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Lecture 5, Part 2: Interior-boundary conditions for multi-time wave functions

Motivation:
- Extend model from lecture 3 to QFT
- Create rigorous example for multi-time QFT
- Show that the multi-time approach is compatible with "interior-boundary conditions" (IBCs).

What is an IBC?

A (linear) condition relating the wave fn. ψ at a boundary point $q \in \partial \Omega$ (Ω: domain) to the wave fn. at an interior point $p \in \Omega$.

Usually: $\psi^{(n+1)}(\underbrace{x_1, \dots \overset{\text{same point}}{x, \dots, x}, \dots x_{n+1}}_{q})$ related to $\psi^{(n)}(x_1, \dots, x, \dots x_n)$

Program by Tumulka & Teufel (& collaborators) Hope that this could help to overcome problems with UV div. in QFT.

IBCs allow to formulate non-rigorous terms (with δ-fns.) resulting from creation operators in the Hamiltonian rigorously. (Similarly as contact int. do with δ-fns. in the Hamiltonian.)

Setting:
- 1 and 2 particle sectors only
- massless Dirac particles
- 1+1 dimensions
- interactions of the type

(One Dirac particle can split up into two, or two can merge into one.)

Wave fn.: $\psi = (\psi^{(1)}, \psi^{(2)})$

with $\psi^{(1)}: \mathbb{R}^2 \to \mathbb{C}^2, \quad (\underbrace{t, z}_{x}) \mapsto \psi^{(1)}(t,z)$

$\psi^{(2)}: \mathring{\mathscr{S}}_1 \to \mathbb{C}^4, \quad (t_1, z_1, t_2, z_2) \mapsto \psi^{(2)}(\dots)$ $\quad \boxed{z_1 < z_2}$

$\hookrightarrow$ has boundary $\mathcal{C}$

Multi-time equations: (Hamiltonian form here)

Source term from 2-part. sector

$$\begin{cases} i\partial_t \psi^{(1)}(t,z) = H^{\text{Dirac}}\psi^{(1)}(t,z,t,z) + A\psi^{(2)}(t,z,t,z) & \text{on } \mathbb{R}^2 \\ i\partial_{t_k}\psi^{(2)}(t_1,z_1,t_2,z_2) = H_k^{\text{Dirac}}\,\psi^{(2)}(\ldots) & \text{on } \mathring{\mathscr{S}}_1 \end{cases}$$

A : complex 2×4 matrix, to be determined.

Boundary condition: $\boxed{\text{IBC}}$

$$\boxed{\psi^{(2)}_{-+}(t,z,t,z) - e^{i\theta}\psi^{(2)}_{+-}(t,z,t,z) = B\psi^{(1)}(t,z)} \quad \text{(IBC)}$$

B: complex 1×2 matrix (row vector), to be determined.
($B=0$ gives the previous B.C.)

Derivation of IBC from local prob. cons.

Prob. cons. here means:

$$P(\Sigma) := \int_\Sigma d\sigma_\mu\, j^\mu + \int_{(\Sigma\times\Sigma)\cap\mathscr{S}_1} d\sigma_{1\mu}\, d\sigma_{2\nu}\, j^{\mu\nu}$$

does not depend on the choice of space-like Cauchy surface Σ.

$j^\mu = \overline{\psi}^{(1)}\gamma^\mu\psi^{(1)}$ – Dirac current for 1 particle

$j^{\mu\nu} = \overline{\psi}^{(2)}\gamma_1^\mu\gamma_2^\nu\psi^{(2)}$ – " " 2 particles

Here: $\partial_\mu j^\mu(x) = -2\,\mathrm{Im}\left(\psi^{(1)\dagger}(x) A\psi^{(2)}(x,x)\right)$ = minus gain of prob. in the 1-particle sector

$\partial_{1\mu} j^{\mu\nu} = 0 = \partial_{2\nu} j^{\mu\nu}$ – that means prob. can only be lost through the boundary ($\mathcal{C}$) here

$$\underset{\text{in sec. 1}}{\text{gain}} = \underset{\text{in sec. 2}}{\text{loss through the boundary}}$$

$$\Longleftrightarrow \quad (\#)\quad \boxed{2\,\mathrm{Im}\left(\psi^{(1)\dagger}(x) A\psi^{(2)}(x,x)\right)} = (j^{01}-j^{10})(x,x) \overset{\text{see lecture 3}}{=} \boxed{2\left(|\psi^{(2)}_{+-}|^2 - |\psi^{(2)}_{-+}|^2\right)(x,x)}$$

This cond. needs to be ensured by the IBC.

Plug the IBC into (#). After some calculations, one finds:

<u>Theorem:</u> Let $\theta \in [0, 2\pi)$. Then (#) holds if A, B have the form

$$A = \begin{pmatrix} 0 & \tilde{A}^{+} & 0 \\ 0 & & 0 \end{pmatrix},$$

$$\tilde{A} = \begin{pmatrix} w_1 & w_2 \\ w_1 e^{i\phi} & w_2 e^{i\phi} \end{pmatrix}, \qquad w_1, w_2 \in \mathbb{C},\ \phi \in [0, 2\pi)$$

and $$B = \frac{1}{2i}(1, e^{i\theta})\tilde{A}.$$

<u>Example:</u> (See Exercise in Session 2) $\qquad \theta = 0 \overset{}{=} \phi, \quad w_1 = w_2 = 1.$

$$\tilde{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \Rightarrow \quad B = \frac{1}{2i}(1,1)\cdot\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{i}(1,1).$$

<u>Relation with creation and annihilation operators:</u> too long to discuss properly?

Hamiltonian on Fock space, at equal times:

$$H = H^{free} + H_{int}$$
$$H_{int} = H_{int}^{ann} + H_{int}^{cre}$$

$$\left(H_{int}^{ann} \psi\right)^{(2)} = 0$$
$$\left(H_{int}^{ann} \psi\right)^{(1)}_{s}(t,z) = -\underset{\text{comb. factor}}{\sqrt{2}} \sum_{t,u=\pm 1} A_s^{tu}\, \psi^{(2)}(t, z, z).$$

The action of H_{int}^{ann} on $\psi^{(n)}$ agrees with

$$H_{int}^{ann} = \int dz \sum_{rst} A_r^{st}\, a_r^{+}(z)\, a_s(z)\, a_t(z)$$

creation / annihilation operators (as defined in previous QFT lectures)

$$\Rightarrow \quad H_{int}^{cre} = \left(H_{int}^{ann}\right)^{+} = \int dz \sum_{rst} (A_r^{st})^{*}\, a_t^{+}(z)\, a_s^{+}(z)\, a_r(z)$$

$$\Rightarrow \quad \left(H_{int}^{cre} \psi\right)^{(2)}_{s_1 s_2}(t, z_1, z_2) = \frac{1}{\sqrt{2}} \sum_r \underbrace{\left[-(A_r^{s_1 s_2})^{*} + (A_r^{s_2 s_1})^{*}\right]}_{= -2 A_r^{s_2 s_1} \text{ (antisym)}} \delta(z_1 - z_2)\, \psi_r^{(1)}(z_1) \quad \Rightarrow \quad \boxed{\phi \overset{!}{=} \pi}$$

maybe just in words

[The δ-fn. should be interpreted by integrating over

$$i\partial_t \psi^{(2)} = H^{free}\psi^{(2)} + \underbrace{\left(H_{int}^{ann}\psi\right)^{(2)}}_{=0} + \left(H_{int}^{cre}\psi\right)^{(2)}$$

in a ε-n.h.d. of $z_1 = z_2$ (and letting $\varepsilon \to 0$). This yields the IBC for $\phi = \pi = \theta$.]

Sketch of how to construct the solution of the model:

Main idea: We know:

a) Given $\psi^{(2)}$, we can solve the eq. for $\psi^{(1)}$,

$$\left(i\partial_t - H^{\mathrm{Dirac}}\right)\psi^{(1)}(t,z) = -\underbrace{A\,\psi^{(2)}(t,z,t,z)}_{f(t,z)}$$

uniquely:

$$\psi^{(1)}(t,z) = \begin{pmatrix} \psi_-^{(1)}(0,z-t) \\ \psi_+^{(1)}(0,z+t) \end{pmatrix} + \int_0^t ds \begin{pmatrix} f_-(s,z-t+s) \\ f_+(s,z+t-s) \end{pmatrix}.$$

b) Given $\psi^{(1)}$, we can solve the eq. for $\psi^{(2)}$ uniquely (using the solution formula from lecture 3) $\hookrightarrow$ (or rather, a slight modification).

Perhaps omit

Let $u_k = z_k - t_k$, $v_k = z_k + t_k$. Then:

$$\psi_{--}^{(2)}(t_1,z_1,t_2,z_2) = \psi_{0,--}^{(2)}(u_1,u_2)$$

$$\psi_{-+}^{(2)}(\dots) = \begin{cases} \psi_{0,-+}^{(2)}(u_1,v_2), & u_1 < v_2 \\ e^{i\theta}\,\psi_{0,+-}^{(2)}(v_2,u_1) + B\,\psi^{(1)}\left(\frac{v_2-u_1}{2}, \frac{v_2+u_1}{2}\right), & u_1 \geq v_2 \end{cases}$$

$$\psi_{+-}^{(2)}(\dots) = \begin{cases} \psi_{0,+-}^{(2)}(v_1,u_2), & v_1 < u_2, \\ e^{-i\theta}\left(\psi_{0,-+}^{(2)}(u_2,v_1) - B\,\psi^{(1)}\left(\frac{v_1-u_2}{2}, \frac{v_1+u_2}{2}\right)\right), & v_1 \geq u_2, \end{cases}$$

$$\psi_{++}^{(2)}(\dots) = \psi_{0,++}^{(2)}(v_1,v_2).$$

Then set up an iteration scheme (solving these eqs. alternatingly), prove that it has a unique fixed point in a suitable Banach space. The fixed point is the solution!

Conclusion:
- Rigorous example for multi-time QFT
- (almost fully) LI
- demonstrates compatibility of IBCs and multi-time wave fns. (→ hope that IBCs could also work for relativistic QFTs)

Outlook:
- up to $N \in \mathbb{N}$ particle sectors
- could perhaps be combined with photons or scalar exchange particles
- (coupling $(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi)$ as in Thirring model?)
- Higher dim ???
  ↳ maybe (Big maybe) in combinations with photons ...

Reference: arXiv: 1808:04192