U Tübingen Winter Term 2018/19

Exercises for the Course
Modular forms

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Exercise sheet 8

Let X be a compact Riemann surface and let $\mathcal{M}(X)$ be the field of meromorphic functions on X. Recall that $\mathrm{Div}(X) = \mathrm{Div}_{\mathbb{Z}}(X)$ denotes the group of divisors of X. Further, for a non-zero meromorphic function $f \in \mathcal{M}(X)$, the principal divisor associated to f is the divisor $(f) := \sum_{P\in X} \mathrm{ord}_P(f) \cdot P \in \mathrm{Div}(X)$, where $\mathrm{ord}_P(f)$ denotes the order of zero (or negative the order of pole) of f at P. For a divisor $D \in \mathrm{Div}(X)$, by $\mathcal{L}(D)$ we denote the set of $f \in \mathcal{M}(X)$ such that $(f) + D \geq 0$, including, by convention, the zero function 0.

Exercise 1

Prove the following assertions:

(a) For non-zero $f, g \in \mathcal{M}(X)$, we have $(fg) = (f) + (g)$ and $(f/g) = (f) - (g)$. Furthermore, the set $\mathrm{PDiv}(X)$ of principal divisors of X is a subgroup of $\mathrm{Div}(X)$.

(b) If $D = \sum_P n_P \cdot P \in \mathrm{Div}(X)$, then $\mathcal{L}(D)$ consists of all meromorphic functions that have at worst a pole of order n_P at P (or a zero of order $-n_P$ or greater, if n_p is negative). The set $\mathcal{L}(D)$ is a $\mathbb{C}$-vector space; by $\ell(D)$ we denote its dimension.

(c) Let $D_1, D_2 \in \mathrm{Div}(X)$. If $D_1 \leq D_2$, then $\mathcal{L}(D_1) \subseteq \mathcal{L}(D_2)$. If $f \in \mathcal{L}(D_1)$ and $g \in \mathcal{L}(D_2)$, then $fg \in \mathcal{L}(D_1 + D_2)$.

(d) The space $\mathcal{L}(0)$ consists only of the constant functions, and $\mathcal{L}(D)$ consists only of 0 if $D < 0$. In particular, $\ell(0) = 1$, and $\ell(D) = 0$ when $D < 0$.

(e) Let $D \in \mathrm{Div}(X)$ and let $P \in X$. Then, $\mathcal{L}(D)$ has codimension at most 1 in $\mathcal{L}(D+P)$.

(f) By (d) and (e), the spaces $\mathcal{L}(D)$ are all finite dimensional, with the dimension $\ell(D)$ increasing by zero or one each time one adds an additional pole to D.

(g) Let $D, D' \in \mathrm{Div}(X)$ be divisors with $D = D' + (g)$ for some $g \in \mathcal{M}(X)$. Then, there is an isomorphism $\mathcal{L}(D) \cong \mathcal{L}(D')$.

Exercise 2

(a) Prove that for any meromorphic 1-form ω on X, the sum of all residues of ω vanishes.

(b) Prove that very principal divisor $(f) \in \mathrm{PDiv}(X)$ has degree zero.

Exercise 3

(a) If $\deg(D) < 0$, show that $\ell(D) = 0$.

(b) If $\deg(D) = 0$, show that $\ell(D) = 0$ is equal to 0 or 1, with the latter occuring if and only if D is principal. Furthermore, any non-zero element of $\mathcal{L}(D)$ has divisor $-D$.

(c) If $\deg(D) \geq 0$, establish the bound $\ell(D) \leq \deg(D) + 1$.

Exercise 4

Compute the following dimensions and verify your results SAGE:

(a) $\dim M_2(\Gamma_0(2))$, $\dim S_{10}(\Gamma_0(2))$.

(b) $\dim M_2(\Gamma_0(4))$, $\dim M_4(\Gamma_0(4))$.

(c) $\dim S_2(\Gamma_0(11))$, $\dim S_4(\Gamma_0(11))$.

(d) $\dim S_2(\Gamma_0(13))$, $\dim S_4(\Gamma_0(13))$.

(e) $\dim S_2(\Gamma_0(35))$, $\dim S_{12}(\Gamma_0(35))$.