U Tübingen Winter Term 2018/19

Exercises for the Course
Modular forms

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Exercise sheet 4

Exercise 1

(a) Prove that

$$\mathrm{PSL}_2(\mathbb{Z})_\infty \cong \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

(b) Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{Z})$ be a subgroup of finite index. Prove that

$$\Gamma_\infty \cong \left\langle \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \right\rangle$$

for some $h \in \mathbb{N}$.

Exercise 2

Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. If $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, show that $\gamma^{-1}\Gamma\gamma$ is again a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Exercise 3

Let $N \in \mathbb{N}$, $N > 1$. Consider the following subsets of $\mathrm{SL}_2(\mathbb{Z})$:

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, a \equiv d \equiv 1, b \equiv c \equiv 0 \mod N \right\}$$
$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, a \equiv d \equiv 1, c \equiv 0 \mod N \right\},$$
$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \mod N \right\}.$$

(a) Prove that

$$\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z}).$$

is chain of subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

(b) Prove that the natural group homomorphism from $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, given by reducing the entries of the matrix modulo N, is surjective with kernel $\Gamma(N)$.

(c) Prove that the map

$$\Gamma_1(N) \longrightarrow \mathbb{Z}/N\mathbb{Z}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto b \mod N \qquad \text{resp.}$$
$$\Gamma_0(N) \longrightarrow (\mathbb{Z}/N\mathbb{Z})^\times, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \mod N$$

is a surjective group homomorphism with kernel $\Gamma(N)$ resp. $\Gamma_1(N)$.

(d) Conclude that

$$\Gamma(N) \trianglelefteq \Gamma_1(N) \trianglelefteq \Gamma_0(N) \qquad \text{and} \qquad \Gamma(N) \trianglelefteq \mathrm{SL}_2(\mathbb{Z})$$

are chains of normal subgroups of $\mathrm{SL}_2(\mathbb{Z})$.