U Tübingen Winter Term 2018/19

Exercises for the Course
Modular forms

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Exercise sheet 1

Exercise 1

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$ and $\tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, we define

$$\gamma\tau := \frac{a\tau + b}{c\tau + d}. \tag{1}$$

(a) Show that

$$\mathrm{Im}(\gamma\tau) = \frac{\mathrm{Im}(\tau)\det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}{|c\tau + d|^2}.$$

(b) Show that (1) defines an action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$, and an action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathbb{H}$.

(c) Show that $\mathrm{SL}_2(\mathbb{R})$ acts transitively on $\mathbb{H}$. To do this, determine for every $z \in \mathbb{H}$ an element $\gamma \in \mathrm{SL}_2(\mathbb{R})$ with $\gamma i = z$.

Exercise 2

Let $\mathrm{SL}_2(\mathbb{Z}) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1\}$ be the modular group. Let

$$E := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(a) Show the identities

$$S^2 = -E;$$

$$T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \qquad (n \in \mathbb{Z});$$

$$(ST)^3 = (TS)^3 = -E.$$

(b) Express the matrix

$$\begin{pmatrix} 4 & 9 \\ 11 & 25 \end{pmatrix}$$

as word in S, S^{-1}, T and T^{-1}.

(c) Let Γ be the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by S and T. Let $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a matrix in $\mathrm{SL}_2(\mathbb{Z})$. Use the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} T^n = \begin{pmatrix} a & b' \\ c & nc+d \end{pmatrix}$$

to show that unless $c = 0$, some matrix $\alpha\gamma$ with $\gamma \in \Gamma$ has bottom row (c, d') with $|d'| \leq |c|/2$. Use the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} S = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix}$$

to show that this process can be iterated until some matrix $\alpha\gamma$ with $\gamma \in \Gamma$ has bottom row $(0, \star)$. Show that in fact the bottom row is $(0, \pm 1)$, and since $S^2 = -E$ it can be taken to be $(0, 1)$. Show that therefore $\alpha\gamma \in \Gamma$ and so $\alpha \in \Gamma$. Thus Γ is all of $\mathrm{SL}_2(\mathbb{Z})$, that is, the matrices S und T generate the group $\mathrm{SL}_2(\mathbb{Z})$.

Exercise 3

Show that the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges absolutely and locally uniformly for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, and thus defines a holomorphic function on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$.

Exercise 4

Show that, for $z \in \mathbb{C} \setminus \mathbb{Z}$, the identity

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} + \frac{1}{z-n} \right)$$

holds. Show that the series on the right-hand side converges absolutely and locally uniformly on $\mathbb{C} \setminus \mathbb{Z}$.