U Tübingen Winter Term 2018/19

Exercises for the Course
Modular forms

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Exercise sheet 7

Exercise 1

Recall that, for $\tau \in \mathbb{H}$, we have

$$P(\tau) := 2\zeta(2) + \sum_{c\in\mathbb{Z}\setminus\{0\}} \sum_{d\in\mathbb{Z}} \frac{1}{(c\tau+d)^2} = 2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \sigma_1(n) q^n.$$

Also, recall that

$$\frac{1}{\tau} + \sum_{d=1}^{\infty} \left(\frac{1}{\tau-d} + \frac{1}{\tau+d} \right) = \pi\cot(\pi\tau) = \pi i - 2\pi i \sum_{m=0}^{\infty} q^m. \tag{1}$$

The aim of this exercise is to show that

$$(P|_2\gamma)(\tau) = P(\tau) - \frac{2\pi i c}{(c\tau+d)} \tag{2}$$

for any $\tau \in \mathbb{H}$ and any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

(a) Suppose that $P(\tau)$ satisfies the identity in (2) for two particular matrices $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$. Show that $P(\tau)$ then satisfies the identity for the product $\gamma_1\gamma_2$ and the inverse γ_1^{-1} as well. Thus to establish (2) it suffices to prove the identity for a set of generators for $\mathrm{SL}_2(\mathbb{Z})$.

(b) Use the Fourier expansion for $P(\tau)$ to show that it satisfies (2) for $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

(c) Consider $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Show that

$$(P|_2\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right))(\tau) = 2\zeta(2) + \sum_{d\in\mathbb{Z}} \sum_{c\in\mathbb{Z}\setminus\{0\}} \frac{1}{(c\tau+d)^2},$$

which differs from the definition of $P(\tau)$ in the reversed order of summation.

(d) Use partial fractions to show that

$$\sum_{c\in\mathbb{Z}\setminus\{0\}} \sum_{d\in\mathbb{Z}} \frac{1}{(c\tau+d)(c\tau+d+1)} = 0.$$

Subtract this from $P(\tau)$ to show that

$$P(\tau) = 2\zeta(2) + \sum_{c\in\mathbb{Z}\setminus\{0\}} \sum_{d\in\mathbb{Z}} \frac{1}{(c\tau+d)^2(c\tau+d+1)},$$

where now the sum is absolutely convergent.

(e) Reverse the order of summation in (d) and show that

$$P(\tau) = \tau^{-2}P(-1/\tau) - \sum_{d\in\mathbb{Z}} \sum_{c\in\mathbb{Z}\setminus\{0\}} \frac{1}{(c\tau+d)(c\tau+d+1)}.$$

The error term is

$$-\lim_{N\to\infty} \sum_{d=-N}^{N-1} \sum_{c\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{c\tau+d} - \frac{1}{c\tau+d+1}\right).$$

Reverse the order and the inner sum telescopes. Manipulate the result into an expression including a sum for $\pi\cot(\pi N/\tau)$ per the first equality of (1). Then, use the other side of (1) to take the limit. Conclude that $P(\tau)$ satisfies (2) for $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

(f) Show that

$$\frac{1}{j(\gamma,\tau)^2\,\mathrm{Im}(\gamma\tau)} = \frac{1}{\mathrm{Im}(\tau)} - \frac{2ic}{c\tau+d}$$

for any $\tau \in \mathbb{H}$ and any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Conclude that the function $P(\tau) - \pi/\mathrm{Im}(\tau)$ has weight 2 w.r.t. $\mathrm{SL}_2(\mathbb{Z})$, but that it is not holomorphic on $\mathbb{H}$.

Exercise 2

Recall that the Dedekind eta function is given, for $\tau \in \mathbb{H}$, by the infinite product

$$\eta(\tau) = e^{2\pi i\tau/24} \prod_{n=1}^{\infty} (1 - e^{2\pi i\tau n}).$$

Prove that η is holomorphic on $\mathbb{H}$.

Hint: Take the logarithm of the product and show that the resulting series converges absolutely and locally uniformly on $\mathbb{H}$.