Exercises for the Course
Modular forms

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Exercise class: 25.01.19

Exercise sheet 9

Let $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ and let $k \in \mathbb{N}_{>0}$ be even. For $n \in \mathbb{N}_{>0}$, let $T(n)$ denote the n-th Hecke operator acting on $\mathcal{M}_k(\Gamma)$.

Exercise 1

Deduce the following assertions from the proof of Proposition 4 given in class:

(a) If $f \in S_k(\Gamma)$, then $f \,|\, T(n) \in S_k(\Gamma)$.

(b) Let $f \in M_k(\Gamma)$ and let $a(m)$ denote the m-th Fourier coefficient of f $(m \in \mathbb{N})$. Let p be a prime number. Then, the m-th Fourier coefficient $a'(m)$ of $f \,|\, T(p)$ $(m \in \mathbb{N})$ satisfies
$$a'(m) = a(mp) + p^{k-1}\, a\Big(\frac{m}{p}\Big);$$
here, $a(m/p) = 0$, if $m/p \notin \mathbb{Z}$.

Exercise 2

(a) Let p be a prime number and let $r, s \in \mathbb{N}$ with $s \geq r$. Prove that
$$T(p^r) \cdot T(p^s) = \sum_{t=0}^{r} p^{t(k-1)}\, T(p^{r+s-2t}).$$

(b) Let $m, n \in \mathbb{N}_{>0}$. Prove that
$$T(m) \cdot T(n) = \sum_{d \,|\, (m,n)} d^{k-1}\, T\Big(\frac{mn}{d^2}\Big).$$

(c) Using (b), prove that
$$\mathcal{H} := \Big\{ \sum_{n \in \mathbb{N}_{>0}} c_n\, T(n) \,\Big|\, c_n \in \mathbb{C}, c_n = 0 \text{ for almost all } n \Big\}$$
has the structure of a commutative algebra over $\mathbb{C}$. This algebra is called *Hecke algebra.*

Exercise 3
Let $f \in \mathcal{S}_k(\Gamma)$, $f \neq 0$, with Fourier expansion

$$f(\tau) = \sum_{n=1}^{\infty} a(n) \cdot q^n,$$

be a normalized Hecke eigenform. Then, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$, the *associated Dirichlet L-series*

$$L(f, s) := \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

converges absolutely. Prove that, for $\mathrm{Re}(s) \gg 0$, we have

$$L(f, s) = \prod_{p \text{ prime}} \left(1 - a(p)p^{-s} + p^{k-1-2s}\right)^{-1}.$$

Exercise 4
Consider the Delta function $\Delta(\tau) := (E_4(\tau)^3 - E_6(\tau)^2)/1728$ with Fourier expansion

$$\Delta(\tau) = \sum_{n=1}^{\infty} \tau(n) \cdot q^n.$$

Prove the following assertions:

(a) $\Delta(\tau)$ is a Hecke eigenform. Is $\Delta(\tau)$ a normalized Hecke eigenform?

(b) For coprime $m, n \in \mathbb{N}_{>0}$, we have

$$\tau(m \cdot n) = \tau(m) \cdot \tau(n).$$

For $r \in \mathbb{N}_{>0}$ and p a prime number, we have

$$\tau(p^r) \cdot \tau(p) = \tau(p^{r+1}) + p^{11}\tau(p^{r-1}).$$

(c) For $\mathrm{Re}(s) \gg 0$, we have

$$L(\Delta, s) = \prod_{p \text{ prime}} \left(1 - \tau(p)p^{-s} + p^{11-2s}\right)^{-1}.$$

Exercise 5
Consider the Eisenstein series $E_k(\tau) \in \mathcal{M}_k(\Gamma)$ for $k > 2$. Prove the following assertions:

(a) $E_k(\tau)$ is a Hecke eigenform. Is $E_k(\tau)$ a normalized Hecke eigenform?

(b) For coprime $m, n \in \mathbb{N}_{>0}$, we have

$$\sigma_{k-1}(m \cdot n) = \sigma_{k-1}(m) \cdot \sigma_{k-1}(n).$$

For $r \in \mathbb{N}_{>0}$ and p a prime number, we have

$$\sigma_{k-1}(p^r) \cdot \sigma_{k-1}(p) = \sigma_{k-1}(p^{r+1}) + p^{k-1}\sigma_{k-1}(p^{r-1}).$$

(c) The associated Dirichlet L-series of the corresponding normalized Hecke eigenform equals $\zeta(s - k + 1)\zeta(s)$.