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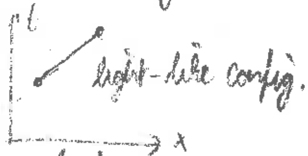
Lecture 7: Multi-time integral equations

Motivation:
- options for multi-time eqs. for N particles too strongly restricted by consistency cond.
- QFTs in 1+3 dim.: UV-divergent & hard to make rigorous in L.I. way
- New idea: direct interactions with time delay (in class. electrodyn. Wheeler-Feynman avoids the UV problem)
- Possible to express such int. using ψ at config. $(t_1, \vec{x}_1, t_2, \vec{x}_2)$ with $t_1 \neq t_2$ (in every frame)

e.g. $t_2 = t_1 - |\vec{x}_1 - \vec{x}_2|$



→ New possibility specific for multi-time wave fns!
↳ not possible using $\varphi(t, \vec{x}_1, \vec{x}_2) = \psi(t, \vec{x}_1, t, \vec{x}_2)$.

Derivation of a suitable integral eq.

Why int. eq.?

S. eq. (S) $\begin{cases} (i\partial_t - H_1^{\text{free}} - H_2^{\text{free}} - V)\, \varphi(t, \vec{x}_1, \vec{x}_2) = 0 \\ \varphi(0, \vec{x}_1, \vec{x}_2) = \varphi_0(\vec{x}_1, \vec{x}_2) \end{cases}$

can be reformulated as the int. eq.

$$(*) \quad \varphi(t, \vec{x}_1, \vec{x}_2) = \varphi^{\text{free}}(t, \vec{x}_1, \vec{x}_2) + \int_0^t dt' \int d^3\vec{x}_1'\, d^3\vec{x}_2'\, G_1^{\text{ret}}(t-t', \vec{x}_1 - \vec{x}_1')\, G_2(t-t', \vec{x}_2 - \vec{x}_2') \times V(t', \vec{x}_1', \vec{x}_2')\, \varphi(t', \vec{x}_1', \vec{x}_2')$$

with: φ^{free}: solution of (S) with $V=0$,
G_k^{ret}: retarded Green's fn. of $(i\partial_t - H_k^{\text{free}})\, G^{\text{ret}}(t-t', \vec{x}_k - \vec{x}_k') = \delta(t-t')\, \delta^{(3)}(\vec{x}_k - \vec{x}_k')$

[Explain in more detail why (S) is equivalent to (*)]

Crucial point: $\boxed{\text{Contrary to (S), (*) has a straightforward relativistic generalization!}}$

→ using $\psi(x_1, x_2)$!

Non-relativistic limit (for $K(x_1,x_2) = \delta((x_1-x_2)^2)$).

Want to show: a solution Ψ of (+) approximately solves (S) at equal times
if time delay of int. is negligible

That means: if the replacement

(A) $$\delta((t_1-t_2)^2 - |\vec{x}_1-\vec{x}_2|) = \frac{1}{2|\vec{x}_1-\vec{x}_2|}\left[\delta(t_1-t_2-|\vec{x}_1-\vec{x}_2|) + \delta(t_1-t_2+|\vec{x}_1-\vec{x}_2|)\right]$$

$$\longrightarrow \frac{1}{|\vec{x}_1-\vec{x}_2|}\,\delta(t_1-t_2)$$ is allowed
(in the integrals over Ψ).

Consider (+) for $t_1=t_2=t$:

$$\Psi(t,\vec{x}_1,t,\vec{x}_2) = \underset{t_1=t_2=t}{\Psi^{free}(\dots)} + \int dt_1'\, d^3\vec{x}_1'\, dt_2'\, d^3\vec{x}_2'\; G_1(t-t_1',\vec{x}_1-\vec{x}_1')\,G_2(t-t_2',\vec{x}_2-\vec{x}_2')$$
$$\times\; \underbrace{\delta((t_1'-t_2')^2-|\vec{x}_1'-\vec{x}_2'|^2)}\,\Psi(t_1',\vec{x}_1',t_2',\vec{x}_2')$$

$\hookrightarrow$ replace by $\frac{1}{|\vec{x}_1-\vec{x}_2|}\delta(t_1-t_2)$ (A)

$$= \Psi^{free}(\dots) + \int dt_1'\, d^3\vec{x}_1'\, d^3\vec{x}_2'\; G_1(t-t_1',\vec{x}_1-\vec{x}_1')\,G_2(t-t_1',\vec{x}_2-\vec{x}_2')$$
$$\frac{1}{|\vec{x}_1'-\vec{x}_2'|}\,\Psi(t_1',\vec{x}_1',t_1',\vec{x}_2')$$

$\rightarrow$ Form of (*) with $V(t,\vec{x}_1,\vec{x}_2) = \frac{1}{|\vec{x}_1-\vec{x}_2|}$ (and $t \leftrightarrow t_1'$).

Conclusion: (*) indeed yields (S) and gives a reason why
V = Coulomb potential!

Action-at-a-distance form of the multi-time eqs.

Let now $K(x_1,x_2) = \gamma_1^\mu \gamma_{2\mu}\,\delta((x_1-x_2)^2)$ and $D_k = i\gamma_k^\mu \partial_{k\mu} - m_k$ (free Dirac operators).

Act on (+) with D_1:

$$\Rightarrow\; D_1\Psi(x_1,x_2) = \int d^4x_2'\; G_2(x_2-x_2')\,\gamma_1^\mu\gamma_{2\mu}\,\delta((x_1-x_2')^2)\,\Psi(x_1,x_2')$$

Analogously:

$$D_2\Psi(x_1,x_2) = \int d^4x_1'\; G_1(x_1-x_1')\,\gamma_1^\mu\gamma_{2\mu}\,\delta((x_1'-x_2)^2)\,\Psi(x_1',x_2).$$

This can be written as:

$$\left[i\gamma_k^\mu(\partial_{k\mu} - i\,\hat{A}_{3-k,\mu}(x_k)) - m_k \right] \Psi(x_1,x_2) = 0\,, \quad k = 1,2 \qquad (\nabla)$$

with $$\hat{A}_{3-k,\mu}(x_k)\Psi(x_1,x_2) = \int d^4x'_{3-k}\, G_{3-k}(x_{3-k}-x'_{3-k})\,\gamma_{3-k,\mu}\,\delta((x_k-x'_{3-k})^2)\,\Psi(\underbrace{x_k, x'_{3-k}}_{\text{order accordingly.}})$$

→ Dirac eq. with field extracted from ^resp.^ other particles' d.o.f. !

Compare with <u>QED model by Dirac</u> (1932): $\Psi = \Psi(x_1,x_2,A)$ ↳ field d.o.f.

$$\left[i\gamma_k^\mu(\partial_{k\mu} - i\,\hat{A}_\mu(x_k)) - m_k \right] \Psi(x_1,x_2) = 0\,, \; k=1,2, \quad \Box \hat{A}(x) = 0$$

↳ field operator

<u>Dirac</u>: "The interaction of the two electrons is due to the motion of each being connected with the same field."

<u>Here (∇)</u>: "The interaction of the two electrons is due to the motion of each being connected with the field generated by the other."

→ Structurally similar to classical action-at-a-distance electrodyn. (Wheeler-Feynman).

<u>Mathematical analysis of the integral eq.</u>

Does one of the previous scenarios apply to (∗) ?

→ Analyze this at a <u>simplified class of eqs.</u>

1. Klein-Gordon (KG) particles (simpler than Dirac eq.) ↳ see talk of Markus!
2. Retarded case (G^{ret})
3. Beginning in time (time cut off before $t=0$).
4. Bounded or only mildly singular interaction kernels K
5. Massless case (for simplicity)

<u>Motivation</u>: 2. & 3. together lead to time integrals only from 0 to t: (→ <u>Volterra structure</u>)
3. motivated by the Big Bang singularity.

Thm: $\quad G_k^{ret}(t_k, \vec{x}_k) = \frac{1}{4\pi} \frac{\delta(t_k - |\vec{x}_k|)}{|\vec{x}_k|}$.

$$\Rightarrow (*) \Leftrightarrow \psi(t_1, \vec{x}_1, t_2, \vec{x}_2) = \psi^{free}(\ldots) + \frac{1}{(4\pi)^2}\int_0^\infty dt_1' \int d^3\vec{x}_1' \int_0^\infty dt_2' \int d^3\vec{x}_2' \frac{\delta(t_1 - t_1' - |\vec{x}_1 - \vec{x}_1'|)}{|\vec{x}_1 - \vec{x}_1'|} \frac{\delta(t_2 - t_2' - |\vec{x}_2 - \vec{x}_2'|)}{|\vec{x}_2 - \vec{x}_2'|} (K\psi)(t_1', \vec{x}_1', t_2', \vec{x}_2')$$

$$= \psi^{free}(\ldots) + \frac{1}{(4\pi)^2} \int_{B_{t_1}(\vec{x}_1)} d^3\vec{x}_1' \int_{B_{t_2}(\vec{x}_2)} d^3\vec{x}_2' \frac{1}{|\vec{x}_1 - \vec{x}_1'|} \frac{1}{|\vec{x}_2 - \vec{x}_2'|} (K\psi)(t_1 - |\vec{x}_1 - \vec{x}_1'|, \vec{x}_1', t_2 - |\vec{x}_2 - \vec{x}_2'|, \vec{x}_2') \qquad (\Delta)$$

Study (Δ) as an eq. on the <u>Banach space</u>

$$\mathcal{B} := L^\infty([0,T]^2, L^2(\mathbb{R}^6))$$

with norm: $\quad \|\psi\| = \operatorname{ess\,sup}_{t_1, t_2 \in [0,T]} \|\psi(t_1, \cdot, t_2, \cdot)\|_{L^2(\mathbb{R}^6)}$.

<u>Main result:</u>

<u>Theorem:</u> Let $K: \mathbb{R}^8 \to \mathbb{C}$ be bounded or of the form

$$K(t_1, \vec{x}_1, t_2, \vec{x}_2) = \frac{f(t_1, \vec{x}_1, t_2, \vec{x}_2)}{|\vec{x}_1 - \vec{x}_2|}$$

with $f: \mathbb{R}^8 \to \mathbb{C}$ bounded.

Then for every $\psi^{free} \in \mathcal{B}$, (Δ) has a unique solution $\psi \in \mathcal{B}$.

<u>Conclusion:</u> Multi-time integral eqs. can indeed be used to define an interacting time evolution for a multi-time wave fn.!

<u>On the Cauchy problem:</u>

Let $\psi^{free} \in \mathcal{B}$ be a solution of $\underbrace{\Box_1 \psi^{free} = 0 = \Box_2 \psi^{free}}_{\text{massless KG eqs.}}$.

Then ψ^{free} is determined by Cauchy data at $t_1 = t_2 = 0$:

$$\begin{cases} \psi^{free}(0, \vec{x}_1, 0, \vec{x}_2) = \psi_0^{free}(\vec{x}_1, \vec{x}_2), \\ \partial_{t_1} \psi^{free}(\ldots) = \psi_1^{free}(\ldots), \\ \partial_{t_2} \psi^{free}(\ldots) = \psi_2^{free}(\ldots), \\ \partial_{t_1}\partial_{t_2} \psi^{free}(\ldots) = \psi_3^{free}(\ldots). \end{cases}$$

As ψ^{free} determines Ψ uniquely, these data determine Ψ.

Moreover, $(\Delta) \Rightarrow \quad \Psi(0, x_1, 0, x_2) = \psi^{free}(\vec{x}_1, \vec{x}_2)$,

so the data (at least for Ψ) are Cauchy data for Ψ as well.

[Think about: what happens to the derivatives?]

Conclusion: $\boxed{\text{$\Psi$ is subject to a Cauchy problem at the initial time!}}$

But: This is not true at other times.

Then, in general, $\psi^{free}(t_1, \cdot, t_2, \cdot) \neq \Psi(t_1, \cdot, t_2, \cdot)$.

OPTIONAL: Idea of the proof:

Let (A) be the integral operator in (Δ).

Show that $\Psi = \sum_{k=0}^{\infty} A^k \psi^{free}$ $(*)$ converges in $\mathcal{B}$ and solves

$\Psi = \psi^{free} + A\Psi$!

Hard part: convergence!

For this: show that

$$\|(A\Psi)(t_1, \cdot, t_2, \cdot)\|^2 \overset{\text{schematically}}{\leq} \|K\|_\infty \underset{\text{polynomial in } t_1, t_2}{P(t_1, t_2)} \int_0^{t_1} d\rho_1 \int_0^{t_2} d\rho_2 \; \|\Psi(\rho_1, \cdot, \rho_2, \cdot)\|^2_{L^2}$$

Iterate this to show

$$\|(A^k \psi^{free})(t_1, \cdot, t_2, \cdot)\|^2 \underset{\substack{\text{(not exactly,} \\ \text{but qualitatively)}}}{\leq} \frac{t_1^k \, t_2^k \, \|K\|^k_\infty}{(k!)^2} \|\psi^{free}\|^2_{\mathcal{B}}$$

This implies absolute convergence of $(*)$, and that A is a bounded operator. Thus:

$$A \sum_{k=0}^{\infty} A^k \psi^{free} = \sum_{k=0}^{\infty} A^{k+1} \psi^{free} = \sum_{k=0}^{\infty} A^k \psi^{free} - \psi^{free}$$

$$\Leftrightarrow \quad A\Psi = \Psi - \psi^{free}, \text{ as desired.}$$

Uniqueness follows from $A(\Psi_1 - \Psi_2) = \Psi_1 - \Psi_2$ for two solutions and from absolute convergence of

$$\sum_{k=0}^{\infty} A^k (\Psi_1 - \Psi_2) = \sum_{k=0}^{\infty} (\Psi_1 - \Psi_2)$$

which can only hold for $\Psi_1 = \Psi_2$! $\square$

[better orally]

Conclusions: • multi-time int. eqs. yield int. rel. quantum dyn. in 1+3 dim.

- no problem with consistency cond.
- natural generalization of integral version of Schröd. eq.
- S. eq. with Coulomb pot. in non-rel. limit (neglecting time delay).
- direct int. at quantum level, quantum version of action-at-a-distance electrodynamics
- rigorous demonstration of math. consistency in certain cases
- unusual role of time evol.: correction to a free solution

Outlook: • Curved spacetimes with Big Bang sing. to justify cutoff at $t=0$ physically

→ work

- Dirac particles: also possible

→ talk by Markus!

- N particles: different suggestions (see also exercise), one seems more appropriate
- "Super consistency condition"

→ certain kind of c.c. for retarded int. & multi-time eqs. with time delay (condition to extend arbitrary histories)

→ automatically satisfied by eqs. coming from multi-time integral eqs.

- Open challenges:
  - very singular interactions → $\delta((x_1-x_2)^2)$?
  - conservation laws
  - particle creation & annihilation ← Dirac sea; Fock space wave fn., coupled int. eqs. for each n

References:

- Introduction: arXiv: 1801.00060
- First math. results: 1803.08772
- Curved spacetimes: 1805.06348
- Dirac particles: 1903.06020

Namely: $$\boxed{\Psi(x_1,x_2) = \psi^{free}(x_1,x_2) + \int d^4x_1'\, d^4x_2'\, G_1(x_1-x_1')\, G_2(x_2-x_2')\, K(x_1',x_2')\, \Psi(x_1',x_2')}$$

↳ [leave somewhere on blackboard!] (+)

Here: ψ^{free} – solution of free multi-time eqs., eg. Dirac eqs. $D_1 \psi^{free}(x_1,x_2) = 0 = D_2 \psi^{free}(x_1,x_2)$.

G_k – Green's fn. of these eqs., $D_k G_k(x_k - x_k') = \delta^{(4)}(x_k - x_k')$.

↳ retarded or other

K – "interaction kernel": responsible for int., analogous to V in (*).

Simultaneous interactions: $K(t_1,\vec{x}_1, t_2,\vec{x}_2) = \delta(t_1-t_2)\, V(t_1,\vec{x}_1,\vec{x}_2)$

Interactions along light cones: $K(x_1,x_2) \propto \delta((x_1-x_2)^2)$

↳ $= (x_1^0-x_2^0)^2 - |\vec{x}_1-\vec{x}_2|^2$

How to understand the time evolution?

Previously: difficulty with consistency cond. (possible conflict of many eqs.)

Now: Just a single integral eq.

Important: Is there an argument that (+) has sufficiently many solutions?

→ Schematic form of (+): $\Psi = \psi^{free} + A\Psi \quad (\#), \quad \Psi \in \mathcal{B}$

↳ Banach space

(Some) Scenarios, where (#) makes sense (defines a time evol.):

a) $\|A\| < 1$: unique solution Ψ for every ψ^{free} (Banach's fixed point thm.)

b) A compact operator and $A\Psi = 0 \Rightarrow \Psi = 0$: same result (Fredholm alternative)

Conclusion: In these cases, there are many solutions of (#).

Can read (#) as determining a correction to ψ^{free} due to interaction.