Spring School on Multi-time Wave Functions

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Lecture 5A: Multi-time Quantum Field Theory 2

I. Heisenberg Picture

Heisenberg picture: time evolution of operators instead of wave fct.s

non-rel. QM: $\langle \psi(t), A\psi(t)\rangle = \langle e^{-iHt}\psi(0), A e^{-iHt}\psi(0)\rangle = \langle \psi(0), \underbrace{e^{iHt} A e^{-iHt}}_{A(t)} \psi(0)\rangle$

$$\Rightarrow i\frac{dA(t)}{dt} = [H, A(t)]$$

QFT: define $a_s^{(+)}(t,\vec{x}) = e^{iHt} a_s^{(+)}(0,\vec{x}) e^{-iHt}$

general idea: $\psi(x_1,\dots,x_N) = \underbrace{\frac{1}{\sqrt{N!}}}_{\text{combinatorial factor}} \langle \emptyset, a(x_1)\dots a(x_N)\psi_0\rangle$

vacuum: $a(x)|\emptyset\rangle = 0$

note: • $\varphi(t,\vec{x}_1,\dots,\vec{x}_N) = \frac{1}{\sqrt{N!}}\langle \emptyset, e^{iHt}a(\vec{x}_1)e^{-iHt}e^{iHt}a(\vec{x}_2)e^{-iHt}\dots e^{iHt}a(\vec{x}_N)e^{-iHt}\psi_0\rangle$

$e^{-iHt}|\emptyset\rangle = |\emptyset\rangle$ if no creation out of vacuum

$$= \frac{1}{\sqrt{N!}}\langle \emptyset, a(\vec{x}_1)\dots a(\vec{x}_N)e^{-iHt}\psi_0\rangle$$

$$= \frac{1}{\sqrt{N!}}\underbrace{\langle a^\dagger(\vec{x}_N)\dots a^\dagger(\vec{x}_1)\emptyset, e^{-iHt}\psi_0\rangle}$$

$$= \int d^3\vec{y}_1\dots\int d^3\vec{y}_N \sum\dots\delta(\vec{y}_1-\vec{x}_1)\dots\varphi(t,\vec{y}_1,\dots,\vec{y}_N)$$

$$= \sqrt{N!}\,\varphi(t,\vec{x}_1,\dots,\vec{x}_N) \quad (\text{initial condition } \varphi(0) = \psi_0)$$

• on collision-free configurations (i.e., $x_i \neq x_j\ \forall i\neq j$):

$\psi(x_1,\dots,x_N) = \frac{1}{\sqrt{N!}}\langle \emptyset, \Phi(x_1)\dots\Phi(x_N)\psi_0\rangle$ with field operators $\Phi(x) = a(x) + a^\dagger(x)$

Since: $\psi = \frac{1}{\sqrt{N!}}\langle \emptyset, (a(x_1)+a^\dagger(x_1))(a(x_2)+a^\dagger(x_2))\dots\psi_0\rangle$

$a(x)|\emptyset\rangle = 0$ $\qquad [a(x_1), a^\dagger(x_2)] = 0$ for $x_1 \neq x_2$

• permutation symm.: $\Psi(x_i \leftrightarrow x_j) = \frac{1}{\sqrt{N!}} \langle \emptyset, \dots \Phi(x_i) \leftrightarrow \Phi(x_j) \dots \Psi_0 \rangle$

$\xleftarrow{\varepsilon^{j-i}}$ $(i<j)$

$\xrightarrow{\varepsilon^{j-i-1}}$ $\quad \varepsilon = \begin{cases} +1, & \text{bosons} \\ -1, & \text{fermions} \end{cases}$

$$= \varepsilon \frac{1}{\sqrt{N}} \Psi(x_i, x_j)$$

<u>Assertion</u>: Let Ψ be sol. to em-ab model with initial cond. Ψ_0 (where all times $= 0$).

Then, on $\mathcal{S}_{xy}$:

$$\Psi_{r_1 \dots r_M s_1 \dots s_N}(x_1, \dots, x_M, y_1, \dots, y_N) = \frac{(-1)^{M(M-1)/2}}{\sqrt{M!N!}} \langle \emptyset, a_{r_1}(x_1) \dots a_{r_M}(x_M) b_{s_1}(x_1) \dots b_{s_N}(x_N) \Psi_0 \rangle$$

"Proof": • use Tomonaga-Schwinger eq. to conclude $a_r(x) = U_{\Sigma \to \Sigma_0} \, a_{\Sigma, r}(x) \, U_{\Sigma_0 \to \Sigma}$, $x \in \Sigma$

full time evolution from $\Sigma \to \Sigma_0$

ann.op. on Σ → satisfies CAR/CCR (with transformed δ-distribution)

• then choose $x^{4M}, y^{4N} \in \Sigma$

$$\Rightarrow \langle \emptyset, a_{r_1}(x_1) \dots a_{r_M}(x_M) b_{s_1}(y_1) \dots b_{s_N}(y_N) \Psi_0 \rangle$$

$$= \langle \emptyset, U^{-1} U a U^{-1} \dots U a U^{-1} \dots U b U^{-1} \underbrace{U \Psi_0}_{\Psi_\Sigma} \rangle$$

no creation out of vacuum → $U|\emptyset\rangle = |\emptyset_\Sigma\rangle$

$$= \langle \emptyset_\Sigma, a_\Sigma \dots a_\Sigma \dots \Psi_\Sigma \rangle$$

$$= \sqrt{M!N!} \, (-1)^{M(M-1)/2} \, \Psi_\Sigma$$

II. Tomonaga–Schwinger

Interaction picture:

non-rel. QM: $H = H^{free} + V$, $i\frac{d}{dt}\varphi(t) = H\varphi(t)$, $\varphi(t)$: Schrödinger picture

$\Rightarrow$ consider $\tilde\varphi(t) = e^{-iH^{free}t}\varphi(t)$, then

$$i\frac{d}{dt}\tilde\varphi(t) = H^{free}\tilde\varphi(t) + e^{-iH^{free}t} H \underbrace{\varphi(t)}_{= e^{iH^{free}t}e^{-iH^{free}t}\varphi(t)} = \underbrace{e^{-iH^{free}t}\, V e^{iH^{free}t}}_{\tilde V(t)}\tilde\varphi(t)$$

Tomonaga–Schwinger:

- wave-fct.s Ψ_Σ on spacelike Σ
- to work in fixed Hilbert space:
  use free evolution $F_{\Sigma\to\Sigma'}$ to identify $\mathcal{H}_\Sigma$ with $\mathcal{H}_{\Sigma'}$
- say, fix $\tilde{\mathcal H} = \mathcal H_{\tilde\Sigma}$, then $\underset{\substack{\uparrow \\ \tilde{\mathcal H}}}{\tilde\Psi_\Sigma} = F_{\Sigma\to\tilde\Sigma}\underbrace{\Psi_\Sigma}_{\in\mathcal H_\Sigma}$
- T-S eq.: $i(\tilde\Psi_{\Sigma'} - \tilde\Psi_\Sigma) = \left(\int_\Sigma^{\Sigma'} d^4x\, \mathcal H_I(x)\right)\tilde\Psi_\Sigma$

$$\mathcal H_I(x) = e^{-iH^{free}x^0}\underbrace{\mathcal H_{int}(\vec x)}_{\text{e.g., } = a^\dagger(\vec x)(b(\vec x)+b^\dagger(\vec x))a(\vec x) \text{ for em-ab}} e^{iH^{free}x^0}$$

- consistency condition $[\mathcal H_I(x), \mathcal H_I(y)] = 0$, x, y spacelike

Assertions for em-ab model:

- MT eq.s def. $U_{\Sigma\to\Sigma'} : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$

- Ψ sol. to MT

  ↳ def. $\Psi_\Sigma(x^{4M}, y^{4N}) := \Psi(x^{4M}, y^{4N})$ for $x^{4M}, y^{4N} \in \Sigma$

  ↳ def. $\tilde{\Psi}_\Sigma := F_{\Sigma\to\Sigma_0} \Psi_\Sigma$

  $\Longrightarrow \tilde{\Psi}_\Sigma$ satisfies TS eq.

- other way around: need $\Psi_\Sigma(x^{4M}, y^{4N}) = \Psi_{\Sigma'}(x^{4M}, y^{4N})$, $x^{4M}, y^{4N} \in \Sigma$, $\forall \Sigma' \ni x^{4M}, y^{4N}$

  This is true for em-ab, i.e., TS sol. Ψ_Σ gives MT wave fct.

  Proof uses in particular:
  - Fock space structure
  - no creation from vacuum
  - finite propagation speed
  - "local" H_I

References:

- Multi-Time Wave Functions for Quantum Field Theory

  S. Petrat and R. Tumulka

  arXiv:1309.0802