Spring School on Multi-Time Wave Functions
Lecture 6: Born's Rule for Arbitrary Cauchy Surfaces

Roderich Tumulka

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1 The Curved Born Rule

Definition. A *Cauchy surface* is a subset Σ of Minkowski space-time $\mathscr{M}$ which is intersected by every inextensible causal [i.e., non-spacelike] curve exactly once.

$\approx$ spacelike hypersurface [but can have kinks and lightlike tangent vectors]

Curved Born rule. If detectors along Σ, then $\rho_\Sigma = |\psi_\Sigma|^2$, suitably interpreted.

For a Dirac wf $\psi_\Sigma(x_1...x_N) \in (\mathbb{C}^4)^{\otimes N}$, I mean $|\cdot|^2$ using, in each spin space, the basis corresponding to the Lorentz frame tangent to Σ [picture]. Equivalently,

$$|\psi_\Sigma(x_1...x_N)|^2 = \overline{\psi_\Sigma}(x_1...x_N)\, \not{n}_1(x_1) \cdots \not{n}_N(x_N)\, \psi_\Sigma(x_1...x_N)\,. \tag{1}$$

Curved collapse rule. If detectors found the configuration in $A \subseteq \Sigma^N$, then ψ_Σ collapses to $1_A\, \psi_\Sigma / \|1_A\, \psi_\Sigma\|$.

Horizontal Born rule. If detectors along $\{x^0 = t\}$, then $\rho_t = |\varphi_t|^2$.

Horizontal collapse rule. If detectors found the configuration in $A \subseteq (\mathbb{R}^3)^N$, then φ_t collapses to $\varphi_{t+} = 1_A\, \varphi_t / \|1_A\, \varphi_t\|$.

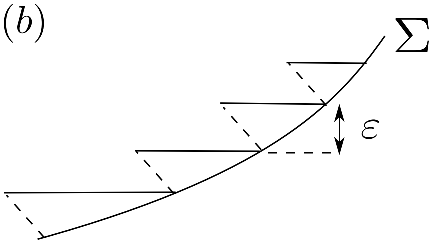
Claim. HBR + HCR determine the statistics of outcomes of *any* experiment.

Reason. Any experiment [e.g., quantum measurement of spin] = unitary interaction with apparatus, followed by reading display of apparatus [picture: time t after Σ] $\square$

Corollary. If we assume HBR + HCR, then CBR and CCR are each either false or a theorem. [That is, we need a proof of CBR and CCR.]

Theorem 1. [Lienert and Tumulka 2017] HBR + HCR + IL + PL $\Rightarrow$ a version of CBR.

(b)



- IL and PL later
- Version: detection process = approximate Σ through horizontal pieces. Limit $\varepsilon \to 0$.
- We didn't prove CCR b/c this detection process yields more info than whether configuration in $A \Rightarrow$ collapses more narrowly than to A.
- Consequence: ρ_Σ can be expressed directly through multi-time wf ψ.
- Consequence: ψ on $\mathscr{S}$ has empirical significance.
- Valid for any N, Fock space, and several species. Presumably, also in curved space-time.

Theorem 2. [Bloch 1934] N non-interacting particles, $i\partial_{t_k}\psi = H_k\psi$. For each k, choose T_k, position measurement of particle k on $\{x^0 = T_k\}$ [picture], result Q_k. Assume HBR + HCR. Then

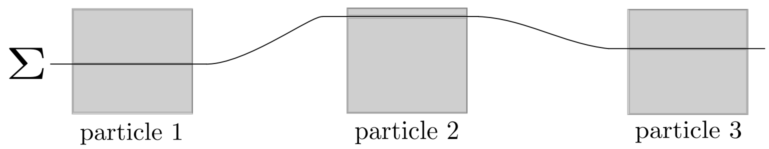
$$P := \text{Prob}\Big(Q_1 \in d^3q_1, \ldots, Q_N \in d^3q_N\Big) = \Big|\psi(T_1, q_1, \ldots, T_N, q_N)\Big|^2 d^3q_1 \cdots d^3q_N.$$

Proof. Compute P using φ, $\mathscr{H} = \otimes_k \mathscr{H}_k$, $H = \sum_k H_k$. Wlog order so that $T_1 \leq T_2 \leq \cdots \leq T_N$. Let P_k = projection in $\mathscr{H}_k$ to position in d^3q_k. $\varphi(0) = \psi(0...0)$.
$\text{Prob}(Q_1 \in d^3q_1) = \|P_1 e^{-iHT_1}\varphi(0)\|^2$ (HBR)
$\varphi(T_1+) = P_1 e^{-iHT_1}\varphi(0)/\|\cdots\|$ (HCR)
$\text{Prob}(Q_2 \in d^3q_2 | Q_1 \in d^3q_1) = \|P_2 e^{-iH(T_2-T_1)}\varphi(T_1+)\|^2$ (HBR)
$\varphi(T_2+) = P_2 e^{-iH(T_2-T_1)}\varphi(T_1+)/\|\cdots\|$ (HCR)
$\vdots$
$P = \|P_N e^{-iH(T_N-T_{N-1})} \cdots P_2 e^{-iH(T_2-T_1)} P_1 e^{-iHT_1}\varphi(0)\|^2$
Using $e^{-iHt} = e^{-iH_1t}e^{-iH_2t}\cdots e^{-iH_Nt}$ and $[P_k, e^{-iH_jt}] = 0$ for $j \neq k$,

$$\begin{aligned} P &= \|P_N e^{-iH_N(T_N-T_{N-1})} \cdots P_1 e^{-iHT_1}\varphi(0)\|^2 && (2)\\ &= \|P_N e^{-iH_NT_N} \cdots P_1 e^{-iH_1T_1}\varphi(0)\|^2 && (3)\\ &= |\psi(T_1, q_1, \ldots, T_N, q_N)|^2\, d^3q_1 \cdots d^3q_N && (4) \end{aligned}$$

as claimed. $\square$

Corollary. Consider N (possibly interacting) particles in spacelike separated regions $V_k \subset \mathscr{M}$, detectors along spacelike Σ. Suppose $\Sigma \cap V_k$ is horizontal for every k. Assume HBR + HCR. Then no interaction, and $\rho_\Sigma = |\psi_\Sigma|^2$. [special case of CBR]



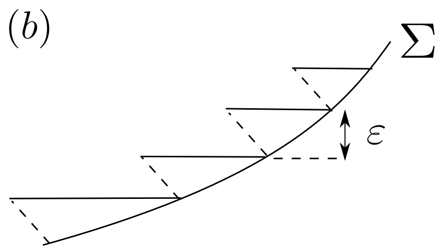
Bohmian argument for CBR for 1 Dirac particle. According to Bohmian mechanics, particles have trajectories = integral curves of j^μ (causal) [picture]. Prob distr of random integral curve C such that

$$\forall \Sigma : \quad \mathrm{Prob}\big(C \cap \Sigma \subset d^3x\big) = j^\mu(x)\, n_\mu(x)\, d^3\sigma(x).$$

Particle gets detected at $C \cap \Sigma$, and the presence of detectors on Σ does not influence C before $\Sigma \Rightarrow$ Prob(detection in d^3x) = Prob($C \cap \Sigma \subset d^3x$ in the absence of detectors) $= \rho_\Sigma(x)\, d^3\sigma(x)$. $\square$

The argument does not work for $N \geq 2$ [b/c BM is nonlocal: detection of particle 1 at x_1 can change trajectory of particle 2 at spacelike separation.]

Proof of CBR from HBR + HCR for 1-particle Dirac eq in 1+1 dim.
Correspondence curved piece $\Sigma_\ell \leftrightarrow$ horizontal piece B_ℓ



Let $A_\ell = \{x^0 = \ell\varepsilon\} \cap \mathrm{past}(\Sigma)$.
Prob(detect in B_ℓ|prior collapses) = flux(j, B_ℓ|collapses) = flux(j, Σ_ℓ|collapses)

$$= \frac{\mathrm{flux}(j_{\psi 1_{A_{\ell-1}}}, \Sigma_\ell)}{\mathrm{Prob(prior\ detection)}} = \frac{\mathrm{flux}(j_\psi, \Sigma_\ell)}{\mathrm{Prob(prior\ detection)}}$$

Thus, Prob(detect in B_ℓ) = flux(j_ψ, Σ_ℓ). $\square$

2 Theorem 1 and Hypersurface Evolution

Want Thm 1 for *every* MT evolution. [MT eqs provide simple formulation of concrete models, but it is hard to say what an *arbitrary* MT evolution is. Better:]

Formalization: "hypersurface evolution." Def: (i)–(vi) below.

(i) $\forall\Sigma:\ \mathscr{H}_\Sigma$

(ii) $\forall\Sigma,\Sigma':\ U_\Sigma^{\Sigma'}:\mathscr{H}_\Sigma\to\mathscr{H}_{\Sigma'}$ unitary such that $U_{\Sigma'}^{\Sigma''}U_\Sigma^{\Sigma'}=U_\Sigma^{\Sigma''}$ and $U_\Sigma^\Sigma=I_\Sigma$

- Let $\Gamma(\Sigma)=\{q\subset\Sigma:\#q<\infty\}$ space of unordered configurations.
- A PVM (projection-valued measure) on a set S associates with $A\subseteq S$ a projection $P(A)$ in $\mathscr{H}$ such that $P(A_1\cup A_2\cup ...)=P(A_1)+P(A_2)+\ldots$ if $A_j\cap A_k=\emptyset$ for $j\neq k$ ("σ-additive") and $P(S)=I$. Examples:
  - On $\mathscr{H}=L^2(S)$, set $P(A)\psi=1_A\,\psi$.
  - Spectral theorem: Self-adjoint [operator] T is associated w/ PVM P on $\mathbb{R}$. For $A\subseteq\mathbb{R}$, $P(A)=$ proj to span(generalized eigenvectors with generalized eigenvalues in A).

(iii) $\forall\Sigma:$ PVM P_Σ on $\Gamma(\Sigma)$ acting on $\mathscr{H}_\Sigma$ (configuration observable)

- Example: bosonic Fock space of $L^2(\Sigma)$ is $L^2(\Gamma(\Sigma))$, equipped with PVM on $\Gamma(\Sigma)$
- Example: bosonic/fermionic Fock space of $L^2(\Sigma,\mathbb{C}^K)$ also automatically with PVM on $\Gamma(\Sigma)$
- Example: $\mathrm{Fock}_1\otimes\mathrm{Fock}_2$
- N-particle sector of $\mathscr{H}_\Sigma$ is range of $P(\Gamma_N(\Sigma))$
- formalizes $\big|\psi_\Sigma(x_1...x_N)\big|^2\,d^3\sigma(x_1)\cdots d^3\sigma(x_N)=\big\|P_\Sigma(d^3x_1\cdots d^3x_N)\,\psi_\Sigma\big\|^2$

(iv) absolutely continuous: $P_\Sigma(A)=0$ for every set A of volume 0

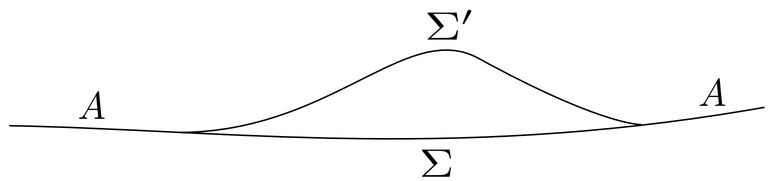
(v) unique vacuum: dim(0-particle sector) = 1

(vi) Factorization: If $A,B\subseteq\Sigma$ are disjoint, then $\mathscr{H}_{A\cup B}=\mathscr{H}_A\otimes\mathscr{H}_B$ and $P_{A\cup B}=P_A\otimes P_B$.

- Example: True of Fock spaces:
- $\Gamma(A\cup B)=\Gamma(A)\times\Gamma(B)$, therefore
- $\mathrm{Fock}(L^2(A\cup B))=\mathrm{Fock}(L^2(A))\otimes\mathrm{Fock}(L^2(B))$ and
- $P_{A\cup B}=P_A\otimes P_B$ for Fock spaces.

3 Interaction locality

(IL): $\forall \Sigma, \Sigma'$ and $A \subseteq \Sigma \cap \Sigma'$: $\quad U_\Sigma^{\Sigma'} = I_A \otimes U_{\Sigma \setminus A}^{\Sigma' \setminus A}$.
Last factor does not depend on A except through $\Sigma \setminus A$ and $\Sigma' \setminus A$.



- Expresses absence of interaction terms between spacelike separated regions.
- Presumably valid in our universe.
- IL $\not\Leftrightarrow$ Bell locality = no influences between events in spacelike separated regions.
- Bell's theorem: Bell locality is violated.
- Presumably, IL $\Leftarrow$ no superluminal signaling

4 Propagation locality

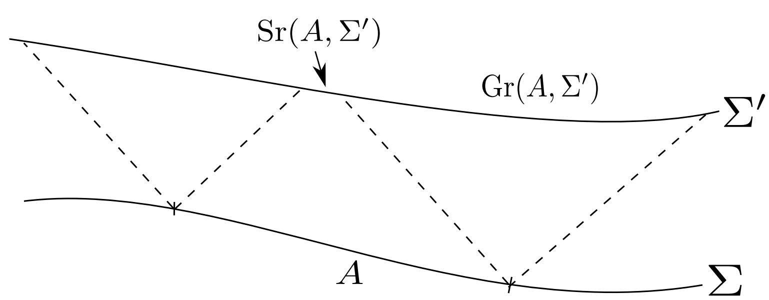
For $R \subseteq \Sigma$ define

$$\forall(R) = \{q \in \Gamma(\Sigma) : q \subseteq R\}. \tag{5}$$

Definition. $\psi_\Sigma \in \mathscr{H}_\Sigma$ is *concentrated in* $A \subseteq \Sigma$ iff $\psi_\Sigma \in \operatorname{range} P_\Sigma(\forall(A))$. Equivalently, $\operatorname{supp}_3(\psi_\Sigma) \subseteq A$.

Definition. $\forall \Sigma, \Sigma'$ and $A \subseteq \Sigma$, the *grown set* is

$$\operatorname{Gr}(A, \Sigma') = \big[\operatorname{future}(A) \cup \operatorname{past}(A)\big] \cap \Sigma' .$$



Propagation locality (PL): Whenever ψ_Σ is concentrated in $A \subseteq \Sigma$, then $\psi_{\Sigma'}$ is concentrated in $\mathrm{Gr}(A, \Sigma')$.

Examples of IL and PL

- Free MT Dirac evolution
- MT Dirac in 1+1 dim w/ zero-range interaction [Lienert 2015]
- MT em-ab model
- MT Dirac in 1+1 dim w/ IBC [Lienert and Nickel 2018]

Theorem 1 again. HBR + HCR + IL + PL $\Rightarrow$ CBR