lecture 1:

\[ \Psi(x_1, x_2) = e^{-iH_1 x_1^0} e^{-iH_2 x_2^0} \Psi((0, x_1^1, 0, x_2^1), (0, x_2^1, 0, x_1^1)) \]

\[ = e^{-iH_1 x_1^0} e^{-iH_2 x_2^0} \Psi((0, x_1^1, 0, x_2^1), (0, x_2^1, 0, x_1^1)) \]

\[ \Rightarrow \text{need} \quad [e^{-iH_1 x_1^0}, e^{-iH_2 x_2^0}] = 0 \quad \iff \quad [H_1, H_2] = 0 \]

\[ \Rightarrow \text{ok for non-interacting particles, i.e.,} \ H_j \text{ acts only on } j^{\text{th}} \text{ variable} \]

\( (H_1 = H_1 \otimes 1_L, \ H_2 = 1_L \otimes H_2) \)

Common solution of \( N \) equations only for some \( H_j \)

\( \Rightarrow \) these have to satisfy certain conditions, like \( [H_1, H_2] = 0 \) \n
Called **consistency conditions**

I. Rigorous Formulations of Consistency Condition

First, necessary condition for solutions

Consider solution to \( MT \) eqs \( \Psi: \mathbb{R}^4 \to \mathbb{C}^N \), \( \Psi \in C^2 \) (twice cont. differentiable)

Then

\[ [i \frac{\partial}{\partial t_j} - H_j, i \frac{\partial}{\partial x_k} - H_k] \Psi = 0 \quad \forall j \neq k \]

\( \text{might act on all variables} \)
Now any given $t_1, \ldots, t_n$ can be times for specifying initial data

$\Rightarrow$ (as $0 \forall \Psi \Rightarrow [\ldots] = 0$ everywhere

Note: $[i \frac{\partial}{\partial x^i}, H_{\text{H}}, i \frac{\partial}{\partial x^j} - H_{\text{K}}] \Psi = 0$

$\Rightarrow \left[ i \frac{\partial}{\partial x^i}, i \frac{\partial}{\partial x^j} - H_{\text{K}} \right] \Psi - \left[ i \frac{\partial}{\partial x^j}, H_{\text{H}} \right] \Psi - \left[ H_{\text{H}}, i \frac{\partial}{\partial x^j} \right] \Psi + \left[ H_{\text{H}}, H_{\text{K}} \right] \Psi = 0$

$= 0 \left( \forall \epsilon \right) = i \left( \frac{\partial H_{\text{H}}}{\partial x^j} \right) \Psi - i (\frac{\partial H_{\text{K}}}{\partial x^i}) \Psi$

$\Rightarrow \left( i \frac{\partial H_{\text{H}}}{\partial x^j} - i \frac{\partial H_{\text{K}}}{\partial x^i} + [H_{\text{H}}, H_{\text{K}}] \right) \Psi = 0

Next, consider Hilbert space framework: $\Psi: \mathbb{R}^N \to \mathcal{H}$, e.g., $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^K)$

Note: • not what we ultimately want: rather $\Psi: \mathbb{R}^N \to \mathbb{C}^K$

• but technically nice to handle set of spacelike configurations

**Theorem:** Let $H_1, \ldots, H_n$ self-adjoint (and time-independent) on $\mathcal{H}$. Then

Solution to $\mathcal{M}$ eqs. exists $\forall$ initial $\Psi(t_1, \ldots, t_n, \ldots, t_n)$

$\iff$

$[H_i, H_k] = 0 \forall i \neq k$

Note: • sol. here means $\Psi(t_1, \ldots, t_n) = e^{-iH_{\text{H}}t} \Psi(t_1, \ldots, t_n, 0, t_{ij}, \ldots, t_{nn})$

• for unbounded $H_j$, $[H_j, H_k] = 0$ is def. via spectral projections

$\left( \left[ A_j(H_j), A_k(H_k) \right] = 0 \forall A_j, A_k \in \mathbb{R} \right)$

**Proof:** $[e^{-iH_{\text{H}}t}, e^{-iH_{\text{H}}t}] = 0 \iff [H_i, H_k] = 0 \quad \square$
Next: time-dependent $H_t = H_j(t_1, \ldots, t_n)$

Theorem: Let $H: H_j : \mathbb{R}^N \rightarrow \mathcal{S}(\mathcal{H})$ be smooth ($\mathcal{S}(\mathcal{H})$-bounded operators $\mathcal{H} \rightarrow \mathcal{H}$). Then

Solution to eqs exists $\forall$ initial $U(1, \bar{q}, t_1, \ldots, q, \bar{q}, \cdots)$

\[
\left[ i \frac{\partial}{\partial x_j} - H_j, i \frac{\partial}{\partial t_k} - H_k \right] = 0 \quad \forall j \neq k
\]

Proof:

recall: $i \frac{d\psi}{dt} = H(t) \psi \quad H$ time-independent

\[
\Rightarrow \psi(t) = e^{-iHt} \psi(0) \quad \text{($H$ self-adjoint $\iff$ $U(t)$ unitary propagator)}
\]

\[
\begin{aligned}
U(t) &= 1 + \sum_{n=1}^{\infty} \frac{(-iHt)^n}{n!} \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-iHt)^n}{n!} \int_0^t \int_0^{T_1} \int_0^{T_2} \ldots \int_0^{T_n} e^{iH(T_1 + \ldots + T_n - T)} dT_1 dT_2 \ldots dT_n \ H^n
\end{aligned}
\]

now: $i \frac{d\psi}{dt} = H(t) \psi(t)$

\[
\Rightarrow \psi(t) = U(t, s) \psi(s) \quad \text{($H(t)$ self-adjoint $\forall t \iff$ $U(t, s)$ unitary)}
\]

Dyson series: $U(t, s) = 1 + \sum_{n=1}^{\infty} \frac{(-iH)^n}{n!} \int_0^t \int_0^{T_1} \int_0^{T_2} \ldots \int_0^{T_n} e^{iH(T_1 + \ldots + T_n - T)} dT_1 dT_2 \ldots dT_n \ H(T_1 + \ldots + T_n - T)
\]

\[
= \int \mathcal{D}H e^{iH(t-s)H}
\]

note: $H$ bounded $\Rightarrow U(t, s)$ bounded

$U(t_1, s) U(s, t_2) = 1$, $U(t_1, s) U(s, t_2) = U(t_1, t_2)$
Similarly for \( \Psi \): take \( N=2 \) : \( \Psi(t_1, t_2) = U(t_{11}, s_1, t_2) \Psi(s_1, t_2) \)

with \( U(t_{11}, s_1, t_2) = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int s_1 \cdots \int s_1 H_1(T_1, t_1) \cdots H_1(T_n, t_n) \) (Dyson series)

write \( \Psi(t_1, t_2) = U(t_{11}, s_1, t_2) U(s_1, t_2, s_2) \Psi(s_1, s_2) \)

\[ = U(t_{11}, t_2, s_2) U(t_{11}, s_1, t_2) \Psi(s_1, s_2) \]

so: existence of solution \( \forall \Psi(s_1, s_2) \) (or \( \forall \Psi(0,0) \) by group property)

\[ \iff \] \( U(t_{11}, s_1, t_2) U(s_1, t_2, s_2) = U(t_{11}, s_1, t_2) U(t_{11}, s_1, t_2) \forall s_1, s_2, t_1, t_2 \in \mathbb{R} \)

\[ \begin{array}{c}
\text{now: consider} \\
\end{array} \]

Taylor expansion for \( (s_1, s_2) \in \mathbb{C} \): \( H_1(s_1, s_2) = H_1(t_{11}, t_2) + \frac{\Delta t}{2} (s_2 - t_{11}) \frac{\partial H_1}{\partial s_2}(t_{11}, t_2) + o(\Delta t) \)

\begin{align*}
\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} & = 0 \\
= & U(t_{11}, t_{11} + \Delta t, t_2) = 1 - i \int s_1 \cdots \int s_1 H_1(T_1, t_1) \\
& \quad - \sum_{n=1}^{\infty} \frac{i^n}{n!} \int s_1 \cdots \int s_1 H_1(T_1, t_1) \cdots H_1(T_n, t_n) \) \( H_1(T_1, t_1) \cdots H_1(T_n, t_n) + O(\Delta t^3) \)
\end{align*}
Computation for other \( U \)'s yields:

\[
(\ast) \implies D = - i [ H_\alpha H_\beta] - i \frac{\partial H_\alpha}{\partial t_\beta} + i \frac{\partial H_\beta}{\partial t_\alpha} \bigg|_{(t_\alpha, t_\beta)} dt_\alpha^2 + o(dt_\beta^2)
\]

Concatenation of \( \square \) proves statement \( \square \)

**Note:** different point of view:

- Define \( U_\gamma = \int e^{-i \int_5^5 H_\delta \, dt} \) for any path in \((t_1, \ldots, t_n)\)-plane

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\[
\begin{array}{c}
\text{consistency} \iff \text{path-independence (fixed start/end points)}
\end{array}
\]
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- Differential geometry language: base manifold \( \mathbb{R}^N \)
  - Fiber space \( \mathcal{H} \) (at each point)
  - Cross-section of vector bundle
  - Covariant derivative
    \[
    D_j := \frac{\partial}{\partial t_j} - i H_j
    \]
  - Parallel transport

\( \Rightarrow \) path-independence \( \iff \) curvature \( F_{jk} = 0 \)

Consistency condition from before
II. Interaction Potentials

Recall: \( \varphi(t, \vec{x}_1, \ldots, \vec{x}_n) = \mathcal{V}(t, \vec{x}_1, \ldots, \vec{x}_1) \)

and \( i \frac{\partial \varphi}{\partial t} = \left\{ \sum_{i=1}^{n} \mathcal{H}_j \mid t \right\} \varphi \)

In non-relativistic quantum mechanics, one often considers

\[
H = \sum_{i=1}^{n} \left( \mathcal{H}_j^{\text{free}} + W(t, \vec{x}_i) \right) + \sum_{i<j} V(\vec{x}_i, \vec{x}_j)
\]

e.g., Coulomb potential \( V(\vec{x}) = \frac{1}{|\vec{x}|} \)

\( \mathcal{H}_j^{\text{free}} = -\Delta_i, \mathcal{H}_j^{\text{Dirac}} \)

\( \Rightarrow \) obvious idea: choose partial Hamiltonians \( \mathcal{H}_j = \mathcal{H}_j^{\text{free}} + \mathcal{V}_j(\vec{x}_1, \ldots, \vec{x}_n) \)

\( \Rightarrow \) Def.: \( \mathcal{H}_j \) interacting \( \iff \) no gauge transformation gives \( \tilde{\mathcal{H}}_j = \mathcal{H}_j^{\text{free}} + \tilde{\mathcal{V}}_j(\vec{x}_j) \)

Theorem: Let \( \mathcal{H}_j^{\text{free}} = \text{free Dirac}, \mathcal{V}_j: \mathbb{R}^4 \to \mathbb{R} \) smooth. Then

Consistency Condition holds \( \iff \) \( \mathcal{H}_j \) not interacting

\( \Rightarrow \) interaction by potentials ruled out (for some class)
Proof: direct computation:

\[ 0 = [H_i, H_j] - i \frac{\partial H_i}{\partial t_i} + i \frac{\partial H_j}{\partial t_j} \]

\[ = \left[ H_i^{\text{free}}, V_j \right] + \left[ V_i, H_j^{\text{free}} \right] - i \frac{\partial V_i}{\partial t_i} + i \frac{\partial V_j}{\partial t_j} \]

\[ = -i \frac{3}{2} \sum_{a=1}^{3} \left( \alpha_i^{(a)} \frac{\partial V_i}{\partial x_i^a} - \alpha_j^{(a)} \frac{\partial V_j}{\partial x_j^a} \right) - i \left( \frac{\partial V_i}{\partial t_i} - \frac{\partial V_j}{\partial t_j} \right) \]

Note: \( \alpha_i^{(a)}, \alpha_j^{(a)} \) linearly independent

\[ \Rightarrow V_j = V_j(\tilde{x}_j, t_1, \ldots, t_n) \]

Next: use \( \frac{\partial V_i}{\partial t_i} = \frac{\partial V_i}{\partial t_j} \) to conclude \( V_i = V_i(\tilde{x}_j, t_j) + \frac{\partial \Theta(t_1, \ldots, t_n)}{\partial t_j} \)

can be removed by gauge transformation

Generalizations:

- Consistency cond. only on \( S \)  
- \( H_j^{\text{free}} = -i \frac{3}{2} \sum_{a=1}^{3} A_j^{(a)}(x_j) \frac{\partial}{\partial x_j^a} + B_j(\mathbf{x}_j) \), \( A_j^{(a)}, 1 \) lin. indep. \( A_j^{(a)} \) smooth  
- \( V_j \) = matrix acting on \( j \)-th spin space  
- \( H_j^{\text{free}} \) second order  

Theorem (Deckert, Nickel 2016):

Let \( V_j \in C^1(\mathbb{R}^4_v \times S \rightarrow \mathbb{C}^{K \times L}) \)

- \( \exists \) sol. \( \Psi \in C^2 \) to MTEqs \( \forall \Psi(0, \mathbf{x}), (0, \tilde{\mathbf{x}}) \in C^0 \)

Then \( V_j \) is not Poincaré invariant

Note: \( \exists \) counter examples with non-trivial spin-dependence, see Exercise Sectin
Proof: sketch. Evaluate condition

- Use linear independence
- $V_i$ spanned by $H_2$ and $f^{(5)}$
- Translation invariance $\implies$ explicit expressions for $V_i,V_\xi$, which are not
  Lorentz-invariant

III. 8-range Interactions

Other idea: "point-like" interactions when $x_i = x_j$, possibly with boundary conditions on $\partial S$

\[ \text{See Lecture 3} \]

Note: for free Dirac in 3+1 dim.:
\[ i\frac{d\psi}{dt} = \sum_{j=1}^{N} \frac{H_{\text{Dirac}}}{\gamma} \psi \quad \text{on } (\mathbb{R}^3)^N \setminus D \]
\[ D = \{ \text{some } x_i = x_j \} \]

but then one can show that some eq. holds on all of $(\mathbb{R}^3)^N$

\[ \text{See Lecture 3: model in 1+1 dim.} \]

Other idea: introduce cutoff $\delta > 0$

- Group particles into families with $t_i = t_j$ $\implies$ partition $P = \{ S_{i_1}, \ldots, S_N \}$, $S_\alpha \subset \{ 1, \ldots, N \}$

\[ \implies S_{\delta,P} = \{ \text{with } S_\alpha : t_i = t_j, \forall \alpha \neq \beta, i \neq j, \text{ and } |x_i - x_j| > |t_i - t_j| + \delta \} \]

\[ \implies \text{on } S_\delta = \bigcup_P S_{\delta,P} \]

\[ \implies \text{MT eq.'s: on } S_{\delta,P} : i\frac{d\psi}{dt} = \left( \sum_{i \in S_\alpha} H_{\text{Dirac}}^{(\alpha)} + \sum_{i \neq j \in S_\alpha} W(x_i - x_j) \right) \psi \]
\[ \kappa = 1, \ldots, L \]
\[ W(x) = 0 \text{ for } |x| \geq \delta \]
**Theorem:** MT eqs consistent for smooth $W$.

**Proof sketch:** induction in $L$

1. **Start:** ok (one-time evolution)
2. **Step:** crucial $U(..., t_c)$ determined by initial data on $B_{t_{L-1}}(x_c)$
   then use consistency condition to show that the new $U$ satisfies MT eqs.

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**References:**

- Multi-Time Schrödinger Equations Cannot Contain Interaction Potentials
  S. Petrat and R. Tumulka
  arXiv:1308.1065

- Consistency of Multi-Time Dirac Equations with General Interaction Potentials
  D. A. Deckert and L. Nickel
  arXiv:1603.02538