Exercises for the Course
Modular forms

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Exercise sheet 2

Exercise 1

Let $q = q(\tau) := e^{2\pi i\tau}$ and consider the theta series

$$\theta(\tau) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

(a) Let k be a positive integer. Prove the identity

$$(\theta(\tau))^k = \sum_{n=0}^{\infty} A_k(n) q^n,$$

where $A_k(n) := \sharp\{(x_1, \ldots, x_k) \in \mathbb{Z}^k \,|\, x_1^2 + \ldots + x_k^2 = n\}$.

(b) Prove the formula

$$A_8(n) = 16 \sum_{d|n, d\geq 1} (-1)^{n-d} d^3,$$

by using similar arguments as in the proof of the formular for $A_4(n)$ given in the lecture.

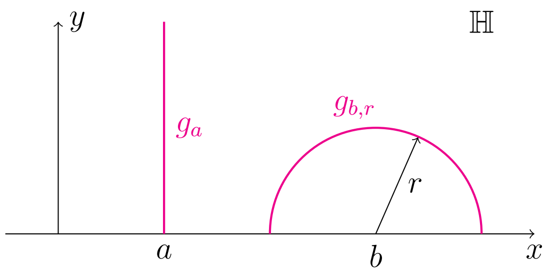
Exercise 2

(a) Let $z, w \in \mathbb{H}$. Show that there exists an element $\gamma \in \mathrm{SL}_2(\mathbb{R})$ such that $\gamma z = ir$ and $\gamma w = it$, where r and t are positive real numbers. (Hint: Observe that $k(2\theta) = \left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right)$ is a hyperbolic rotation at i.)

(b) Let $a, b \in \mathbb{R}$ and $r > 0$, and define the subsets $g_a, g_{b,r}$ of $\mathbb{H}$ as follows

$$g_a := \{x + iy \in \mathbb{H} \,|\, x = a\} \quad \text{and} \quad g_{b,r} := \{x + iy \in \mathbb{H} \,|\, (x-b)^2 + y^2 = r^2\}.$$

The collection of *hyperbolic lines (geodesics)* in the upper-half plane model $\mathbb{H}$ (endowed with the hyperbolic metric) is $\{g_a \,|\, a \in \mathbb{R}\} \cup \{g_{b,r} \,|\, b \in \mathbb{R}, r > 0\}$. Prove that, under the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$, lines are sent to lines.



Exercise 3

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, $\gamma \neq \pm \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and let $\mathrm{tr}(\gamma) := a + d$ denote the trace of γ. We say that γ is

- *parabolic*, if $|\mathrm{tr}(\gamma)| = 2$;
- *hyperbolic*, if $|\mathrm{tr}(\gamma)| > 2$;
- *elliptic*, if $|\mathrm{tr}(\gamma)| < 2$.

Prove the following equivalences:

(i) γ is parabolic if and only if γ has exactly one fixed point on $\mathbb{R} \cup \{\infty\}$;

(ii) γ is hyperbolic if and only if γ has exactly two fixed points on $\mathbb{R} \cup \{\infty\}$;

(iii) γ is elliptic if and only if γ has exactly one fixed point on $\mathbb{H}$.

Exercise 4

Let $z = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$, $y > 0$. Recall that the hyperbolic volume $\mathrm{vol}_{\mathrm{hyp}}(\mathcal{F})$ of a connected subset $\mathcal{F} \subset \mathbb{H}$ is defined by

$$\mathrm{vol}_{\mathrm{hyp}}(\mathcal{F}) = \int_{\mathcal{F}} \frac{dxdy}{y^2},$$

as long as the integral exists. Consider $\mathcal{F} = \{z \in \mathbb{H} : |\mathrm{Re}(z)| < \frac{1}{2}, |z| > 1\}$.

(a) Calculate $\mathrm{vol}_{\mathrm{hyp}}(\mathcal{F})$.

(b) Let Δ be a hyperbolic triangle with angles α, β, γ. Gauß-Bonnet's theorem states that $\mathrm{vol}_{\mathrm{hyp}}(\Delta) = \pi - \alpha - \beta - \gamma$. Using this result, determine $\mathrm{vol}_{\mathrm{hyp}}(\mathcal{F})$ again.