Spring School on Multi-time Wave Functions April 10, 2019

Lecture 2: Consistency Conditions and Interaction Potentials

consider multi-time wave function $\psi(x_1, \dots, x_N)$, $x_i = (x_i^0, x_i^1, x_i^2, x_i^3) = (x_i^0, \vec{x}_i) \in \mathbb{R}^4$

N multi-time equations $i \frac{\partial \psi}{\partial x_j^0} = H_j \psi$, $j = 1, \dots, N$

recall example from lecture 1: $\psi(x_1, x_2) = e^{-iH_1 x_1^0} e^{-iH_2 x_2^0} \psi((0, \vec{x}_1), (0, \vec{x}_2))$

$$= e^{-iH_2 x_2^0} e^{-iH_1 x_1^0} \psi((0, \vec{x}_1), (0, \vec{x}_2))$$

$\Rightarrow$ need $[e^{-iH_1 x_1^0}, e^{-iH_2 x_2^0}] = 0 \Leftrightarrow [H_1, H_2] = 0$

$\Rightarrow$ ok for non-interacting particles, i.e., H_j acts only on j-th variable

$(H_1 = H_1 \otimes \mathbb{1}, H_2 = \mathbb{1} \otimes H_2)$

Common solution of N equations only for some H_j

$\hookrightarrow$ these have to satisfy certain conditions, like $[H_1, H_2] = 0$, called consistency conditions

I. Rigorous Formulations of Consistency Condition

first, necessary condition for solutions

consider solution to MT eq.s $\psi: \mathbb{R}^{4N} \to \mathbb{C}^K$, $\psi \in C^2$ (twice cont. differentiable)

Then $[i \frac{\partial}{\partial t_j} - \underbrace{H_j}_{\text{might act on all variables}}, i \frac{\partial}{\partial t_k} - H_k] \psi = 0 \quad \forall j \neq k$

now any given $t_1, \dots, t_N$ can be times for specifying initial data

$\Rightarrow$ lhs $= 0 \ \forall \psi \Rightarrow [\dots, \dots] = 0$ everywhere

note: $\left[i\frac{\partial}{\partial t_j} - H_j, i\frac{\partial}{\partial t_k} - H_k \right] \psi = 0$

$$\Rightarrow \underbrace{\left[i\frac{\partial}{\partial t_j}, i\frac{\partial}{\partial t_k} \right] \psi}_{=0 \ (\psi \in C^2)} - \underbrace{\left[i\frac{\partial}{\partial t_j}, H_k \right] \psi}_{= i\left(\frac{\partial H_k}{\partial t_j}\right)\psi} - \underbrace{\left[H_j, i\frac{\partial}{\partial t_k} \right] \psi}_{-i\left(\frac{\partial H_j}{\partial t_k}\right)\psi} + [H_j, H_k]\psi = 0$$

$$\Rightarrow \left(i\frac{\partial H_j}{\partial t_k} - i\frac{\partial H_k}{\partial t_j} + [H_j, H_k] \right) \psi = 0$$

next, consider Hilbert space framework: $\psi : \mathbb{R}^N \to \mathcal{H}$, e.g., $\mathcal{H} = L^2(\mathbb{R}^{3N}, \mathbb{C}^K)$

$\mathcal{H}$: Hilbert space; $\mathbb{C}^K$: spin space

note: • not what we ultimately want: rather $\psi : \underbrace{\mathcal{S}}_{\text{set of spacelike configurations}} \to \mathbb{C}^K$

• but technically nice to handle

<u>Theorem:</u> Let $H_1, \dots, H_N$ self-adjoint (and time-independent) on $\mathcal{H}$. Then

Solution to MT eq.s exists $\forall$ initial $\psi((0,\vec{x}_1), \dots, (0,\vec{x}_N))$

$\Longleftrightarrow$

$[H_j, H_k] = 0 \ \forall \ j \neq k$

note: • sol. here means $\psi(t_1, \dots, t_N) = e^{-iH_j t_j} \psi(t_1, \dots, t_{j-1}, 0, t_{j+1}, \dots, t_N)$

• for unbounded H_j, $[H_j, H_k] = 0$ is def. via spectral projections

$\left([\mathbb{1}_{A_j}(H_j), \mathbb{1}_{A_k}(H_k)] = 0 \ \forall A_j, A_k \subset \mathbb{R} \right)$

<u>Proof:</u> $[e^{-iH_j t_j}, e^{-iH_k t_k}] = 0 \Longleftrightarrow [H_j, H_k] = 0$ $\square$

Next: time-dependent $H_j = H_j(t_1, \dots, t_N)$

<u>Theorem:</u> Let $\mathcal{H} \ni H_j : \mathbb{R}^N \to \mathcal{B}(\mathcal{H})$ be smooth ($\mathcal{B}(\mathcal{H})$ = bounded operators $\mathcal{H} \to \mathcal{H}$). Then

Solution to MT eq.s exists $\forall$ initial $\psi((0,\vec{x}_1), \dots, (0,\vec{x}_N))$

$\Longleftrightarrow$

$$\left[i\frac{\partial}{\partial t_j} - H_j, i\frac{\partial}{\partial t_k} - H_k \right] = 0 \quad \forall j \neq k$$

<u>Proof:</u>

recall: $i\frac{d\varphi}{dt} = H\varphi$, H time-independent

$$\Rightarrow \varphi(t) = \underbrace{e^{-iHt}}_{U(t)} \varphi(0) \quad (H \text{ self-adjoint} \Leftrightarrow U(t) \text{ unitary propagator})$$

$$\hookrightarrow U(t) = 1 + \sum_{n=1}^{\infty} \frac{(-iHt)^n}{n!} = 1 + \sum_{n=1}^{\infty} (-i)^n \underbrace{\int_0^t dT_1 \int_0^{T_1} dT_2 \dots \int_0^{T_{n-1}} dT_n}_{= \frac{t^n}{n!}} H^n$$

now: $i\frac{d\varphi}{dt} = H(t)\varphi(t)$

$$\Rightarrow \varphi(t) = U(t,s)\varphi(s) \quad (H(t) \text{ self-adjoint } \forall t \Leftrightarrow U(t,s) \text{ unitary})$$

$$\text{Dyson series: } U(t,s) = 1 + \sum_{n=1}^{\infty} (-i)^n \int_s^t dT_1 \int_s^{T_1} dT_2 \dots \int_s^{T_{n-1}} dT_n \, H(T_1) \cdots H(T_N)$$

$$=: \mathcal{T} e^{-i\int_s^t H(T)dT}$$

note: • H bounded $\Rightarrow$ $U(t,s)$ bounded

• $U(t,s)U(s,t) = 1$, $U(t,s)U(s,r) = U(t,r)$

similarly for Ψ: take $N=2$: $\Psi(t_1,t_2) = U(t_1,s_1;t_2)\,\Psi(s_1,t_2)$

$$\text{with } U(t_1,s_1;t_2) = 1 + \sum_{n=1}^{\infty} (-i)^n \int_{s_1}^{t_1} dT_1 \int_{s_1}^{T_1} dT_2 \ldots \int_{s_1}^{T_{n-1}} dT_n\, H_1(T_1,t_2) \cdots H_1(T_n,t_2)$$

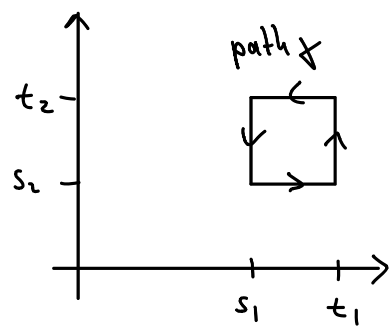
(Dyson series)

$$\text{write } \Psi(t_1,t_2) = U(t_1,s_1;t_2)\,U(s_1;t_2,s_2)\,\Psi(s_1,s_2)$$

$$= U(t_1;t_2,s_2)\,U(t_1,s_1;s_2)\,\Psi(s_1,s_2)$$

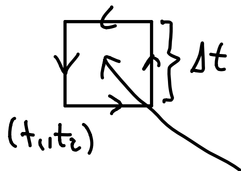
so: existence of solution $\forall\, \Psi(s_1,s_2)$ (or $\forall\, \Psi(0,0)$ by group property)

$$\Longleftrightarrow U(t_1,s_1;t_2)\,U(s_1;t_2,s_2) = U(t_1;t_2,s_2)\,U(t_1,s_1;s_2) \quad \forall s_1,s_2,t_1,t_2 \in \mathbb{R} \qquad (*)$$



$U_\gamma = 0$

now: consider



Taylor expansion for $(s_1,s_2) \in$: $H_j(s_1,s_2) = H_j(t_1,t_2) + \sum_{k=1}^{2} (s_k - t_k) \frac{\partial H_j}{\partial t_k}(t_1,t_2) + o(\Delta t)$

$$\frac{o(\Delta t)}{\Delta t} \xrightarrow{\Delta t \to 0} 0$$

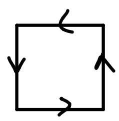
$$\Rightarrow U(t_1+\Delta t, t_1; t_2) = 1 - i \int_{t_1}^{t_1+\Delta t} dT_1\, H_1(T_1,t_2)$$

$$- \int_{t_1}^{t_1+\Delta t} dT_1 \int_{t_1}^{T_1} dT_2\, H_1(T_1,t_2)\,H_1(T_2,t_2) + O(\Delta t^3)$$

$$= 1 - i H_1(t_1,t_2)\Delta t - \frac{i}{2}\frac{\partial H_1}{\partial t_1}(t_1,t_2)\Delta t^2 - \frac{1}{2} H_1(t_1,t_2)^2 \Delta t^2 + o(\Delta t^2)$$

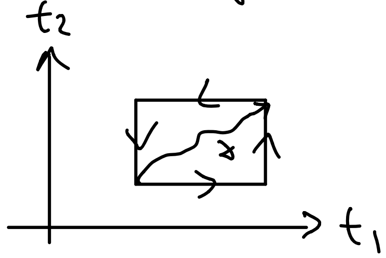
computation for other U's yields

$$(*) \iff 0 = \left(-[H_1, H_2] - i \frac{\partial H_1}{\partial t_2} + i \frac{\partial H_2}{\partial t_1} \right)\Big|_{(t_1,t_2)} \Delta t^2 + o(\Delta t^3)$$

concatentation of  proves statement $\square$

Note: different point of view:

- define $U_\gamma = \mathcal{T} e^{-i \int_\gamma \sum_j H_j dt_j}$ for any path in $(t_1, \dots, t_n)$-plane



consistency $\iff$ path-independence (fixed start/end points)

- differential geometry language:
  - base manifold $\mathbb{R}^N$
  - fiber space $\mathcal{H}$ (at each point)
  - ψ cross-section of vector bundle
  - covariant derivative $D_j := \frac{\partial}{\partial t_j} - i H_j$
  - U_γ parallel transport

$\Rightarrow$ path-independence $\iff$ curvature $\underbrace{F_{jk} = 0}_{\text{consistency condition from before}}$

II. Interaction Potentials

recall: $\varphi(t,\vec{x}_1,\dots,\vec{x}_N) := \Psi((t,\vec{x}_1),\dots,(t,\vec{x}_N))$

and $i\frac{\partial\varphi}{\partial t} = \underbrace{\left(\sum_{j=1}^{N} H_j\Big|_t\right)}_{H}\varphi$

in non-relativistic quantum mechanics, one often considers

$$H = \underbrace{\sum_{i=1}^{N}\left(H_i^{\text{free}} + W(t,\vec{x}_i)\right)}_{\text{non-interacting part}} + \underbrace{\sum_{1\le i<j\le N} v(\vec{x}_i - \vec{x}_j)}_{\text{interaction}}$$

, e.g., Coulomb potential $v(\vec{x}) = \frac{\lambda}{|\vec{x}|}$, $H_i^{\text{free}} = -\Delta_i$, $H_i^{\text{free}} = H_i^{\text{Dirac}}$

$\Rightarrow$ obvious idea: choose partial Hamiltonians $H_j = H_j^{\text{free}} + V_j(x_1,\dots,x_N)$

note: • preferably $\left(i\frac{\partial}{\partial x_j^0} - H_j\right)\Psi = 0$ should be Lorentz invariant

• $H_j = H_j^{\text{free}} + V_j(x_j)$, $V_j \in \mathbb{R}$ clearly non-interacting

but gauge transformation $\tilde{\varphi} := e^{i\theta(x_1,\dots,x_N)}\varphi$ satisfies $i\frac{\partial\tilde{\varphi}}{\partial x_j^0} = (H_j^{\text{free}} + \tilde{V}_j)\tilde{\varphi}$

$\tilde{V}_j(x_1,\dots,x_N)$ in general

↳ see Exercise Session

$\Rightarrow$ **Def.:** H_j interacting $\Leftrightarrow$ no gauge transformation gives $\tilde{H}_j = H_j^{\text{free}} + \tilde{V}_j(x_j)$

Theorem: Let H_j^{free} = free Dirac, $V_j : \mathbb{R}^{4N} \to \mathbb{R}$ smooth. Then

Consistency Condition holds $\Leftrightarrow$ H_j not interacting

$\Rightarrow$ $\boxed{\text{Interaction by potentials ruled out (for some class)}}$

Proof: direct computation:

$$0 = [H_i, H_j] - i\frac{\partial H_j}{\partial t_i} + i\frac{\partial H_i}{\partial t_j}$$

$$= [H_i^{free}, V_j] + [V_i, H_j^{free}] - i\frac{\partial V_j}{\partial t_i} + i\frac{\partial V_i}{\partial t_j}$$

$$= -i\sum_{a=1}^{3}\left(\alpha_i^{(a)}\frac{\partial V_j}{\partial x_i^a} - \alpha_j^{(a)}\frac{\partial V_i}{\partial x_j^a}\right) - i\left(\frac{\partial V_j}{\partial t_i} - \frac{\partial V_i}{\partial t_j}\right)$$

note: $1, \alpha_i^{(a)}, \alpha_j^{(a)}$ linearly independent

$$\Rightarrow V_j = V_j(\vec{x}_j, t_1, \dots, t_N)$$

next: use $\frac{\partial V_j}{\partial t_i} = \frac{\partial V_i}{\partial t_j}$ to conclude $V_j = \tilde{V}_j(\vec{x}_j, t_j) + \underbrace{\frac{\partial \theta(t_1, \dots, t_N)}{\partial t_j}}_{\text{can be removed by gauge transformation}}$

$\square$

Generalizations:

- consistency cond. only on $\overset{\circ}{\Sigma}$ ✓
- $H_j^{free} = -i\sum_{a=1}^{3} A_j^a(x_j)\frac{\partial}{\partial x_j^a} + B_j(x_j)$, $A_j^a, 1$ lin. indep., A_j^a, B_i smooth ✓
- V_j = matrix acting on j-th spin space ✓
- H_j^{free} second order ✓

Theorem (Deckert, Nickel 2016):

Let
- $V_j \in C^1(\mathbb{R}^{4N}$ or $\Sigma \to \mathbb{C}^{K\otimes K})$
- $\exists$ sol. $\psi \in C^2$ to MT eq.s $\forall\ \psi((0,\vec{x}_1),(0,\vec{x}_2)) \in C_c^\infty$

Then V_j is not Poincaré invariant

note: $\exists$ counter examples with non-trivial spin-dependence, see Exercise Session

Proof: sketch: • evaluate condition

• use linear independence

• V_1 spanned by $\mathbb{1}_2$ and $\gamma_2^{(5)}$

• translation invariance $\Rightarrow$ explicit expressions for V_1, V_2, which are not Lorentz-invariant

III. δ-range Interactions

Other idea: "point-like" interactions when $x_i = x_j$, possibly with boundary conditions on $\partial \mathcal{S}$

↳ see lecture 3

Note: • for free Dirac in 3+1 dim.: $i\frac{d\varphi}{dt} = \sum_{j=1}^{N} H_j^{Dirac}\,\varphi$ on $(\mathbb{R}^3)^N \setminus D$

↳ $D = \{\text{some } x_i = x_j\}$

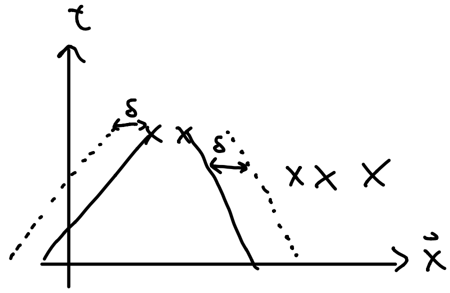
but then one can show that same eq. holds on all of $(\mathbb{R}^3)^N$

↳ lecture 3: model in 1+1 dim.

Other idea: introduce cutoff $\delta > 0$

• group particles into families with $t_i = t_j$ $\Rightarrow$ partition $P = \{S_1, \dots, S_L\}$, $S_\alpha \subset \{1, \dots, N\}$

$\Rightarrow$ $\mathcal{S}_{\delta,P} = \{\forall i,j \in S_\alpha : t_i = t_j,\ \forall \alpha \neq \beta, i \in S_\alpha, j \in S_\beta : |\vec{x}_i - \vec{x}_j| > |t_i - t_j| + \delta\}$



$\Rightarrow \psi$ on $\mathcal{S}_\delta = \bigcup_P \mathcal{S}_{\delta,P}$

$\Rightarrow$ MT eq.s: on $\mathcal{S}_{\delta,P}$: $i\frac{\partial \psi}{\partial t_\alpha} = \left(\sum_{j \in S_\alpha} H_j^{Dirac} + \sum_{i \neq j \in S_\alpha} W(\vec{x}_i - \vec{x}_j)\right)\psi$

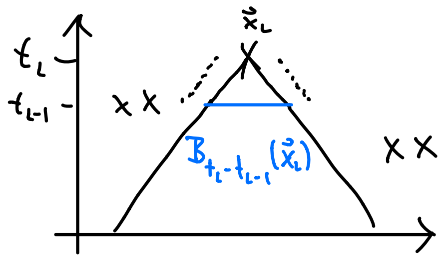
$\alpha = 1, \dots, L$ $\quad W(\vec{x}) = 0$ for $|\vec{x}| \geq \delta$

Theorem: MT eq.s consistent for smooth W.

Proof sketch: induction in L

↳ start: ok (one-time evolution)

↳ step:



crucial $\psi(\dots t_L)$ determined by initial data on $B_{t_L - t_{L-1}}(\vec{x}_L)$

then use consistency condition to show that the new ψ satisfies MT eq.s

References:

- Multi-Time Schrödinger Equations Cannot Contain Interaction Potentials
  S. Petrat and R. Tumulka
  arXiv:1308.1065
- Consistency of Multi-Time Dirac Equations with General Interaction Potentials
  D.-A. Deckert and L. Nickel
  arXiv:1603.02538