Spring School on Multi-time Wave Functions

April 11, 2019

Lecture 4: Multi-time Quantum Field Theory

Interacting MT equations: • potentials? $\rightarrow$ mostly ruled out

• δ-like interactions? $\rightarrow$ not in 3+1 dim.

here: int. by exchange of particles, as in QFT

I. Fock Space

so far: • $\varphi(t, \underbrace{\vec{x}_1, \dots, \vec{x}_N}_{\in (\mathbb{R}^3)^N = \text{configuration space}}) \in (\mathbb{C}^4)^N = \text{spin space}$; better $\varphi_{\underbrace{s_1 \dots s_N}_{\text{spin indices}}}(t, \vec{x}_1, \dots, \vec{x}_N)$

• $\varphi \in L^2_{s/as}(\mathbb{R}^3, \mathbb{C}^4)^{\otimes N} = \text{Hilbert space}$

$\uparrow$ symmetrized (antisymm.) for bosons (fermions)

• scalar product $\langle \varphi, \tilde{\varphi} \rangle_N = \int d^3\vec{x}_1 \cdots \int d^3\vec{x}_N \, \varphi^*(\vec{x}_1, \dots, \vec{x}_N) \tilde{\varphi}(\vec{x}_1, \dots, \vec{x}_N)$

now: • $\varphi = \begin{pmatrix} \varphi(t) \\ \varphi(t, \vec{x}_1) \\ \varphi(t, \vec{x}_1, \vec{x}_2) \\ \vdots \\ \varphi(t, \vec{x}_1, \dots, \vec{x}_N) \\ \vdots \end{pmatrix}$, configuration space $\bigcup_{N=0}^{\infty} (\mathbb{R}^3)^N =: \Gamma(\mathbb{R}^3)$

$\leftarrow$ N-particle sector, sometimes: $\varphi^{(N)}$

• $\varphi \in \bigoplus_{N=0}^{\infty} L^2(\mathbb{R}^3, \mathbb{C}^4)^{\otimes N} =: \mathcal{F} = \text{Fock space (still a Hilbert space)}$

• scalar product $\langle \varphi, \tilde{\varphi} \rangle = \sum_{N=0}^{\infty} \langle \varphi^{(N)}, \tilde{\varphi}^{(N)} \rangle_N$

Connection between different sectors:

• annihilation operator $a_s(\vec{x})$:

$$\underbrace{\left(a_s(\vec{x})\varphi\right)^{(N)}_{s_1\dots s_N}(\vec{x}_1,\dots,\vec{x}_N)}_{N\text{-particle sector}} = \sqrt{N+1}\,\varepsilon^N\,\underbrace{\varphi^{(N+1)}_{s_1\dots s_N s}(\vec{x}_1,\dots,\vec{x}_N,\vec{x})}_{(N+1)\text{-particle sector}}$$

$\sqrt{N+1}$: convenient combinatorial factor

$\varepsilon = \begin{cases} +1, & \text{bosons} \\ -1, & \text{fermions, to preserve antisymm.} \end{cases}$

- creation operator $a_s^\dagger(\vec{x})$ = adjoint of $a_r(\vec{x})$

$$\left(a_s^\dagger(\vec{x})\varphi\right)^{(N)}_{s_1\dots s_N}(\vec{x}_1,\dots,\vec{x}_N) = \frac{1}{\sqrt{N}}\underbrace{\sum_{j=1}^N \varepsilon^{j+1}}_{\text{to preserve symm./antisymm.}}\underbrace{\delta_{ss_j}\overbrace{\delta^{(3)}(\vec{x}_j-\vec{x})}^{=\delta(x_j^1-x^1)\delta(x_j^2-x^2)\delta(x_j^3-x^3)}}_{\text{"wave fct. of created particle"}}\varphi_{\underbrace{s_1\dots s_N\setminus s_j}_{\equiv \hat{s}_j}}(\underbrace{\vec{x}_1,\dots,\vec{x}_N\setminus\vec{x}_j}_{\equiv \vec{x}_1,\dots,\hat{\vec{x}}_j,\dots,\vec{x}_N})$$

note: • CCR/CAR: $\left[a_s^\dagger(\vec{x}), a_{s'}^\dagger(\vec{y})\right]_\varepsilon = 0 = \left[a_s(\vec{x}), a_{s'}(\vec{y})\right]_\varepsilon$,

$\left[a_s(\vec{x}), a_{s'}^\dagger(\vec{y})\right]_\varepsilon = \delta_{ss'}\,\delta^{(3)}(\vec{x}-\vec{y})$, $[\cdot,\cdot]_{\varepsilon=1}$ commutator, $[\cdot,\cdot]_{\varepsilon=-1}$ anti-commutator

- $\mathcal{N} := \int_{\mathbb{R}^3} d^3\vec{x} \sum_{s=1}^4 a_s^\dagger(\vec{x})a_s(\vec{x})$ called number operator

  $\hookrightarrow \mathcal{N}\varphi^{(N)} = N\varphi^{(N)}$

- $\underbrace{\int_{\mathbb{R}^3} d^3\vec{x} \sum_{s=1}^4 a_s^\dagger(\vec{x})\, H_{\vec{x}}\, a_s(\vec{x})}\,\varphi^{(N)} = \sum_{j=1}^N H_{\vec{x}_j}\varphi^{(N)}$, see Exercise Session

  = "second quantization" of $H_{\vec{x}}: L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)$ (differential + multiplication operator)

  = Hamiltonian on $\mathcal{F}$ (non-int.)

- field operator: $\Phi_s(\vec{x}) = a_s(\vec{x}) + a_s^\dagger(\vec{x})$
- $a_s(\vec{x}), a_s^\dagger(\vec{x})$ operator-valued distributions (δ-like wave fct.s are created)

II. Emission-Absorption Model

- fix N: N fermions (x-particles) can emit/absorb bosons (y-particles)
- free evolution apart from emission/absorption
- both x and y are Dirac particles (y: bosonic Dirac particles, contrary to spin-statistic thm.)

Hilbert space $\mathcal{F}_{X,Y} = \mathcal{F}_X \otimes \mathcal{F}_Y$

$\hookrightarrow$ antisymm. $\nwarrow$ symm.

Schrödinger eq.: $i\frac{d\varphi}{dt} = H\varphi(t)$

$$\text{Hamiltonian } H = \underbrace{\int d^3\vec{x} \sum_{r,r'=1}^{4} a_r^\dagger(\vec{x})\, H_{\vec{x}}^{(\text{Dirac})\,r,r'}\, a_{r'}(\vec{x})}_{H_X} + \underbrace{\int d^3\vec{y} \sum_{s,s'=1}^{4} b_s^\dagger(\vec{y})\, H_{\vec{y}}^{(\text{Dirac})\,s,s'}\, b_{s'}(\vec{y})}_{H_Y}$$

X-particle ann. op.

Y-particle ann. op.

$$+ \underbrace{\int d^3\vec{x} \sum_{r,s=1}^{4} a_r^\dagger(\vec{x}) \left(g_s^* b_s(\vec{x}) + g_s b_s^\dagger(\vec{x}) \right) a_r(\vec{x})}_{H_{int}} \quad , g \in \mathbb{C}^4 \text{ given}$$

in each sector: $i\frac{d\varphi}{dt}(\underbrace{\vec{x}_1,\dots,\vec{x}_M}_{=:X^{3M}}, \underbrace{\vec{y}_1,\dots,\vec{y}_N}_{=:Y^{3N}}) = (H\varphi)(X^{3M}, Y^{3N})$

$$= \sum_{j=1}^{M} H_{x_j}^{\text{Dirac}} \varphi(X^{3M}, Y^{3N}) + \sum_{k=1}^{N} H_{y_k}^{\text{Dirac}} \varphi(X^{3M}, Y^{3N})$$

$$+ \sqrt{N+1} \sum_{j=1}^{M} \sum_{s_{N+1}=1}^{4} g_{s_{N+1}}^* \varphi_{s_{N+1}}(X^{3M}, (Y^{3N}, \vec{x}_j))$$

$$+ \frac{1}{\sqrt{N}} \sum_{j=1}^{M} \sum_{k=1}^{N} g_{s_k} \delta^{(3)}(\vec{y}_k - \vec{x}_j)\, \varphi_{\hat{s}_k}(X^{3M}, (Y^{3N} \setminus \vec{y}_k))$$

"change in N-particle sector related to $(N+1)$ and $(N-1)$-particle sectors"

note:
- indeed interacting! (see, e.g., non-relativistic limit)
- H ill-defined: ultraviolet divergent
  - $\hookrightarrow$ introduce cutoff: replace $\delta^{(3)}(\vec{x})$ by $\Lambda(\vec{x}) \in C_c^\infty$ (supp $\Lambda \to 0$, renormalization)
  - $\hookrightarrow$ IBCs (lectures 3, 5)
  - $\hookrightarrow$ here: just ignore
- g not Lorentz invariant
- there are negative energy solutions

III. MT em-ab Model

spacelike configurations $\mathcal{S}_{xy} = \bigcup_{M,N=0}^{\infty} \{ \text{all } x^{4M}, y^{4N} \text{ spacelike or equal} \}$

$$
\left.
\begin{aligned}
i\frac{\partial \psi}{\partial x_j^0}(x^{4M},y^{4N}) = H_{x_j}^{Dirac}\psi(x^{4M},y^{4N}) \\
+\sqrt{N+1}\sum_{s_{N+1}=1}^{4} g^*_{s_{N+1}} \psi_{s_{N+1}}(x^{4M},(y^{4N},x_j)) \\
+\frac{1}{\sqrt{N}}\sum_{k=1}^{N} G_{s_k}(y_k-x_j)\,\psi_{\hat{s}_k}(x^{4M},y^{4N}\setminus y_k)
\end{aligned}
\right\} \text{int. at collision configurations}
$$

$$
i\frac{\partial \psi}{\partial y_k^0}(x^{4M},y^{4N}) = H_{y_k}^{Dirac}\psi(x^{4M},y^{4N})
$$

with G sol. to $i\frac{\partial G}{\partial t} = H_y^{Dirac} G$ with $G_s(0,\vec{y}) = g_s\,\delta^{(3)}(\vec{y})$

note:
- meaning at e.g., $x_j = y_k$: directional derivative

  $i\left(\frac{\partial}{\partial x_j^0} + \frac{\partial}{\partial y_k^0}\right)\psi = \dots$
- if all times equal, we recover one-time Hamiltonian from II
- $G(y_k - x_j) = 0$ if y_k spacelike to x_j

  $\Rightarrow$ could also add $\sum_j G_{s_k}\dots$ to y_k^0 eq.

  ↳ no difference on $\mathcal{S}_{xy}$ when $x_j \neq y_k$

  ↳ when $x_j = y_k$, use directional derivative anyway

  $\Rightarrow$ same sol. on $\mathcal{S}_{xy}$ (but different sol. on $T(\mathbb{R}^4)^2$)

- 0-particle sector has no time-variable in MT formulation, so need one-time theory with time-indep. 0-particle sector ("no creation out of vacuum")
- preserves permutation symm. for space-time points:

$$\psi_{r_{\pi(1)}\dots r_{\pi(M)} s_{\rho(1)}\dots s_{\rho(N)}}(x_{\pi(1)},\dots,x_{\pi(M)},y_{\rho(1)},\dots,y_{\rho(N)}) = (-1)^{\sigma}\psi_{s_1\dots s_M r_1\dots r_N}(x_1,\dots,x_M,y_1,\dots,y_N)$$

for permutations π,ρ

__Assertion:__ Ignoring UV divergence, sol.s to em-ab model on $\mathcal{S}_{xy}$ exist and are unique $\forall$ initial cond. (model is consistent).

note: • special case when $g^\dagger\beta g = 0$ or mass $m_y = 0$, then sol.s exist on all of $\Gamma(\mathbb{R}^4)^2$

__Proof sketch:__ • compute consistency conditions (see also Exercise Sessions)

$$\left[i\frac{\partial}{\partial x_i^0} - H_{x_i}, i\frac{\partial}{\partial x_j^0} - H_{x_j}\right] = \sum_{s=1}^{4} g_s^* \left(G_s(x_i-x_j) - G_s(x_j-x_i)\right)$$

$$= m_y g^\dagger \beta g \underbrace{\Delta(x_i-x_j)}_{=0 \text{ for } x_i \text{ spacelike to } x_j \text{ or } x_i = x_j}$$

• generalize domain of dependence to variable particle number

• similar to δ-range model (end of lecture 2): partition into families, induction

IV. Pair Creation Model

• three species of Dirac particles: $x, \bar{x}, y$

• $x + \bar{x} \to y$ (pair annihilation), $y \to x + \bar{x}$ (pair creation)

• $H_{int} = \int d^3\vec{x} \sum_{r,\bar{r},s=1}^{4} \left(g_{r\bar{r}s}\, a_r^\dagger(\vec{x})\, \bar{a}_{\bar{r}}^\dagger(\vec{x})\, b_s(\vec{x}) + g_{r\bar{r}s}^*\, a_r(\vec{x})\, \bar{a}_{\bar{r}}(\vec{x})\, b_s^\dagger(\vec{x}) \right)$

• same remarks as in III apply

• consistency conditions: terms $(\varepsilon_x \varepsilon_{\bar{x}} \varepsilon_y - 1)$ appear (see also lecture 5A)

$\Rightarrow$ consistency iff • all bosons

• two fermions, one boson

$\Rightarrow$ "fermion number conservation"

References:

- Multi-Time Wave Functions for Quantum Field Theory

  S. Petrat and R. Tumulka

  arXiv:1309.0802
- Multi-Time Formulation of Pair Creation

  S. Petrat and R. Tumulka

  arXiv:1401.6093