U Tübingen Winter Term 2018/19

Exercises for the Course
Modular forms

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Exercise sheet 3

Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a subgroup. A fundamental domain for Γ is a connected subset $\mathcal{F}_\Gamma \subseteq \mathbb{H}^\star$ such that there is a bijection $\mathcal{F}_\Gamma \cong \Gamma\backslash\mathbb{H}^\star$. Note that this definition slightly differs from the definition given in class.

Exercise 1

(a) Prove that

$$\begin{aligned}\mathcal{F}_{\mathrm{SL}_2(\mathbb{Z})} :=&\{\tau \in \mathbb{H} \,|\, |\tau| > 1, -1/2 \leq \mathrm{Re}(\tau) < 1/2\} \cup \\ &\{\tau \in \mathbb{H} \,|\, |\tau| = 1, -1/2 \leq \mathrm{Re}(\tau) \leq 0\} \cup \{\infty\}\end{aligned}$$

is a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$.

(b) Prove that, for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $\gamma\mathcal{F}_{\mathrm{SL}_2(\mathbb{Z})}$ is a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$. Draw $\gamma\mathcal{F}_{\mathrm{SL}_2(\mathbb{Z})}$ for

$$\gamma \in \left\{E,\, T,\, TS,\, ST^{-1}S,\, S,\, ST,\, STS,\, T^{-1}S,\, T^{-1}\right\},$$

where

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Exercise 2

For $N \in \mathbb{N}$, $N > 1$, we define the subgroup

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \mod N \right\}$$

of $\mathrm{SL}_2(\mathbb{Z})$. Let E, S, and T be as in Exercise 1 and let p be a prime number. For $j \in \mathbb{N}$, $0 \leq j \leq p$, we set

$$\alpha_j := \begin{cases} ST^j, & \text{if } 0 \leq j \leq p-1; \\ E, & \text{if } j = p. \end{cases}$$

(a) Prove that

$$\mathrm{SL}_2(\mathbb{Z}) = \dot{\bigcup}_{j=0}^{p} \alpha_j^{-1}\Gamma_0(p) = \dot{\bigcup}_{j=0}^{p} \Gamma_0(p)\alpha_j.$$

(b) Using (a), conclude that there is a fundamental domain $\mathcal{F}_{\Gamma_0(p)}$ for $\Gamma_0(p)$ satisfying

$$\mathcal{F}_{\Gamma_0(p)} \subseteq \bigcup_{j=0}^{p} \alpha_j \mathcal{F}_{\mathrm{SL}_2(\mathbb{Z})}. \tag{1}$$

Can one replace "$\subseteq$" by "$=$" in (1)?

(c) Draw the fundamental domain $\mathcal{F}_{\Gamma_0(2)}$ and determine all the cusps of $\Gamma_0(2)$, i.e., the orbits $\Gamma_0(2)[r:s]$ with $[r:s] \in \mathbb{P}^1(\mathbb{Q})$.

(d) Find a fundamental domain drawer in the internet and plot the fundamental domain $\mathcal{F}_{\Gamma_0(N)}$ for different N.