U Tübingen Winter Term 2018/19

Exercises for the Course
Modular forms

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Exercise sheet 5

Let $\Gamma := \mathrm{SL}_2(\mathbb{Z})$, $q := e^{2\pi i\tau}$, and $\varrho := e^{2\pi i/3}$.

Exercise 1

(a) Let $f \in \mathcal{M}_k(\Gamma)$, $f \neq 0$. Use formula

$$\nu_\infty(f) + \frac{\nu_i(f)}{2} + \frac{\nu_\varrho(f)}{3} + \sum_{\substack{\Gamma\tau\in\Gamma\backslash\mathbb{H} \\ \tau\notin\Gamma i,\Gamma\varrho}} \nu_\tau(f) = \frac{k}{12}$$

to prove that there is exactly one possibility for the rows in the following table and complete the table.

k	$\frac{k}{12}$	$\nu_\infty(f)$	$\nu_i(f)$	$\nu_\varrho(f)$	$\nu_\tau(f)$ $(\Gamma\tau \in \Gamma\backslash\mathbb{H};\tau \notin \Gamma i, \Gamma\varrho)$
4	$\frac{1}{3}$				
6	$\frac{1}{2}$				
8	$\frac{2}{3}$				
10	$\frac{5}{6}$				
14	$\frac{7}{6}$				

(b) Prove that $\mathcal{M}_k(\Gamma) = \mathbb{C}\cdot E_k$ for $k = 4, 6, 8, 10, 14$.

(c) Prove that $\mathcal{S}_k(\Gamma) = \{0\}$ for $k = 4, 6, 8, 10, 14$.

(d) Compute $E_4(\varrho)$ and $E_6(i)$.

Exercise 2

(a) Consider the Delta function $\Delta(\tau) := (E_4(\tau)^3 - E_6(\tau)^2)/1728$. Prove that $\Delta(\tau)$ has a Fourier expansion of the form

$$\Delta(\tau) = q - 24q^2 + 252q^3 + O(q^4) \in \mathbb{Z}[[q]]\,.$$

(b) Prove that $\Delta(\tau) \in \mathcal{S}_{12}(\Gamma)$. Prove that $\nu_\infty(\Delta) = 1$ and conclude that $\Delta(\tau)$ is non-vanishing on $\mathbb{H}$.

Exercise 3

(a) Consider the j-function $j(\tau) := E_4(\tau)^3/\Delta(\tau)$. Prove that $j(\tau)$ has a Fourier expansion of the form

$$j(\tau) = \frac{1}{q} + 744 + 196884q + O(q^2) \in \mathbb{Z}[[q]]\,.$$

Prove that $\nu_\infty(j) = -1$, and that $j(\tau)$ is holomorphic on $\mathbb{H}$. Prove that $j(\tau)$ is a modular function of weight 0 for Γ.

(b) For $c \in \mathbb{C}$, consider the function $f_c(\tau) := E_4(\tau)^3 - c\Delta(\tau)$. Prove that $f_c(\tau) \in \mathcal{M}_{12}(\Gamma)$, and that $f_c(\tau) = 0$ if and only if $j(\tau) = c$. Prove that $\nu_\infty(f_c) = 0$.

(c) Use (b) to prove that the j-function induces a bijection

$$j : \Gamma\backslash\mathbb{H} \to \mathbb{C}.$$

(d) Compute $j(i)$ and $j(\varrho)$.

Exercise 4

(a) Visit the website http://www.sagemath.org. Use the link to CoCalc (SageMath-Cloud) and create an account or sign in with your account.

(b) Use Sage to compute the first ten coefficients in the q-series of the functions E_4, E_6, Δ, and j.