U Tübingen Winter Term 2018/19

Exercises for the Course
Modular forms

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Exercise class: 14.12.18

Exercise sheet 6

Exercise 1

(a) Prove a relation between $E_4(\tau)^2$ and $E_8(\tau)$.

(b) Use (a) to prove the far from obvious identity

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m).$$

Exercise 2

(a) Let $N \in \mathbb{N}_{\geq 1}$. Prove that all the cusps of $\Gamma_0(N)$ are regular.

(b) Let $N \in \mathbb{N}_{\geq 5}$. Prove that all the cusps of $\Gamma_1(N)$ are regular.

(c) Prove that the cusp $[1 : 2]$ is an irregular cusp of $\Gamma_1(4)$.

Exercise 3

Let $k \in \mathbb{Z}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$. We let

$$j(\gamma, \tau) := c\tau + d$$

and we define the weight-k operator $|_k$ on functions $f : \mathbb{H} \to \mathbb{C}$ by

$$(f|_k\gamma)(\tau) = \det(\gamma)^{k/2} j(\gamma, \tau)^{-k} f(\gamma\tau) \qquad (\tau \in \mathbb{H}).$$

For $\gamma, \gamma' \in \mathrm{GL}_2^+(\mathbb{R})$ and $\tau \in \mathbb{H}$, prove the following identities

(i) $j(\gamma\gamma', \tau) = j(\gamma, \gamma'\tau) j(\gamma', \tau)$.

(ii) $j(\gamma^{-1}, \tau) = j(\gamma, \gamma^{-1}\tau)^{-1}$.

(iii) $\frac{d(\gamma\tau)}{d\tau} = \det(\gamma) j(\gamma, \tau)^{-2}$.

(iv) $f|_k(\gamma\gamma') = (f|_k\gamma)|_k\gamma'$.

Exercise 4

(a) Prove that the group $\Gamma(2)$ is generated by the following matrices

$$T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad -E = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad R := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

and verify this result with SAGE. For the proof, proceed as follows:

(1) Prove that $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(2)$ if and only if a, d are odd, b, c are even, and $ad - bc = 1$. In particular, $(a, b) = 1$ and $(a, c) = 1$.

(2) Prove that every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(2)$ can be transformed to

$$\begin{pmatrix} \star & 0 \\ \star & \star \end{pmatrix},$$

after a suitable multiplication from the right by T^2 and R.

(3) Prove that every matrix of the form $\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right) \in \Gamma(2)$ is a word in R and $-E$.

(b) Consider the group Γ_θ, which is defined to be the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by T^2 and S, with S as in Sheet 1. Prove that Γ_θ is a congruence group by showing that $\Gamma(2) \subseteq \Gamma_\theta$.