Lecture 3 – Relativistic point interactions in 1+1 dimensions

11.4.2019
Matthias Lienert

Motivation: • first example showing that an interacting, Lorentz invariant (LI) dynamics is possible for multi-time wave fns

- point interactions: delta-interaction, zero-range int.: effect only when two particles meet.
  Mathematically implemented via boundary condition (BC), otherwise free dynamics
  $\rightarrow$ no problem with consistency condition!

Setting:
- 2 Dirac particles (phys. most relevant)
- only 1+1 spacetime dimensions (then point int. is feasible mathematically)
- massless case (then Dirac eq. is easily solvable explicitly)

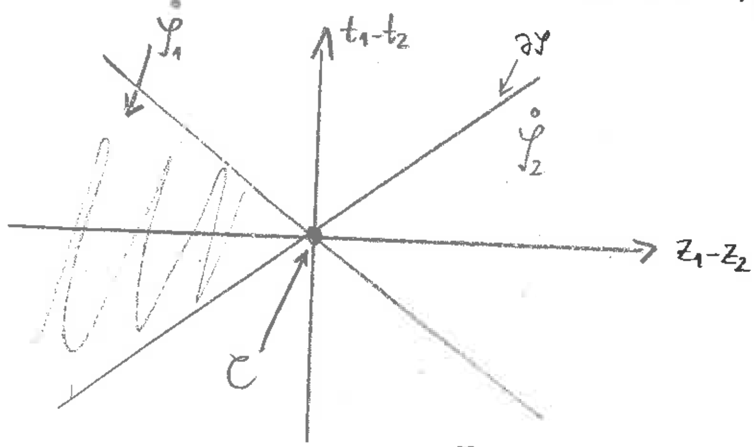
Definition of the model:

- $$\psi : \mathring{\mathscr{I}}_1 \subset \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4,$$
  $$(\underbrace{t_1, z_1}_{x_1}, \underbrace{t_2, z_2}_{x_2}) \longmapsto \begin{pmatrix} \psi_{--} \\ \psi_{-+} \\ \psi_{+-} \\ \psi_{++} \end{pmatrix} (t_1, z_1, t_2, z_2).$$

- In 1+1 dimensions: $$\mathring{\mathscr{I}} = \mathring{\mathscr{I}}_1 \sqcup \mathring{\mathscr{I}}_2$$ (disjoint union)

  with $$\mathring{\mathscr{I}}_{1/2} = \{ (t_1, z_1, t_2, z_2) \in \mathbb{R}^2 \times \mathbb{R}^2 : |t_1 - t_2| < |z_1 - z_2| \text{ and } z_1 \lessgtr z_2 \}$$

  In relative coordinates:



[leave diagram somewhere on the blackboard throughout the talk]

Consequence: Can set up a model only on the reduced domain $\mathring{\mathcal{I}}_1$ $(z_1 < z_2)$.

Boundary: $\partial \mathring{\mathcal{I}}_1 \supset \mathcal{C} = \{ (t_1, z_1, t_2, z_2) \in \mathbb{R}^2 \times \mathbb{R}^2 : t_1 = t_2, \; z_1 = z_2 \}$

"coincidence points"

- Multi-time eqs: free Dirac eqs. on $\mathring{\mathcal{I}}_1$:

$$(*) \qquad i \gamma_k^\mu \partial_{k,\mu} \psi(x_1, x_2) = 0, \qquad k = 1, 2, \; (\mu = 0, 1) \text{ sum over}$$

Here: representation $\gamma^0 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^1 = \sigma^1 \sigma^3$ $\left(\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$

- Initial conditions: $\psi(0, z_1, 0, z_2) = \psi_0(z_1, z_2)$ $\boxed{z_1 < z_2}$

given C^1 function — important!

- Boundary conditions:

linear relation between the components of ψ on $\mathcal{C}$. $\left(\text{could be written as } M \psi(t, z, t, z) = 0 \right)$

$\rightarrow$ to be specified later! — limit towards the boundary

(Should be so that local probability conservation holds.)

General solution:

Multiply $(*)$ with γ_k^0 from the left: $\quad ((\gamma^0)^2 = \mathbb{1})$

$$\Rightarrow \quad (*) \quad \Leftrightarrow \quad (\partial_{t_k} + \sigma_k^3 \partial_{z_k}) \psi = 0$$

As $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, this is diagonal. Best seen using the notation $\psi = (\psi_{s_1 s_2})$, $s_i = \mp 1$:

$$(*) \Leftrightarrow \quad (\partial_{t_k} - s_k \partial_{z_k}) \psi_{s_1 s_2} = 0, \quad k = 1, 2$$

Examples: $(\partial_{t_1} + \partial_{z_1}) \psi_{--} = 0, \quad (\partial_{t_2} + \partial_{z_2}) \psi_{--} = 0,$

$(\partial_{t_1} - \partial_{z_1}) \psi_{+-} = 0, \quad$ etc.

$$\Rightarrow \quad \psi_{--} = f_{--}(z_1 - t_1, z_2 - t_2) \qquad \text{for some } C^1\text{-fn. } f_{--}$$

In general: $\psi_{s_1 s_2} = f_{s_1 s_2}(z_1 + s_1 t_1, \; z_2 + s_2 t_2)$.

easy to remember!

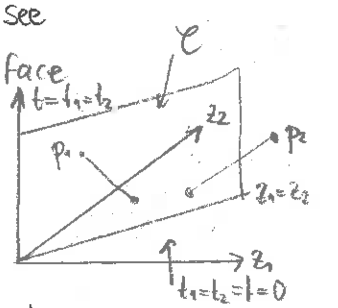
Conclusions:

- If we know $f_{s_1 s_2}$, we know $\psi_{s_1 s_2}$
- $f_{s_1 s_2}$ should be determined via initial cond. or B.C.
- $\psi_{s_1 s_2}$ is constant along " 2d-surfaces in $\mathbb{R}^2 \times \mathbb{R}^2$:

$$z_1 + s_1 t_1 = \text{const}, \qquad z_2 + s_2 t_2 = \text{const}.$$

$\rightsquigarrow$ "multi-time characteristics"

So: If we want to know ψ in $p = (t_1, z_1, t_2, z_2)$, we have to see where the respective multi-time characteristic intersects a surface where ψ is already known.



1st candidate: Initial data surface: $t_1 = t_2 = 0$.

We have: $\psi_{s_1 s_2}(0, z_1, 0, z_2) = f_{s_1 s_2}(z_1 + 0, z_2 + 0) \overset{!}{=} \psi_{0, s_1 s_2}(z_1, z_2)$, $\boxed{z_1 < z_2}$

This determines $f_{s_1 s_2}(x,y)$ for $x < y$. (Half the values of f.)

Do we need more values of $f_{s_1 s_2}$?

We have $(t_1, z_1, t_2, z_2) \in \mathcal{S}_1$ (i.e. $z_1 < z_2$, $|t_1 - t_2| < |z_1 - z_2|$.)

a) ψ_{--}: $f_{--}(z_1 - t_1, z_2 - t_2)$

Can $z_1 - t_1 > z_2 - t_2$ occur?

$\Leftrightarrow \quad t_2 - t_1 > \underbrace{z_2 - z_1}_{>0} = |z_1 - z_2| \quad \Rightarrow \quad |t_2 - t_1| > |z_1 - z_2|$ ↯ cannot occur (time-like config.)

Similarly for ψ_{++}.

b) ψ_{-+}: $f_{-+}(z_1 - t_1, z_2 + t_2)$

Can $z_1 - t_1 > z_2 + t_2$ occur? $\Leftrightarrow \quad -(t_1 + t_2) > z_2 - z_1 > 0$

Consider e.g. $t_1 = t_2 = t$. Then (t, z_1, t, z_2) is always spacelike.

$$z_1 - t > z_2 + t \quad \Leftrightarrow \quad -t > \frac{z_2 - z_1}{2} (> 0).$$

This can occur for negative times!

Similarly: ψ_{+-} is not determined for $(t_1 + t_2) > z_2 - z_1 > 0$.

This suggests:

- ψ_{--}, ψ_{++} should not be subject to a B.C.
- ψ_{-+}, ψ_{+-} should (alternatingly, for positive / negative times) be subject to a B.C.

How a B.C. works:

Say, for $t < 0$:

$$\psi_{-+}(t,z,t,z) = h_1(t,z) \quad \text{(given fn.)}$$

$$\Leftrightarrow \quad f_{-+}(\underbrace{z-t}_{a}, \underbrace{z+t}_{b}) = h_1(t,z)$$

$$\Leftrightarrow \quad f_{-+}(a,b) = h_1\left(\tfrac{b-a}{2}, \tfrac{a+b}{2}\right).$$

But how to choose these B.C. (or fn. h_1 etc.)?

→ physical considerations needed!

[maybe here would be a good point for a break.]

Boundary conditions from local probability conservation (P.C.)

P.C. for a domain $\Omega \subset \mathbb{R}^2 \times \mathbb{R}^2$ means: $P(\Sigma) = P(\Sigma')$ $\forall$ space-like Cauchy surfaces Σ, Σ',

$$P(\Sigma) = \int_{(\Sigma\times\Sigma)\cap\Omega} d\sigma_{1\mu}\, d\sigma_{2\nu}\ \underbrace{\overline{\Psi}\gamma_1^\mu\gamma_2^\nu\Psi}_{j^{\mu\nu}}.$$

For multi-time eqs. on $\Omega = \mathbb{R}^2\times\mathbb{R}^2$, we know already that this holds, as

$$\partial_{1\mu} j^{\mu\nu} = 0 = \partial_{2\nu} j^{\nu\mu}$$

and because of the div-thm.

Here, (we need to consider that) prob. can get lost through the boundary.

→ Repeat construction for div. thm (using Stokes' thm. this time).

Possible shortcut (if time is scarce): Just say that one needs to avoid a loss of P. through $\mathcal{C}$ (only effective boundary, as $(\Sigma\times\Sigma)\cap\mathcal{Y}_1$ only has a boundary $\subset \mathcal{C}$).

This leads to $\boxed{(j^{01}-j^{10})(t,z,t,z) = 0}$

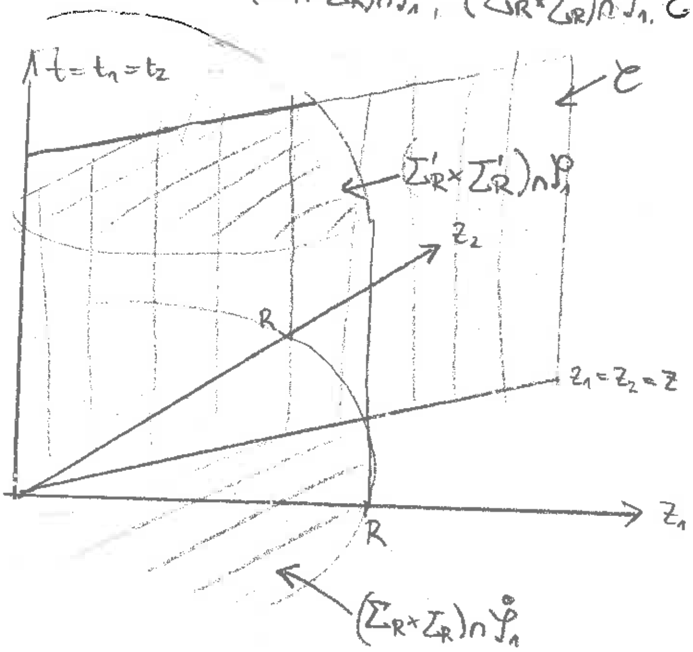
Let Ψ be s.t. $\Psi(t_1, z_1, t_2, z_2) = 0$ if $|z_1|$ or $|z_2| > R$ for some $R>0$.

"compactly supported in space"

Define $\Sigma_R^{(1)} = \{ (t_1, z_1, t_2, z_2) \in \Sigma^{(1)} : |z_1|, |z_2| < R \}$

Then construct a closed 2-dim. surface $S_R \subset \mathbb{R}^2 \times \mathbb{R}^2$ s.t. $(\Sigma_R \times \Sigma_R) \cap \mathring{\mathscr{Y}}_1, \ (\Sigma'_R \times \Sigma'_R) \cap \mathring{\mathscr{Y}}_1 \subset S_R$

Visualization for equal times:



S_R will have the form

$$S_R = \left[(\Sigma_R \times \Sigma_R) \cap \mathring{\mathscr{Y}}_1 \right] \cup \left[(\Sigma'_R \times \Sigma'_R) \cap \mathring{\mathscr{Y}}_1 \right] \cup M_R \cup M_{\mathscr{C}}$$

with $(t_1, z_1, t_2, z_2) \in M_R \Rightarrow |z_1| = R$ or $|z_2| = R$, i.e. $\Psi|_{M_R} = 0$

and $M_{\mathscr{C}} \subset \mathscr{C}$.

Let $\omega_j = d\sigma_{1\mu}\, d\sigma_{2\nu}\, j^{\mu\nu}$ (2-form)

"current density form"

$$\omega_j = \sum_{\mu,\nu=0}^{1} (-1)^{\mu+\nu} j^{\mu\nu} \, dx_1^0 \wedge \underset{\swarrow \text{omitted}}{\widehat{dx_1^\mu}} \wedge dx_2^0 \wedge \widehat{dx_2^\nu}$$

We have (Exercise?): $d\omega_j = 0$ (exterior derivative).

$\Rightarrow$ Stokes thm. yields: (Let V_R be s.t. $\partial V_R = S_R$):

$$\int_{S_R} \omega_j = \int_{V_R} d\omega_j = 0$$

$$\Rightarrow \int_{(\Sigma_R \times \Sigma_R) \cap \mathring{\mathscr{Y}}_1} \omega_j = \int_{(\Sigma'_R \times \Sigma'_R) \cap \mathring{\mathscr{Y}}_1} \omega_j + \underbrace{\int_{M_R} \omega_j}_{\substack{= 0, \text{ as} \\ \Psi|_{M_R} = 0}} + \int_{M_{\mathscr{C}}} \omega_j$$

Conclusion: We get P.C. if

$$\int_{M_{\mathcal{E}}} \omega_j = 0.$$

Our B.C. needs to ensure that. We demand:

$$\omega_j|_{\mathcal{C}} = 0$$

Evaluate this cond. in rel. coords.: $z = z_1 - z_2$, $Z = z_1 + z_2$, $\tau = t_1 - t_2$, $T = t_1 + t_2$

$$\omega_j = \tfrac{1}{2} j^{00}\, dz \wedge dZ - \tfrac{1}{4}(j^{10} - j^{01})\, d\tau \wedge dZ + \tfrac{1}{4}(j^{10} - j^{01})\, d\tau \wedge dz$$
$$- \tfrac{1}{4}(j^{10} - j^{01})\, dT \wedge dZ - \tfrac{1}{4}(j^{10} + j^{01})\, dz \wedge dT + \tfrac{1}{2} j^{11}\, d\tau \wedge dT$$

On $\mathcal{C}$: $z = 0 = t$.

$$\Rightarrow \quad \omega_j|_{\mathcal{C}} = -\tfrac{1}{4}(j^{10} - j^{01})\, dT \wedge dZ$$

Vanishes if:

$$\boxed{(j^{10} - j^{01})(t,z,t,z) = 0 \quad \forall\, t,z \in \mathbb{R}.} \qquad (+)$$

This is our cond. ("local prob. cons.")

How to convert this into a B.C. for Ψ?

Write out $j^{\mu\nu} = \bar{\Psi} \gamma_1^\mu \gamma_2^\nu \Psi$ in comp. of Ψ:

$$j^{\mu\nu} = |\Psi_{--}|^2 + (-1)^\nu |\Psi_{-+}|^2 + (-1)^\mu |\Psi_{+-}|^2 + (-1)^{\mu+\nu} |\Psi_{++}|^2$$

Thus: $$j^{01} - j^{10} = 2\left(|\Psi_{+-}|^2 - |\Psi_{-+}|^2\right)$$

and $(+) \iff \underbrace{\boxed{\Psi_{-+} = e^{i\theta}\, \Psi_{+-}}}_{\text{on } \mathcal{C}}$ for some $\theta \in [0, 2\pi)$.

(In principle, θ could be a fn. of t, z but then the dynamics would not be translation invariant.)

This is our B.C.!

Main result: (perhaps abbreviate in the lecture.)

Theorem: Let $\theta \in [0, 2\pi)$. Then our model defined by

$$\begin{cases} i\gamma_k^\mu \partial_{k,\mu} \psi = 0 & \text{on } \mathring{\mathcal{S}}_1, \\ \psi(0, z_1, 0, z_2) = \psi_0(z_1, z_2), \quad z_1 < z_2, & (\psi_0 \in C^1(\mathbb{R}^2)) \\ \psi_{-+} = e^{i\theta}\psi_{+-} & \text{on } \mathcal{C} \end{cases}$$

has the unique solution

$$\begin{pmatrix} \psi_{--} \\ \psi_{-+} \\ \psi_{+-} \\ \psi_{++} \end{pmatrix}(t_1, z_1, t_2, z_2) = \begin{pmatrix} \psi_{0,--}(z_1 - t_1, z_2 - t_2), \\ \begin{cases} \psi_{0,-+}(z_1 - t_1, z_2 + t_2), & z_1 - t_1 \le z_2 + t_2, \\ \psi_{0,+-}(z_2 + t_2, z_1 - t_1) & \text{else} \end{cases} \\ \begin{cases} \psi_{0,+-}(z_1 + t_1, z_2 - t_2), & z_1 + t_1 \le z_2 - t_2, \\ \psi_{0,-+}(z_2 - t_2, t_1 + t_1), & \text{else} \end{cases} \\ \psi_{0,++}(z_1 + t_1, z_2 + t_2) \end{pmatrix}.$$

This solution is continuous if ψ_0 satisfies the B.C. and continuously differentiable if in addition

$$\begin{aligned} &\text{(i)} \quad (\partial_1 \psi_{0,-+})(z, z) = e^{i\theta}(\partial_2 \psi_{0,+-})(z, z) \\ &\text{(ii)} \quad (\partial_2 \psi_{0,-+})(z, z) = e^{i\theta}(\partial_1 \psi_{0,+-})(z, z) \end{aligned} \qquad \forall z \in \mathbb{R}.$$

Moreover, if ψ_0 is compactly supported, then ψ is compactly supported in space for all t_1, t_2 and prob. cons. in the sense of

$$\int_{(\Sigma \times \Sigma) \cap \mathring{\mathcal{S}}_1} d\sigma_{1\mu}\, d\sigma_{2\nu}\, j^{\mu\nu} = \int_{(\Sigma' \times \Sigma') \cap \mathring{\mathcal{S}}_1} d\sigma_{1\mu}\, d\sigma_{2\nu}\, j^{\mu\nu}$$

for all Cauchy surfaces $\Sigma, \Sigma' \in \mathbb{R}^2$ holds.

Lorentz invariance (LI)

Apart from the B.C., LI is already manifest.
To check the LI of the BC, note that under Λ proper L.T.:

$$\psi'(t_1', z_1', t_2', z_2') = S[\Lambda] \otimes S[\Lambda]\, \psi(t_1, z_1, t_2, z_2)$$

with $S[\Lambda] = \exp(\omega\, \gamma^0 \gamma^1 / 2) \underset{\text{calculation with matrix exponential}}{=} \cosh(\omega/2)\, \mathbb{1}_2 + \sinh(\omega/2)\, \sigma^3$

$$= \begin{pmatrix} \cosh(\omega/2) + \sinh(\omega/2) & 0 \\ 0 & \cosh(\omega/2) - \sinh(\omega/2) \end{pmatrix}$$

From this it follows that

$$\psi'_{-+} = (\cosh(\omega/2) + \sinh(\omega/2))(\cosh(\omega/2) - \sinh(\omega/2))\psi_{-+} \overset{\cosh^2 - \sinh^2 = 1}{=} \psi_{-+}$$

and $$\psi'_{+-} = (\cosh(\omega/2) - \sinh(\omega/2))(\cosh(\omega/2) + \sinh(\omega/2))\psi_{+-} = \psi_{+-}$$

Thus, as (t_1', z_1', t_2', z_2') is again in $\mathcal{C}$, the B.C. is indeed! LI.

Interaction

The goal was to construct an interacting model.
How do we check that?

Meaning of interaction: There are some initial product wave fns. which get entangled with time.
("Free" would mean: every initial product wave fn. remains a product wave fn. also for later times.)

Here: Our model is indeed interacting. (One can see this from the solution formula.)

Conclusion: The model indeed gives a first (rigorous) example of a manifestly LI, interacting multi-time model compatible with a prob. interpretation.

Outlook:

- N particles → done
- mass → certainly possible but a lot more indirect & complicated
- QFT-model in a similar spirit: see lecture 5, part 2
- interacting electron & photon: current work, soon completed
- Higher dimensions: probably not possible
  (→ Svendsen's result: impossibility of point interactions for Dirac particles in 2 or 3 spatial dimensions)

References:
arXiv: 1411.2833
: 1502.00917