ASYMPTOTICS OF CONTINUOUS-TIME DISCRETE STATE SPACE BRANCHING PROCESSES FOR LARGE INITIAL STATE

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Abstract

Scaling limits for continuous-time branching processes with discrete state space are provided as the initial state tends to infinity. Depending on the finiteness or non-finiteness of the mean and/or the variance of the offspring distribution, the limits are in general time-inhomogeneous Gaussian processes, time-inhomogeneous generalized Ornstein–Uhlenbeck type processes or continuous-state branching processes. We also provide transfer results showing how specific asymptotic relations for the probability generating function of the offspring distribution carry over to those of the one-dimensional distributions of the branching process.

Keywords: Branching process; generalized Mehler semigroup; Neveu’s continuous-state branching process; Ornstein–Uhlenbeck type process; self-decomposability; stable law; time-inhomogeneous process; weak convergence

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1 Introduction

Suppose that the lifetime of each individual in some population is exponentially distributed with a given parameter \( a \in (0, \infty) \) and that at the end of its life each individual gives birth to \( k \in \mathbb{N}_0 := \{0, 1, \ldots\} \) individuals with probability \( p_k \), independently of the rest of the population. Assuming that the population consists of \( n \in \mathbb{N} := \{1, 2, \ldots\} \) individuals at time \( t = 0 \) we denote with \( Z^{(n)}_t \) the random number of individuals alive at time \( t \geq 0 \). The process \( Z^{(n)}_t := (Z^{(n)}_t)_{t \geq 0} \) is a classical continuous-time branching process with discrete state space \( \mathbb{N}_0 \cup \{\infty\} \) and initial state \( Z^{(n)}_0 = n \). These processes have been studied extensively in the literature. For fundamental properties of these processes we refer the reader to the classical books of Harris [22, Chapter V], Athreya and Ney [4, Chapter III] and Sewastjanow [43]. Define \( Z_t := Z^{(1)}_t \) and \( Z := Z^{(1)} \) for convenience. By the branching property, \( Z^{(n)}_t \) is distributed as the sum of \( n \) independent copies of \( Z \). The literature thus mainly focuses on the situation \( n = 1 \) and most results focus on the asymptotic behavior of these processes as the time \( t \) tends to infinity.

In contrast we are interested in the asymptotic behavior of \( Z^{(n)}_t \) as the initial state \( n \) tends to infinity. To the best of the authors knowledge this question has not been discussed rigorously in the literature for continuous-time discrete state space branching processes. Related questions for discrete-time Bienaymé–Galton–Watson processes have been studied extensively in the literature (see for example Lamperti [29,30] or Green [20]), however in this situation time is usually scaled as well, which make these approaches different from the continuous-time case. The article of Sagitov [42] contains related results, however the critical case is considered and again an additional time scaling is used.

The asymptotics as the initial state \( n \) tends to infinity may in some sense be viewed as a non-natural question in branching process theory, however this question has fundamental applications, for example in coalescent theory. It is well known that the block counting process of any exchangeable coalescent, restricted to a sample of size \( n \), has a Sieg mund dual process, called the fixation line. For the Bolthausen–Sznitman coalescent the fixation line is (see, for example, [27]) a continuous-time discrete state space branching process \( Z^{(n)}_t \) with offspring distribution \( p_k = 1/(k(k-1)), \, k \in \{2, 3, \ldots\} \). In this context the parameter \( n \) is the sample size and hence the
question about its asymptotic behavior when the sample size $n$ size becomes large is natural and important. In fact, this example was the starting point to become interested in the asymptotical behavior of branching processes for large initial value.

The convergence results are provided in Section 2. We provide a convergence result for the finite variance case (Theorem 1), another result for the situation when the process has still finite mean but infinite variance (Theorem 2), and for the situation when even the mean is infinite but the process still does not explode in finite time (Theorem 3). The limiting processes arising in Theorem 1 are (time-inhomogeneous) Gaussian processes whereas those in Theorem 2 are (time-inhomogeneous) Ornstein–Uhlenbeck type processes. In Theorem 3 continuous-state branching processes arise in the limit as $n \to \infty$. For all three regimes typical examples are provided. The basic idea to obtain convergence results of this form is relatively obvious. For fixed time $t$, since $Z_t^{(n)}$ is a sum of $n$ independent copies of $Z_t$, we can apply central limit theorems, leading to the convergence of the one-dimensional distributions. We refer the reader exemplary to the books of Petrov [35, 36] and Ibragimov and Linnik [24] and the article of Geluk and De Haan [18] for classical limiting results on sums of independent and identically distributed random variables. However, we prove not only convergence of the marginals or the finite-dimensional distributions.

We provide functional limiting results for the sequence of processes $(Z_t^{(n)})_{n \in \mathbb{N}}$. Their proofs require some additional efforts. We think that the arising limiting processes are quite interesting. For example, since the centering or scaling of the space in Theorem 1 and Theorem 2 in general explicitly depends on the time $t$, the limiting processes are in general time-inhomogeneous.

The convergence results are as well based on crucial transfer results showing how particular asymptotic relations for the probability generating function (pgf) of the offspring distribution carry over to the pgf of $Z_t$. Results of this form are for example provided in Lemma 1, Lemma 2 and Lemma 3 and are of its own interest. Despite the fact that there is a vast literature on continuous-time branching processes, we have not been able to trace these results.

Throughout the article $\xi$ denotes a random variable taking values in $\mathbb{N}_0$ with probability $p_k := \mathbb{P}(\xi = k)$, $k \in \mathbb{N}_0$. For a space $E$ equipped with a $\sigma$-algebra we denote with $B(E)$ the space of all bounded measurable functions $g : E \to \mathbb{R}$. For a topological space $X$ and $K \subset \{\mathbb{R}, \mathbb{C}\}$ we denote by $\hat{C}(X, K)$ the space of continuous functions $g : X \to K$ vanishing at infinity and also write $\hat{C}(X)$ for $\hat{C}(X, \mathbb{R})$.

## 2 Results

Let $f$ denote the pgf of $\xi$, i.e. $f(s) := \mathbb{E}(s^\xi) = \sum_{k \geq 0} p_k s^k$ and define $u(s) := a(f(s) - s)$ for $s \in [0,1]$. Let $r \geq 1$. It is well known (see, for example, Athreya and Ney [4], p. 111, Corollary 1) that $m_r(t) := \mathbb{E}(\xi^r) < \infty$ for all $t > 0$ if and only if $\mathbb{E}(\xi^2) = \sum_{k \geq 0} k^r p_k < \infty$. Moreover $m(t) := m_1(t) = e^{\lambda t}$ with $\lambda := u'(1) = a(\mathbb{E}(\xi) - 1)$ and

$$m_2(t) = \begin{cases} \tau^2 \lambda^{-1} e^{\lambda t} (e^{\lambda t} - 1) + e^{\lambda t} & \text{if } \lambda \neq 0, \\ \tau^2 t + 1 & \text{if } \lambda = 0, \end{cases}$$

(1)

with $\tau := u''(1) - a f''(1) = a \mathbb{E}(\xi (\xi - 1))$. Note that (1) slightly corrects Eq. (5) on p. 109 in [4], which accidently provides the formula for the second descending factorial moment $\mathbb{E}(Z_t(Z_t - 1))$ instead of the second moment $\mathbb{E}(Z_t^2)$. In particular, if $m_2(t) < \infty$, then

$$\sigma^2(t) := \text{Var}(Z_t) = \begin{cases} (\tau^2 - \lambda) e^{\lambda t} (e^{\lambda t} - 1)/\lambda & \text{if } \lambda \neq 0, \\ \tau^2 t & \text{if } \lambda = 0. \end{cases}$$

### 2.1 The finite variance case

Assume that the second moment $\mathbb{E}(\xi^2) = \sum_{k \geq 0} k^2 p_k$ of the offspring distribution is finite or, equivalently, that $\text{Var}(Z_t) < \infty$ for all $t \geq 0$. In the following $a \wedge b := \min\{a, b\}$ denotes the minimum of $a, b \in \mathbb{R}$. We furthermore use for $\mu \in \mathbb{R}$ and $\sigma^2 \geq 0$ the notation $N(\mu, \sigma^2)$ for the normal distribution with mean $\mu$ and variance $\sigma^2$ with the convention that $N(\mu, 0)$ is the Dirac measure at $\mu$. Our first fluctuation result (Theorem 1) clarifies the asymptotic behavior of $Z_t^{(n)}$ as the initial state $n$ tends to infinity. The proof of Theorem 1 is provided in Section 3.
Remarks.

1. (Continuity of X) Let $s, t \geq 0$ and $x \in \mathbb{R}$. Conditional on $X_s = x$ the random variable $X_{s+t} - X_s$ has a normal distribution with mean $\mu := xm(t) - x = x(m(t) - 1)$ and variance $\sigma^2 := m(s)\sigma^2(t)$. Thus, $\mathbb{E}((X_{s+t} - X_s)^4 | X_s = x) = 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4 = 3m^2(s)\sigma^4(t) + 6x^2(m(t) - 1)^2m(s)\sigma^2(t) + x^4(m(t) - 1)^4$ or, equivalently,

$$\mathbb{E}((X_{s+t} - X_s)^4) = 3m^2(s)\sigma^4(t) + 6X_s^2(m(t) - 1)^2m(s)\sigma^2(t) + X_s^4(m(t) - 1)^4.$$

Taking expectation yields

$$\mathbb{E}((X_{s+t} - X_s)^4) = 3m^2(s)\sigma^4(t) + 6\mathbb{E}(X_s^2)(m(t) - 1)^2m(s)\sigma^2(t) + \mathbb{E}(X_s^4)(m(t) - 1)^4 = 3m^2(s)\sigma^4(t) + 6\sigma^2(s)(m(t) - 1)^2m(s)\sigma^2(t) + 3\sigma^4(s)(m(t) - 1)^4.$$

From this formula it follows that for every $T > 0$ there exists a constant $K = K(T) \in (0, \infty)$ such that $\mathbb{E}((X_s - X_t)^4) \leq K(s-t)^2$ for all $s, t \in [0, T]$. By Kolmogorov’s continuity theorem (see, for example, Kallenberg [26, p. 57, Theorem 3.23]) we can therefore assume that $X$ has continuous paths.

2. (Generator) For $\lambda \neq 0$ the Gaussian process $X$ is time-inhomogeneous. Note that $T_{s,t}g(x) := \mathbb{E}(g(X_{s+t}) | X_s = x) = \mathbb{E}(g(xm(t) + \sqrt{m(s)}X_t))$, $s, t \geq 0$, $g \in C^2(\mathbb{R})$, $x \in \mathbb{R}$. Let $C^2(\mathbb{R})$ denote the space of real valued twice continuously differentiable functions on $\mathbb{R}$. For $s \geq 0$, $g \in C^2(\mathbb{R})$ and $x \in \mathbb{R}$ it follows that

$$A_sg(x) := \lim_{t \to 0} \frac{T_{s,t}g(x) - g(x)}{t} = \lambda xg'(x) + \frac{\sigma^2}{2}m(s)g''(x),$$

where $\sigma^2 := \lim_{t \to 0}\sigma^2(t)/t = \tau^2 - \lambda = a\mathbb{E}(\xi - 1)^2$. For $\lambda = 0$ (critical case) the process $X$ is a time-homogeneous Brownian motion with generator $Ag(x) = (\tau^2/2)g''(x)$, $g \in C^2(\mathbb{R})$, $x \in \mathbb{R}$, where $\tau^2 = a\text{Var}(\xi)$.

3. (Doob–Meyer decomposition) Define the process $C := (C_t)_{t \geq 0}$ via $C_t := \lambda \int_0^t X_s \, ds$, $t \geq 0$. Let $\mathcal{F}_t := \sigma(X_s, s \leq t)$, $t \geq 0$. For all $0 \leq s \leq t$,

$$\mathbb{E}(C_t - C_s | \mathcal{F}_s) = \lambda \mathbb{E} \left( \int_s^t X_u \, du \mid \mathcal{F}_s \right) = \lambda \int_s^t \mathbb{E}(X_u | \mathcal{F}_s) \, du = \lambda \int_s^t m(u-s)X_u \, du = X_s \int_s^t \lambda e^{u-s} \, du = X_s(e^{\lambda(t-s)} - 1) = X_s m(t-s) - X_s = \mathbb{E}(X_t | \mathcal{F}_s) - X_s.$$ 

Thus, the compensated process $M := (M_t)_{t \geq 0} := (X_t - C_t)_{t \geq 0}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$. For $\lambda = 0$ the process $X$ itself is hence a martingale. Clearly, $X = M + C$ is the Doob–Meyer decomposition of $X$. The process $C$ is not monotone, but decomposes into $C = C^+ - C^-$, where $C^+ := (C^+_t)_{t \geq 0}$ and $C^- := (C^-_t)_{t \geq 0}$, defined via $C^+_t := \lambda \int_0^t X^+_s \, ds$ and $C^-_t := \lambda \int_0^t X^-_s \, ds$ for all $t \geq 0$, both have non-decreasing paths.
4. (Positive semi-definiteness) The limiting process $X$ in Theorem 1 is Gaussian. For any finite number $k$ of time points $0 \leq t_1 < \cdots < t_k < \infty$ it follows that $(X_{t_1}, \ldots, X_{t_k})$ has a multivariate normal distribution with positive semi-definite covariance matrix $\Sigma := (\sigma_{i,j})_{i,j \in \{1,\ldots,k\}}$ having entries $\sigma_{i,j} = \text{Cov}(X_{t_i}, X_{t_j}) = m(|t_i - t_j|)\sigma^2(t_i \wedge t_j)$, $i,j \in \{1,\ldots,k\}$. For $\lambda = 0$ (critical case) it follows that the matrix $(t_{i \wedge t_j})_{i,j \in \{1,\ldots,k\}}$ is positive semi-definite.

Examples. (i) Let $\xi$ be geometrically distributed with parameter $p \in (0,1)$. Define $q := 1 - p$. Then all descending factorial moments $E((\xi)_j) = j!(q/p)^j$, $j \in \mathbb{N}_0$, are finite. Theorem 1 is hence applicable with $\lambda = aE(\xi) - 1 = a(q/p - 1)$ and $\tau^2 = aE((\xi)_2) = 2a(q/p)^2$. For $p = 1/2$ (critical case) the process $X$ is a Brownian motion with generator $Af(x) = af''(x)$, $f \in C^2(\mathbb{R})$, $x \in \mathbb{R}$.

(ii) If $\xi$ is Poisson distributed with parameter $\mu \in (0, \infty)$, then again all descending factorial moments $E((\xi)_j) = \mu^j$, $j \in \mathbb{N}_0$, are finite. Theorem 1 is applicable with $\lambda = aE(\xi) - 1 = a(\mu - 1)$ and $\tau^2 = aE((\xi)_2) = a\mu^2$. For $\mu = 1$ (critical case) the process $X$ is a Brownian motion with generator $Af(x) = a(2/3)f''(x)$, $f \in C^2(\mathbb{R})$, $x \in \mathbb{R}$.

(iii) Let $a_1, a_2 \geq 0$ with $a_1 + a_2 > 0$. Theorem 1 is applicable for birth and death processes with rates $na_1$ and $na_2$ for birth and death respectively if the process is in state $n$. In this case we have $a = a_1 + a_2$, $f(s) = (a_2 + a_1 s^2)/a$, $u(s) = a_2 + a_1 s^2 - a s$, $\lambda = a_1 - a_2$ and $\tau^2 = 2a_1$. For $a_1 = a_2$ (critical case) the process $X$ is a Brownian motion with generator $Af(x) = a_1 f''(x)$, $f \in C^2(\mathbb{R})$, $x \in \mathbb{R}$.

2.2 The finite mean infinite variance case

In this subsection it is assumed that $m := E(\xi) < \infty$. Since $f$ is convex on $[0,1]$ the inequality $1 - f(s) \leq m(1 - s)$ holds for all $s \in [0,1]$. In order to state appropriate limiting results it is usual to control the difference between $m(1 - s)$ and $1 - f(s)$. A typical assumption of this form is the following.

Assumption A. There exists a constant $\alpha \in (1,2]$ and a function $L : [1, \infty) \to (0, \infty)$ slowly varying (at infinity) such that

$$1 - f(s) = m(1 - s) - (1 - s)^\alpha L((1 - s)^{-1}), \quad s \in [0,1].$$

(3)

Since $f$ is differentiable, Assumption A in particular implies that $L$ is differentiable. Define $F(s,t) := E(s^{2t})$ for $s \in [0,1]$ and $t \geq 0$. The following lemma clarifies the structure of $F(s,t)$ under Assumption A. Recall that $m(t) := E(Z_t) = e^{\lambda t} < \infty$.

Lemma 1 If the offspring pgf $f$ satisfies Assumption A then, for every $t \geq 0$,

$$1 - F(s,t) = m(t)(1 - s) - c(t)(1 - s)^\alpha L((1 - s)^{-1})(1 + o(1)), \quad s \uparrow 1,$$

(4)

where

$$c(t) := \begin{cases} \frac{m(at) - m(t)}{(\alpha - 1)(m - 1)} & \text{if } \lambda = 0 \text{ (critical case)}, \\ ae^{-\lambda t} \frac{e^{(\alpha - 1)t} - 1}{(\alpha - 1)\lambda} & \text{if } \lambda \neq 0 \text{ (non-critical case)}. \end{cases}$$

(5)

Remark. Although we are in this subsection mainly interested in the infinite variance case, Lemma 1 holds in particular for the finite variance case. In this case, expansion of $f$ for $s \uparrow 1$ shows that (5) holds with $\alpha = 2$ and $L((1 - s)^{-1}) \sim f''(1)/2 = E(\xi(\xi - 1))/2$ as $s \to 1$. Moreover, $c(t)f''(1 - s) = E(Z_t(Z_t - 1)) = F''(1 - t)$, where $F''(s,t)$ denotes the second derivative of $F(s,t)$ with respect to $s$.

In the following we are however interested in the infinite variance situation, so we assume that $E(\xi^2) = \infty$. We are now able to state our second main convergence result.

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Theorem 2 Assume that $m : = \mathbb{E}(\xi) < \infty$ and $\mathbb{E}(\xi^2) = \infty$. Suppose that Assumption A holds. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers satisfying $a_n \geq 1$ for all sufficiently large $n$ and $L(a_n) \sim a_n^\alpha/(\alpha n)$ as $n \to \infty$. Then the process $X^{(n)} := (X^{(n)}_t)_{t \geq 0}$, defined via

$$X^{(n)}_t := \frac{Z^{(n)}_t - nm(t)}{a_n}, \quad n \in \mathbb{N}, t \geq 0,$$

converges in $D_0[0, \infty]$ as $n \to \infty$ to a limiting process $X = (X_t)_{t \geq 0}$ with state space $\mathbb{R}$ and initial state $X_0 = 0$, whose distribution is characterized as follows. Conditional on $X_s = x$ the random variable $X_{s+t}$ is distributed as $x \alpha$-stable with characteristic function $u \mapsto \mathbb{E}(e^{iuX_s}) = \exp(c(t)(-iu)^\alpha/\alpha)$, $s, t \geq 0$, $u \in \mathbb{R}$, and Laplace transform $\eta \mapsto \mathbb{E}(e^{-\alpha X_t}) = \exp(c(t)(\eta^\alpha/\alpha), \eta, t \geq 0$. Note that $\mathbb{E}(X_t) = 0, t \geq 0$. The variance of $X_t$ is equal to $c(t)$ for $\alpha = 2$ whereas $\text{Var}(X_t) = \infty$ for $t > 0$ and $\alpha \in (1, 2)$.

Remark. As in Theorem 1 the limiting process $X$ in Theorem 2 is time-homogeneous if and only if $\lambda = 0$. We have $T_{s,t}g(x) := \mathbb{E}(g(X_{s+t}) \mid X_s = x) = \mathbb{E}(g(xm(t) + (m(s))^{1/\alpha}X_t))$ for $s, t \geq 0$, $g \in \mathcal{B}(\mathbb{R})$ and $x \in \mathbb{R}$. Note that $T_{s,t}g(x)$ is well defined even for some functions $g$ which are not bounded. For example, for Laplace test functions of the form $g = g_\eta$, defined via $g_\eta(x) := e^{-\eta x}$ for all $x \in \mathbb{R}$ and $\eta \geq 0$, we obtain the explicit formula

$$A_\alpha g_\eta(x) := \lim_{t \to 0} T_{s,t}g(x) - g(x) = \lim_{t \to 0} \left[ \frac{e^{-m(t)\eta x + c(t)m(s)\eta^\alpha/\alpha}}{t} - e^{-\eta x} \right]$$

$$= \lim_{t \to 0} \left( - m'(t)\eta x + c'(t)m(s)\frac{\eta^\alpha}{\alpha} \right) e^{-m(t)\eta x + c(t)m(s)\eta^\alpha/\alpha} \eta^\alpha/\alpha e^{-\eta x}$$

$$= \left( - \lambda x + am(s)\frac{\eta^\alpha}{\alpha} \right) e^{-\eta x}, \quad s, \eta \geq 0, x \in \mathbb{R}. \quad (6)$$

For $\alpha = 2$ and $g \in C^2(\mathbb{R})$ it follows from (6) that

$$A_\alpha g(x) := \lim_{t \to 0} T_{s,t}g(x) - g(x) = \lambda x g'(x) + \frac{a}{2} m(s) g''(x), \quad s \geq 0, x \in \mathbb{R},$$

showing that for $\alpha = 2$ the process $X$ has the same structure as in Theorem 1 with $\sigma^2$ replaced by the constant $a$.

Assume now that $\alpha \in (1, 2)$. Then, from (6), a straightforward calculation based on the formula

$$\int_0^\infty \frac{e^{-\eta h} - 1 + \eta h}{h^{\alpha+1}} dh = \frac{\Gamma(2 - \alpha)}{\alpha(\alpha - 1)} \eta^\alpha = \Gamma(-\alpha)\eta^\alpha, \quad \eta \geq 0, \alpha \in (1, 2),$$

yields

$$A_\alpha g(x) = \lambda x g'(x) + am(s) \frac{\alpha - 1}{\Gamma(2 - \alpha)} \int_0^\infty \frac{g(x + h) - g(x) - h g'(x)}{h^{\alpha+1}} dh, \quad s \geq 0, x \in \mathbb{R},$$

first for $g = g_\eta$ and, hence, for other classes of functions $g$, for example for $g \in C^2(\mathbb{R})$. These formulas for the semigroup and the generator show that $X$ is a time-inhomogeneous Ornstein–Uhlenbeck type process [14]. For fundamental results on such processes and related generalized Mehler semigroups we refer the reader to [10].

For $\alpha = 2$ we have $a_n^\alpha/n \sim aL(a_n) \to \infty$ as $n \to \infty$, in contrast to the situation in Theorem 1 where $a_n = \sqrt{n}$ and, hence, $a_n^\alpha/n = 1$. For $\alpha = 2$ the limiting random variable $X_t$ has a normal distribution with mean 0 and variance $c(t)$ given via (5) with $\alpha = 2$.

Two examples are now provided, one with $\alpha = 2$ and the other with $\alpha \in (1, 2)$. In the first example the underlying branching process is supercritical whereas in the second example it is critical. In the first example $F(s, t)$ can be expressed in terms of the Lambert $W$ function. In the second example $F(s, t)$ is known explicitly.
Example 1 Suppose that \( p_k = 4/((k-1)k(k+1)) \) for \( k \in \{2, 3, \ldots \} \), i.e. \( f(s) = \sum_{k=2}^\infty p_k s^k = 2s^{-1}(1-s)^2(-\log(1-s)) - 2 + 3s \), \( s \in (0, 1) \). Note that (8) holds with \( \alpha = 2 \), \( m := \mathbb{E}(\xi) = 3 \), \( L(1) := 2 \) and \( L(x) := 2(\log x)/(1-1/x) \) for \( x > 1 \). Clearly, \( L(x) \sim 2 \log x \) as \( x \to \infty \). Moreover, \( \lambda = 2a \), \( m(t) := \mathbb{E}(Z_t) = e^{2at} \) and \( \text{Var}(Z_t) = \infty \) for \( t > 0 \). The sequence \((a_n)_{n \in \mathbb{N}}\), defined via \( a_1 := 1 \) and \( a_n := \sqrt{2n \log n} \) for \( n \in \mathbb{N} \setminus \{1\} \), satisfies \( L(a_n) \sim 2 \log a_n \sim \log n = a_n^2/(2n) \) as \( n \to \infty \). By Theorem 2, the process \((Z^{(n)} - ne^{2at})/\sqrt{2n \log n})_{t \geq 0} \) converges in \( D_{\mathbb{R}}[0, \infty) \) as \( n \to \infty \) to a time-inhomogeneous process \( X = (X_t)_{t \geq 0} \) with distribution as described in Theorem 2. In particular, for every \( t > 0 \) the random variable \( X_t \) has a normal distribution with mean 0 and variance \( c(t) = 2e^{2at}(e^{2at} - 1) \). The pgf \( F(. \mid t) \) of \( Z_t \) is computed as follows. From the backward equation
\[
\begin{align*}
t & = \int s \frac{F(s, t)}{u(x)} \, dx = \frac{1}{\alpha} \int s \frac{F(s, t)}{2(1-x)((x-1)\log(1-x)-x)} \, dx = \frac{1}{2\alpha}[u(x)]_{s}^{F(s, t)}
\end{align*}
\]
with \( v(x) := \log(1-x) - \log(x + (1-x)\log(1-x)) \), \( x \in (0, 1) \), we conclude that
\[
F(s, t) = v^{-1}(2at + v(s)),
\]
where \( v^{-1} : \mathbb{R} \to (0, 1) \) denotes the inverse of \( v \), which turns out to be of the form \( v^{-1}(y) = 1 + 1/W(h) \), where \( h := \exp(-1-e^{-y}) \in (0, 1/e) \) and \( W = W_1 \) denotes the lower branch of the Lambert W function satisfying \( W(h)e^{W(h)} = h \) and being real valued on \( [-1/e, 0) \). Expansion of (7) shows that
\[
F(s, t) = 1 - e^{2at}(1-s) + e^{2at}(e^{2at} - 1)(1-s)^2 \log((1-s)^{-1}) + O((1-s)^2), \quad s \to 1,
\]
in agreement with (8), since \( c(t) = 2e^{2at}(e^{2at} - 1) \) and \( L(x) \sim 2 \log x \) as \( x \to \infty \).

Example 2 Let \( \alpha \in (1, 2) \). Assume that \( f(s) = s + (1-s)^{n-\alpha}/\alpha \), \( s \in (0, 1) \). Note that \( p_0 = 1/\alpha \), \( p_1 = 0 \) and \( p_k = (-1)^k\binom{n-\alpha}{k}/\alpha \) for \( k \in \{2, 3, \ldots \} \). In particular, \( p_k \sim 1/((\alpha-\gamma)\kappa^{\alpha-1}) \) as \( k \to \infty \). Moreover, \( f'(s) = 1 - (1-s)^{-\alpha} \) and, therefore, \( m := \mathbb{E}(\xi) = f'(1) = 1 \). Thus, the underlying branching process is critical, the extinction probability is \( q = 1 \) and (3) holds with \( L \equiv 1/\alpha \). Note that \( u(s) = a(1-s)^{-\alpha}/\alpha \). Theorem 3 is applicable with \( a_n := n^{-\alpha} \). It follows that \( (n^{-\alpha}(Z^{(n)}_{\alpha}) - n))_{t \geq 0} \) converges in \( D_{\mathbb{R}}[0, \infty) \) as \( n \to \infty \) to a process \( X \) with distribution as described in Theorem 4. In particular, for every \( t \geq 0 \) the random variable \( X_t \) has characteristic function \( u \mapsto \exp(-at(-iu)^{\alpha}/\alpha) \), \( u \in \mathbb{R} \). From
\[
\begin{align*}
at & = \int s \frac{F(s, t)}{f(x)-x} \, dx = \int s \frac{F(s, t)}{\alpha(1-x)^{-\alpha}} \, dx = \frac{\alpha}{\alpha-1}((1-F(s, t))^{1-\alpha} - (1-s)^{1-\alpha})
\end{align*}
\]
it follows that \( F(s, t) = 1 - ((1-s)^{-\alpha}ta + (1-s)^{1-\alpha})^{1/(1-\alpha)} \). Note that
\[
1 - F(s, t) = (1-s) - \frac{at}{\alpha}(1-s)^{\alpha} + \frac{a^2t^2}{2\alpha}(1-s)^{2\alpha-1} + O((1-s)^{3\alpha-2}), \quad s \to 1,
\]
in agreement with (4), since \( c(t) = at \) and \( L \equiv 1/\alpha \).

2.3 The infinite mean case with non-explosion

In this subsection it is assumed that \( m := \mathbb{E}(\xi) = \infty \) or, equivalently, that \( m(t) := \mathbb{E}(Z_t) = \infty \) for all \( t > 0 \). In order to state the result it is convenient to define the function \( L : [1, \infty) \to (0, \infty) \) via
\[
L(x) := x(1-f(1-x^{-1})), \quad x \geq 1.
\]
The substitution \( s = 1 - x^{-1} \) shows that this definition is equivalent to
\[
1 - f(s) = (1-s)L((1-s)^{-1}), \quad s \in (0, 1).
\]
Non-explosion is assumed throughout this section, which is equivalent to (see, for example, Harris [22] Chapter V, Section 9, p. 106, Theorem 9.1)

\[ \int_{\varepsilon}^{1} \frac{1}{s - f(s)} \, ds = \int_{(1-\varepsilon)^{-1}}^{\infty} \frac{1}{x(L(x) - 1)} \, dx = \infty \]

for all \( \varepsilon \in (q, 1) \), where \( q \) denotes the extinction probability. For the theory of stable distributions and their domains of attraction we refer the reader to Geluk and de Haan [18]. For the moment let \( t > 0 \) be fixed. Then \( Z_{t}^{(n)} \), suitably normalized, converges in distribution as \( n \to \infty \) to a non-degenerate limit, that is, \( Z_{t} \) is in the domain of attraction of a stable law, if and only if the following condition is satisfied. There exists \( \alpha(t) \in (0, 1) \) and a slowly varying function \( L_{t} : [1, \infty) \to (0, \infty) \) such that

\[ \mathbb{P}(Z_{t} > x) \sim x^{-\alpha(t)}L_{t}(x), \quad x \to \infty. \]  

(10)

And, if \( \alpha(t) = 1 \), then \( L_{t}(x) \to \infty \) as \( x \to \infty \). In this subsection only the case \( \alpha(t) < 1 \) is investigated. Recall that \( F(s,t) = \mathbb{E}(s^{Z_{t}}) \) for \( s \in [0, 1] \) and \( t \geq 0 \). It follows from Bingham and Doney [8] that (10) is then equivalent to

\[ 1 - F(s,t) = (1 - s)^{\alpha(t)}L_{t}((1 - s)^{-1}), \quad s \in [0, 1), \]

(11)

where, to be precise, the function \( \alpha(t) \) replaces \( \Gamma(1 - \alpha(t))L_{t} \). Then,

\[ \alpha(t) = \frac{\log \frac{1 - F(s,t)}{L_{t}(1 + s^{-1})}}{\log(1 - s)}, \quad t \geq 0, s \in [0, 1). \]  

(12)

Since \( L_{t} \) is slowly varying and hence satisfies \( \log L_{t}(x)/\log x \to 0 \) as \( x \to \infty \), it follows from (12) that

\[ \alpha(t) = \lim_{s \to 1} \frac{\log(1 - F(s,t))}{\log(1 - s)}, \quad t \geq 0. \]  

(13)

In particular, \( \alpha(t) \) is uniquely determined by the pgf \( F(.,t) \). Note that (11) always holds for \( t = 0 \) with \( \alpha(0) = 1 \) and \( c(0) = 1 \) because of the boundary condition \( F(s,0) = s \).

Suppose (11) holds for all \( t \geq 0 \). From the iteration formula \( F(s,t+u) = F(F(s,t),u) \) it follows that

\[ (1 - s)^{\alpha(t+u)}L_{t+u}((1 - s)^{-1}) = 1 - F(s,t+u) = 1 - F(F(s,t),u) = (1 - F(s,t))^{u}L_{u}((1 - F(s,t))^{-1}) = (1 - s)^{\alpha(t)\alpha(u)}L_{t}^{\alpha(u)}((1 - s)^{-1})L_{u}((1 - s)^{-\alpha(t)}L_{t}^{-1}((1 - s)^{-1})), \quad s \in [0, 1). \]

Since all terms depending on \( L \) are slowly varying, \( \alpha(.) \) has to be multiplicative, i.e. \( \alpha(t+u) = \alpha(t)\alpha(u) \) for all \( t, u \geq 0 \). The map \( k : [0, \infty) \to [0, \infty) \), defined via \( k(t) := -\log \alpha(t) \) for all \( t \geq 0 \), is hence additive, so it satisfies the Cauchy functional equation. By Aczel [2, p. 34, Theorem 1], \( k(t) = Ct \) and, hence, \( \alpha(t) = e^{-Ct} \) for all \( t \geq 0 \), where \( C := k(1) = -\log \alpha(1) \in [0, \infty) \). Clearly, either \( \alpha(t) = 1 \) for all \( t \geq 0 \), or \( \alpha(t) < 1 \) for all \( t > 0 \), depending on whether \( C = 0 \) or \( C > 0 \). Also, the map \( t \mapsto L_{t}(x) \) is continuously differentiable and satisfies the equation

\[ L_{t+u}((1 - s)^{-1}) = L_{t}^{\alpha(u)}(1 - s)^{-1}L_{u}((1 - s)^{-\alpha(t)}L_{t}^{-1}((1 - s)^{-1})), \quad t, u \geq 0, s \in [0, 1), \]

or \( L_{t+u}(x) = L_{t}^{\alpha(u)}(x)L_{u}(x^{\alpha(t)}L_{t}^{-1}(x)) \) for all \( t, u \geq 0 \) and all \( x \geq 1 \). The following result (Lemma 2) relates (11) to the pgf \( f \) of the offspring distribution of the branching process. The map \( s \mapsto L((1 - s)^{-1}) = \frac{1 - f(s)}{1 - s} \) has derivative \( s \mapsto \frac{1 - f(s)}{(1 - s)^{2}} - f'(s) \), which is strictly positive on \( [0, 1) \) since \( f \) is strictly convex. Thus, \( L \) is strictly increasing on \( [1, \infty) \). We also have \( L(x) \to \infty \) as \( x \to \infty \) since \( m = \infty \). The proof of Lemma 2 is provided in Section 5.

**Lemma 2** If \( m := f'(1-) = \infty \) then the following conditions are equivalent.
(i) For every $t > 0$ there exists $\alpha(t) \in (0, 1)$ and a slowly varying function $L_t : [1, \infty) \to (0, \infty)$ such that (17) holds.

(ii) For every $t > 0$ the limit

$$\alpha(t) := \lim_{s \to 1} \alpha(s, t) \in (0, 1)$$

exists, where $\alpha(s, t) := (1 - s)((1 - s) F(s, t))/(1 - F(s, t))$ for all $s \in [0, 1)$.

(iii) The limit

$$A := \lim_{x \to \infty} \frac{L(x)}{\log x} = \lim_{s \to 1} \frac{1 - f(s)}{(1 - s) \log((1 - s)^{-1})} \in (0, \infty)$$

exists.

In this case $\alpha(t) = e^{-aAt}$ for all $t \geq 0$.

Remark. Note that

$$aA = a \lim_{s \to 1} \frac{f(s) - 1}{(1 - s) \log(1 - s)} = \lim_{s \to 1} \frac{u(s) - a(1 - s)}{(1 - s) \log(1 - s)} = \lim_{s \to 1} \frac{u(s)}{(1 - s) \log(1 - s)}.$$

Thus, $\alpha(t) = e^{-aAt}$ can be alternatively computed from the function $u(.)$.

Suppose $m = \infty$ and that the limit $A := \lim_{x \to \infty} L(x)/\log x \in (0, \infty)$ in Lemma 2 exists. Recall that, by Lemma 2, the existence of the limit $A$ is equivalent to the existence of constants $\alpha(t) \in (0, 1)$ and of slowly varying functions $L_t$ such that (11) holds, i.e. $1 - F(s, t) = (1 - s)^{\alpha(t)} L_t((1 - s)^{-1})$. In the following we focus on the particular situation that the limit

$$\beta(t) := \lim_{x \to \infty} L_t(x) = \lim_{s \to 1} L_t((1 - s)^{-1}) = \lim_{s \to 1} \frac{1 - F(s, t)}{(1 - s)^{\alpha(t)}} \in (0, \infty)$$

exists for each $t \geq 0$ and is neither 0 nor $\infty$. We know already that $\alpha(t) = e^{-aAt}$. If (15) holds, then we must have $A > 0$, since otherwise $\alpha(t) = 1$ and hence $\beta(t) = m(t) = \infty$, in contradiction to (15). The following result relates (15) to the offspring’s pgf $f$ and provides an explicit formula for $\beta(t)$. The proof of Lemma 3 is provided in Section 6.

**Lemma 3** Suppose $m = \infty$ and that (17) holds. If the limit $B := \lim_{x \to \infty} (L(x) - A \log x) \in \mathbb{R}$ exists, then (15) holds for all $t \geq 0$. In this case

$$\beta(t) = \exp \left( \frac{B - 1}{A} (1 - \alpha(t)) \right), \quad t \geq 0. \quad (16)$$

We are now able to provide the third main convergence result. In the following the notation $E := [0, \infty)$ is used.

**Theorem 3** Suppose that $m = \infty$ and let $L$ be defined via (8) such that (see (6)) the relation $1 - f(s) = (1 - s)L((1 - s)^{-1})$ holds for all $s \in [0, 1)$. Assume that both limits

$$A := \lim_{x \to \infty} \frac{L(x)}{\log x} \in (0, \infty) \quad \text{and} \quad B := \lim_{x \to \infty} (L(x) - A \log x) \in \mathbb{R}$$

exist. For $t \geq 0$ define

$$\alpha(t) := e^{-aAt} \quad \text{and} \quad \beta(t) := \exp \left( \frac{B - 1}{A} (1 - \alpha(t)) \right). \quad (17)$$

Then, as $n \to \infty$, the scaled process $X^{(n)} := (X^{(n)}_t)_{t \geq 0}$, defined via

$$X^{(n)}_t := n^{-1/\alpha(t)} Z^{(n)}_t, \quad t \geq 0,$$

converges in $D_{E}[0, \infty)$ to a limiting continuous-state branching process $X = (X_t)_{t \geq 0}$, whose distribution is characterized as follows.
i) For every $t \geq 0$ the marginal random variable $X_t$ is $\alpha(t)$-stable with Laplace transform $\lambda \mapsto \exp(-\beta(t)\lambda^{\alpha(t)})$, $\lambda \geq 0$.

ii) The semigroup $(T_t)_{t \geq 0}$ of $X$ satisfies $T_t g(x) = \mathbb{E}(g(x^{1/\alpha(t)} X_t))$, $x, t \geq 0, g \in B(E)$, i.e. conditional on $X_0 = x$ the random variable $X_{s+t}$ has the same distribution as $x^{1/\alpha(t)} X_t$.

The proof of Theorem 3 is provided in Section 5. We now provide three examples. In the first two examples the distribution of $Z_t$ is known explicitly.

Example 3 Assume that $\xi$ has distribution $p_k := \mathbb{P}(\xi = k) := 1/(k(k-1))$, $k \in \{2, 3, \ldots\}$. Note that $\xi = [X]$, where $X$ has density $x \mapsto 1/(x-1)^2$, $x \geq 2$, so $X$ has a shifted Pareto distribution with parameter 1. Then, $f(s) = s + (1-s) \log(1-s) = 1 - (1-s)L((1-s)^{-1})$ with $L(x) := 1 + \log x$ and $u(s) := a(f(s) - s) = a(1-s) \log(1-s)$. Note that $A := \lim_{x \to \infty} L(x)/\log x = 1$ and $B := \lim_{x \to \infty} (L(x) - \log x) = 1$. From the backward equation $(\partial/\partial t)F(s, t) = u(F(s, t))$ it follows that

$$t = \int_s^T F(s, t) \frac{1}{u(x)} \, dx = \frac{1}{a} \left[ -\log(-\log(1-x)) \right]_s^T = \frac{1}{a} \log \left( \frac{1}{\log(1 - F(s, t))} \right).$$

Thus, $F(s, t) = 1 - (1-s)e^{-ut}$ showing that $Z_t$ is Sibuya distributed (see, for example, Christoph and Schreiber [22], Eq. (21)) with parameter $e^{-ut}$. The Sibuya distribution and similar distributions occur for example in Gnedenk [12], p. 84, Eq. (9)], Huillet and Möhle [23], p. 9], Iksanov and Schreiber [12, Eq. (2)] (with parameter $e^{-xt}$) for the fixation line of the Bolthausen–Sznitman branching process.

Example 4 Example 3 is easily generalized as follows. Fix two constants $b > 0$ and $c \geq 0$ with $b + c \leq 1$ and assume that $p_0 := c$, $p_1 := 1 - b - c$ and $p_k := b/(k(k-1))$ for $k \geq 2$. Then $f(s) = s + (1-s)(c + b \log(1-s)) = 1 - (1-s)(c - b \log(1-s))$, $u(s) = a(f(s) - s) = a(1-s)(c + b \log(1-s))$ and $L(x) = 1 - c + b \log x$. For $b = 1$ and $c = 0$ we are back in Example 3. Note that $A := \lim_{x \to \infty} L(x)/\log x = 1 > 0$ and $B := \lim_{x \to \infty} (L(x) - b \log x) = 1 - c \in (0, 1]$. The same argument as in Example 3 leads to $F(s, t) = 1 - (1-s)e^{-ut}$ with $u(s) = \exp(c b^{-1}(e^{-b t} - 1))$. Thus, Theorem 3 is applicable with $\alpha(t) := e^{-b t}$ and $\beta(t) := b \exp(c b^{-1}(e^{-b t} - 1))$, $t \geq 0$. Clearly, these formulas for $\alpha(t)$ and $\beta(t)$ are in agreement with those from Lemma 2 and Lemma 3, namely $\alpha(t) = e^{-\alpha At} = e^{-ub t}$ and $\beta(t) = \exp((B-1)(A^{-1}(1 - \alpha(t)))) = \exp(c b^{-1}(e^{-b t} - 1))$, $t \geq 0$.

Example 5 (Discrete Luria–Delbrück distribution) Assume that $\xi$ has a discrete Luria–Delbrück distribution with parameter $b \in (0, \infty)$, i.e. $f(s) = (1-s)^{s(1-s)}/s$, $s \in (0, 1)$. Note that $f(0) = e^{-b}$ and $f(s) = 1 - (1-s)L((1-s)^{-1})$ for $s \in [0, 1]$, where $L(1) := 1 - e^{-b}$ and $L(x) := x(1-x)/(1-x^2)$ for $x \in [1, \infty)$. Note that $A := \lim_{x \to \infty} L(x)/\log x = b$ and $B := \lim_{x \to \infty} (L(x) - b \log x) = 0$. Let $q = q(b)$ denote the extinction probability, i.e. the smallest fixed point of $f$ in the interval $[0, 1]$. For all $\varepsilon \in (q, 1)$,

$$\int_{\varepsilon}^{1} \frac{1}{s - f(s)} \, ds = \int_{(1-\varepsilon)^{-1}}^{\infty} \frac{1}{x(L(x) - 1)} \, dx = \infty,$$

since $L(x) \sim b \log x$ as $x \to \infty$. By the explosion criterion the associated branching process $Z = (Z_t)_{t \geq 0}$ does not explode. The functions $\alpha(.)$ and $\beta(.)$ are obtained as follows. By Lemma 2 $\alpha(t) = e^{-\alpha At} = e^{-ub t}$, $t \geq 0$. Furthermore,

$$\beta(t) = \exp \left( \frac{B}{A} (1 - \alpha(t)) \right) = \exp \left( \frac{e^{-b t} - 1}{b} \right), \quad t \geq 0.$$
By Theorem 3 as $n \to \infty$, the scaled process $X^{(n)} := (Z^{(n)}_t/n^{c+\epsilon})_{t \geq 0}$ converges in $D_E[0,\infty)$ to a limiting process $X = (X_t)_{t \geq 0}$ such that $X_t$ has Laplace transform $\lambda \mapsto \exp(-\beta(t)\lambda^{c+\epsilon})$, $\lambda \geq 0$, and the semigroup $(T_t)_{t \geq 0}$ of $X$ satisfies $T_t g(x) = \mathbb{E}(g(x + X_t))$, $x, t \geq 0, g \in B(E)$.

The previous three examples are summarized in the following table.

<table>
<thead>
<tr>
<th>Example 3</th>
<th>Example 4</th>
<th>Example 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters</td>
<td>$b &gt; 0$, $c \geq 0$, $b + c \leq 1$</td>
<td>$0 &lt; b &lt; \infty$</td>
</tr>
<tr>
<td>$\log f(s)$</td>
<td>$s + (1-s)\log(1-s)$</td>
<td>$s + (1-s)(c + b \log(1-s))$</td>
</tr>
<tr>
<td>$L(x)$</td>
<td>$1 + \log x$</td>
<td>$1 + c + b \log x$</td>
</tr>
<tr>
<td>$\alpha(t)$</td>
<td>$e^{-at}$</td>
<td>$e^{-at}$</td>
</tr>
<tr>
<td>$\beta(t)$</td>
<td>1</td>
<td>$\exp((e^{-abt} - 1)/b)$</td>
</tr>
</tbody>
</table>

Remark. Theorem 3 does not cover the situation when the limit $A := \lim_{x \to \infty} L(x)/\log x$ is either 0 or $\infty$. We leave the analysis of the two boundary cases $A = 0$ and $A = \infty$ for future work, but provide two concrete examples.

Example 6 An example satisfying $A = 0$ (and $\mathbb{E}(\xi) = \infty$) is obtained as follows. Define $L(1) := 1$, $L(x) := 1 + \log x - \log(1 - 1/x)$ for $x > 1$ and $f(s) := 1 - (1-s)L((1-s)^{-1})$ for $s \in (0,1)$. Clearly, $L(x) \sim \log x$ as $x \to \infty$. Hence, $A := \lim_{x \to \infty} L(x)/\log x = 0$. In the following it is clarified that $f$ is a pgf. It is not hard to check that the function $g : [0,1) \to \mathbb{R}$, defined via $g(0) := 0$ and $g(s) := \log(-\log(1-s)) - \log s$ for $s \in (0,1)$, has the Taylor expansion $g(s) = \sum_{n \geq 1} g_n s^n$ with coefficients $g_n := (n!n)^{-1} \int_0^1 [x]_n \, dx$, $n \in \mathbb{N}$, where $[x]_n := x(x+1) \cdots (x+n-1)$, i.e. $g_1 = 1/2, g_2 = 5/24, g_3 = 1/8$, and so on. Thus, $f(s) = 1 - (1-s)(1 + g(s)) = s - (1-s)g(s)$ has the Taylor expansion $f(s) = \sum_{n \geq 1} p_n s^n$ with coefficients $p_1 = 1 - g_1 = 1/2$ and

$$p_n = g_{n-1} - g_n = \frac{n-1}{(n-1)(n-2)!} \int_0^{1} [x]_{n-1} \, dx - \frac{1}{n!} \int_0^{1} [x]_{n} \, dx$$

$$= \frac{1}{n!} \int_0^{1} [x]_{n-1} \left( \frac{1-x}{n} + \frac{1}{n-1} \right) \, dx, \quad n \in \mathbb{N} \setminus \{1\}.$$ 

In particular, $p_n > 0$ for all $n \in \mathbb{N}$. Thus, $f$ is the pgf of some (offspring) random variable $\xi$ taking values in $\mathbb{N}$. Note that $\mathbb{E}(\xi) = \infty$, since $\lim_{x \to \infty} L(x) = \infty$. From $L(x) \sim \log x$ as $x \to \infty$ it follows that the associated continuous-time branching process $Z = (Z_t)_{t \geq 0}$ does not explode.

The asymptotics of $p_n$ as $n \to \infty$ is obtained as follows. It is easily seen that $f''(s) \sim (1-s)^{-1}\ell((1-s)^{-1})$ as $s \to 1$, where $\ell(u) := 1/\log u$. Moreover, the sequence

$$a_n := [s^n]f''(s) = (n+1)(n+2)p_{n+2} = \frac{1}{n!} \int_0^{1} [x]_{n+1} \left( \frac{1-x}{n+2} + \frac{1}{n+1} \right) \, dx, \quad n \in \mathbb{N}_0,$$

is strictly decreasing, since, by straightforward calculations,

$$a_{n-1} - a_n = \frac{1}{n!(n+2)} \int_0^{1} [x]_{n}(1-x)(2-x) \, dx > 0, \quad n \in \mathbb{N}.$$

From Karamata’s Tauberian theorem for power series (apply, for example, Bingham, Goldie and Teugels [4] p. 40, Corollary 1.7.3) with $A := f''$, $c := 1$ and $\rho := 1$ it follows that $a_n = [s^n]f''(s) \sim \ell(n) = 1/\log n$ as $n \to \infty$. Thus, $p_n \sim 1/(n^2 \log n)$ as $n \to \infty$.

Example 7 A fruitful example satisfying $A = \infty$ is the following. Define $L(x) := 1 + (1 + \log x) \log(1 + \log x)$ for $x \geq 1$ and $f(s) := 1 - (1-s)L((1-s)^{-1})$, i.e. $f(s) = s - (1-s)(1 - \log(1-s)) \log(1 - \log(1-s))$, $s \in [0,1]$. Clearly, $f(1) = 1 = f(1)$ and $f'(s) = (-\log(1-s)) \log(1 - \log(1-s))$, $s \in [0,1)$. By Lemma 8 provided in the appendix, $f'$ is absolutely monotone and $f(0) = f'(0) = f''(0) = 0$, which implies
that \( f \) is the pgf of some (offspring) random variable \( \xi \) taking values in \( \{3, 4, \ldots\} \). Note that \( A = \infty \) implies \( \lim_{x \to \infty} L(x) = \infty \), which is equivalent to \( \mathbb{E}(\xi) = \infty \). Nevertheless, the associated continuous-time branching process \( Z = (Z_t)_{t \geq 0} \) does not explode. The pgf \( F(., t) \) of \( Z_t \) is even explicitly known. More precisely, solving the Kolmogorov backward equation

\[
\begin{aligned}
\dot{a}_t &= \int_{F(s, t)}^1 \frac{1}{u - f(u)} \, du = \int_{F(s, t)}^1 \frac{1}{(1 - u)(1 - \log(1 - u)) \log(1 - \log(1 - u))} \, du \\
&= \left[ \log(\log(1 - \log(1 - u))) \right]_{F(s, t)}^{1} = \log \frac{\log(1 - \log(1 - s))}{\log(1 - \log(1 - F(s, t)))}
\end{aligned}
\]

yields the solution \( F(s, t) = 1 - \exp(1 - (1 - \log(1 - s))^{e^{-\alpha t}}) \), \( s \in [0, 1] \), \( t \geq 0 \). In particular, for each \( \alpha \in (0, 1) \) the map \( s \mapsto 1 - \exp(1 - (1 - \log(1 - s))^\alpha) \), \( s \in [0, 1] \), is a pgf, which does not seem to be straightforward to verify directly.

### 2.4 The explosive case

We briefly comment on the situation when the branching process may explode in finite time. Note that explosion implies that \( A := \lim_{x \to \infty} L(x)/\log x = \infty \). Thus, Theorem 5 is not applicable. We have \( F(1, t) < 1 \) for all \( t > 0 \). For \( t \geq 0 \) let \( G(., t) \) denote the pgf of \( Z_t \) conditioned on \( Z_t < \infty \), i.e.

\[
G(s, t) := \frac{F(s, t)}{F(1, t)}, \quad s \in [0, 1], t \geq 0,
\]

In this situation a convergence result in the spirit of the previous theorems, but with \( F \) replaced by \( G \), is obtained as follows. For \( t > 0 \) we have \( \mathbb{E}(Z_t \mid Z_t < \infty) = G'(1, t) = F'(1, t)/F(1, t) = \infty \). Thus, it is natural to assume that \( 1 - G(s, t) = (1 - s)^{\alpha(t)} L_t((1 - s)^{-1}) \) for some \( \alpha(t) \in (0, 1] \) and some slowly varying function \( L_t \). Assume now furthermore that the limits

\[
\beta(t) := \lim_{x \to \infty} L_t(x) \in (0, \infty), \quad t \geq 0,
\]

exist. Then \( \alpha(t) < 1 \) for all \( t > 0 \). Now, for \( t \geq 0 \) and \( n \in \mathbb{N} \) choose \( a_n(t) \) such that \( L_t(a_n(t)) \sim (a_n(t))^{\alpha(t)}/(n(\alpha(t))) \) as \( n \to \infty \). Then \( Z_t(n)/a_n(t) \), conditioned on \( Z_t < \infty \), converges to \( X_t \) in distribution as \( n \to \infty \), where \( X_t \) has Laplace transform \( \lambda \mapsto \exp(-\beta(t) \lambda^{\alpha(t)}) \), \( \lambda \geq 0 \). Example 8 below, going back at least to Sewastjanow [43, Chapter 1, Section 8, Example 6], turns out to be in that regime.

**Example 8** Suppose that \( \xi \) is Sibuya distributed with parameter \( \alpha \in (0, 1) \), i.e. \( f(s) = 1 - (1 - s)^\alpha, \ s \in [0, 1] \). Note that \( f \) has the Taylor expansion \( f(s) = \sum_{n \geq 1} p_n s^n \) with coefficients

\[
p_n := \binom{\alpha}{n}(-1)^{n-1} = \frac{\alpha}{\Gamma(1 - \alpha)} \frac{\Gamma(n - \alpha)}{\Gamma(1 - \alpha)}, \quad n \in \mathbb{N}.
\]

In particular, \( p_n \sim (\alpha/\Gamma(1 - \alpha))n^{-\alpha-1} \) as \( n \to \infty \). Moreover, \( f(s) = 1 - (1 - s)^R((1 - s)^{-1}) \), where \( R(x) := x^{1-\alpha} \) is regularly varying of index \( 1 - \alpha \). The backward equation

\[
\begin{aligned}
\dot{a}_t &= \int_{F(s, t)}^1 \frac{1}{f(x) - x} \, dx = \int_{F(s, t)}^1 \frac{1}{1 - x - (1 - x)^\alpha} \, dx \\
&= \left[ \frac{-\log(1 - (1 - x)^{1-\alpha})}{1 - \alpha} \right]_{F(s, t)}^1 = \frac{1}{1 - \alpha} \log \frac{1 - (1 - s)^{1-\alpha}}{1 - (F(s, t))^{1-\alpha}}, \quad t \geq 0,
\end{aligned}
\]

yields the explicit solution (see [43, p. 26, Eq. (19)])

\[
F(s, t) = 1 - \left( 1 - e^{-(1-\alpha)at}(1 - (1 - s)^{1-\alpha}) \right)^{\frac{1}{1-\alpha}}, \quad s \in [0, 1], t \geq 0.
\]

We have \( \mathbb{P}(Z_t = \infty) = 1 - F(1, t) = (1 - e^{-(1-\alpha)at})^{\frac{1}{1-\alpha}} \) for \( t \geq 0 \), so \( 0 < \mathbb{P}(Z_t = \infty) < 1 \) for all \( t > 0 \). The time \( T := \inf\{t > 0 : Z_t = \infty\} \) of explosion satisfies \( \mathbb{P}(T < \infty) = \lim_{t \to \infty} \mathbb{P}(Z_t = \infty) = 1 \), so \( Z \) explodes in finite time almost surely. Note that \( T \) has mean

\[
\mathbb{E}(T) = \int_0^\infty \mathbb{P}(T > t) \, dt = \int_0^\infty \mathbb{P}(Z_t < \infty) \, dt = \int_0^\infty (1 - (1 - e^{-(1-\alpha)at})^{\frac{1}{1-\alpha}}) \, dt.
\]
The substitution \( x = 1 - e^{-(1-\alpha)t} \) yields
\[
\mathbb{E}(T) = \frac{1}{a(1-\alpha)} \int_0^1 \frac{1-x^{1-\alpha}}{1-x} \, dx = \frac{1}{a(1-\alpha)} \left( \Psi\left(\frac{2-\alpha}{1-\alpha}\right) + \gamma \right),
\]
where \( \Psi = \Gamma'/\Gamma \) denotes the logarithmic derivative of the gamma function and \( \gamma \) is the Euler–Mascheroni constant.

Let \( t > 0 \) in the following. Expansion of (18) yields
\[
F(s, t) = F(1, t) - \frac{1}{1-\alpha}(1-e^{-(1-\alpha)t}) \int_0^\infty e^{-(1-\alpha)t} (1-s)^{1-\alpha} + O((1-s)^{2(1-\alpha)}), \quad s \to 1. \tag{19}
\]

Rewriting (19) in the form
\[
1 - G(s, t) = 1 - \frac{F(s, t)}{F(1, t)} = \frac{(1-e^{-(1-\alpha)t})^{1-\alpha} e^{-(1-\alpha)t}}{(1-\alpha)(1-(1-e^{-(1-\alpha)t})^{1-\alpha})(1-s)^{1-\alpha} + O((1-s)^{2(1-\alpha)}), \quad s \to 1,
\]
yields \( \alpha(t) = 1-\alpha \) for all \( t > 0 \) and
\[
\beta(t) := \lim_{x \to \infty} L_t(x) = \frac{(1-e^{-(1-\alpha)t})^{a(t)} e^{-(1-\alpha)t}}{(1-\alpha)(1-(1-e^{-(1-\alpha)t})^{1-\alpha})}, \quad t > 0.
\]

Thus, the sequence \( a_n(t) := (n\alpha(t))^{1/\alpha(t)} \) satisfies \( L_t(a_n(t)) \sim (a_n(t))^{\alpha(t)}/(n\alpha(t)) \) as \( n \to \infty \)
and it follows that \( X_t^{(n)} := Z_t^{(n)}/a_n(t) \), conditioned on \( Z_t < \infty \), converges to \( X_t \) in distribution as \( n \to \infty \), where \( X_t \) has Laplace transform \( \lambda \mapsto \exp(-\beta(t)\lambda^{\alpha(t)}) \), \( \lambda \geq 0 \).

We leave the study of further examples of branching processes with explosion similar to those of Example 8 to the interested reader. One may for instance study the pgf \( f(s) := \frac{s}{2} \arcsin s \), \( s \in [0, 1] \), occurring in Pakes \[24\] p. 276, Example 4.5. A further example is the offspring distribution \( p_k = \frac{\sqrt{\pi}}{k(3/2)!} \Gamma(k+3/2), k \in \mathbb{N} \), in which case the offspring pgf has the form \( f(s) = 1 - \sqrt{(1-s)/s} \arcsin \sqrt{s} \).

Let us finally discuss the situation when
\[
1 - G(s, t) = (1-s)L_t((1-s)^{1-\alpha}), \quad t \geq 0, \tag{20}
\]
for some slowly varying function \( L_t \). Note that (see, for example, Bingham and Doney [8], Theorem A) \[20\] is equivalent to \( \sum_{k=0}^{\infty} \mathbb{P}(Z_t > k | Z_t < \infty) \sim L_t(n) \) as \( n \to \infty \), which is Condition (ii) in Rogozin’s relative stability theorem (see, for example, Bingham, Goldie and Teugels [9], Theorem 8.8.1]). Let \( (a_n(t))_{n \in \mathbb{N}} \) be a sequence such that \( L_t(a_n(t)) \sim a_n(t)/n \) as \( n \to \infty \). Then, by Theorem 8.8.1 of [9], \( Z_t^{(n)}/a_n(t)|_{Z_t < \infty} \to 1 \) in probability as \( n \to \infty \). Thus, in this situation we cannot have a non-degenerate limit. The following example fits into this regime. In this example the limits
\[
\gamma(t) := \lim_{x \to \infty} \frac{L_t(x)}{\log x} \in (0, \infty), \quad t \geq 0,
\]
exist.

**Example 9** Define \( f(0) := 0, f(1) := 1 and \)
\[
f(s) := 1 + \frac{s}{\log(1-s)}, \quad s \in (0, 1).
\]

It is easily seen that \( f \) has the Taylor expansion \( f(s) = \sum_{n \geq 1} p_n s^n \) with positive coefficients
\[
p_n := (-1)^n-1 \int_0^1 \frac{x^n}{n} \, dx = \frac{1}{n!} \int_0^1 x^{(n-1)} \frac{\Gamma(n-x)}{\Gamma(1-x)} \, dx > 0, \quad n \in \mathbb{N}.
\]

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Thus, $f$ is the pdf of some random variable $\xi$ taking values in $\mathbb{N}$. Note that $p_n = (-1)^{n-1} b_n / n!$ for all $n \in \mathbb{N}$, where $b_n := \int_0^1 x^n \, dx$ denotes the $n$-th Bernoulli number of the second kind (see, e.g., Roman [41, p. 114]). Here $(x)_n := x(x-1) \cdots (x-n+1)$, $n \in \mathbb{N}$, denotes the $n$-th descending factorial of $x \in \mathbb{R}$. From $p_0 = 0$ it follows that the associated continuous-time branching process $Z = (Z_t)_{t \geq 0}$ has extinction probability $q = 0$. Note that $f(s) = 1 - (1 - s) R((1 - s)^{-1})$, where $R(x) := (x-1)/\log x$, $x > 1$, is regularly varying of index 1. For all $\varepsilon \in (q, 1) = (0, 1)$,

$$
\int_\varepsilon^1 \frac{1}{s - f(s)} \, ds = \int_\varepsilon^1 \frac{1}{s - 1 - \log(1 - s)} \, ds = \int_\varepsilon^1 \frac{1}{\log(1 - s)} \, ds
$$

$$
= - \log(\varepsilon + (1 - \varepsilon) \log(1 - \varepsilon)) < \infty,
$$

which shows that $Z$ explodes. It is also known (see, for example, Flajolet and Sedgewick [17, p. 387]) that $p_n \sim 1/(n \log^2 n)$ as $n \to \infty$. Thus, $p_n$ tends slower to 0 than in Example 8. In this sense $Z$ is strongly explosive. The Kolmogorov backward equation is

$$
\begin{align*}
\frac{\partial}{\partial s} F(s, t) + (1 - F(s, t)) \log(1 - F(s, t)) &= e^{-at} (s + (1 - s) \log(1 - s)) =: h(s, t).
\end{align*}
$$

It is straightforward to check that this equation has the solution

$$
F(s, t) = 1 - \exp \left(1 + W \left( \frac{h(s, t) - 1}{e} \right) \right), \quad s \in [0, 1], t \geq 0,
$$

where $W = W_{-1}$ denotes the lower branch of the Lambert $W$ function satisfying $W(h) e^{W(h)} = h$ and being real valued on $[-1/e, 0)$. Note that $\mathbb{P}(Z_t = \infty) = 1 - F(1, t) = \exp(1 + W((-e^{-at} - 1)/e))$ for $t \geq 0$, so $0 < \mathbb{P}(Z_t = \infty) < 1$ for $t > 0$. The time $T := \inf\{t > 0 : Z_t = \infty\}$ of explosion satisfies $\mathbb{P}(T < \infty) = \lim_{t \to \infty} \mathbb{P}(Z_t = \infty) = \exp(1 + W(-1/e)) = \exp(0) = 1$, so $Z$ explodes in finite time almost surely. Note that $T$ has mean

$$
\mathbb{E}(T) = \int_0^\infty \mathbb{P}(Z_t < \infty) \, dt = \int_0^\infty \left(1 - \exp \left(1 + W \left( \frac{e^{-at} - 1}{e} \right) \right) \right) \, dt.
$$

The substitution $x = 1 - e^{-at} \Rightarrow t = -\frac{1}{a} \log(1 - x)$ and $\frac{dt}{dx} = \frac{1}{a(1-x)}$ leads to

$$
\mathbb{E}(T) = \frac{1}{a} \int_0^1 \frac{1 - \exp(1 + W(-x/e))}{1 - x} \, dx.
$$

The function below the integral has a singularity at $x = 1$. From $1 + W(-x/e) \sim \sqrt{2(1 - x)}$ as $x \to 1$ it follows that the function below the integral behaves asymptotically as $\sqrt{2(1 - x)}$ as $x \to 1$, which yields $\mathbb{E}(T) < \infty$. Numerical computations show that $\mathbb{E}(T) \approx 2.45/a$.

Let $G(s, t) := F(s, t)/F(1, t)$ denote the pgf of $Z_t$ conditioned on $Z_t < \infty$. A somewhat tedious but straightforward calculation shows that $1 - G(s, t) = (1-s)L_t((1-s)^{-1})$, where $L_t$ is slowly varying with

$$
\gamma(t) := \lim_{x \to \infty} \frac{L_t(x)}{\log x} = \frac{w}{(w + 1)(1 - (w + 1) e^{at})}
$$

with $w := W((-e^{-a t})/e^{-1})$. For $t \geq 0$ let $(a_n(t))_{n \in \mathbb{N}}$ be a sequence such that $L_t(a_n(t)) \sim a_n(t)/n$ as $n \to \infty$. Then, as explained before, for every $t \geq 0$, conditional on $Z_t < \infty$, $Z_{t(n)}^{(n)}/a_n(t) \to 1$ in probability as $n \to \infty$. A concrete sequence $(a_n(t))_{n \in \mathbb{N}}$ is $a_n(t) := \gamma(t) n \log n$, since, in this case, $L_t(a_n(t)) = L_t(\gamma(t) n \log n) \sim L_t(n \log n) \sim \gamma(t) n \log(n \log n) \sim \gamma(t) n \log n = a_n(t)/n$ as $n \to \infty$. 

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3 Proof of Theorem 1

The proof of Theorem 1 is quite natural and can be summarised as follows. An application of the multivariate central limit theorem yields the convergence of the finite-dimensional distributions. The convergence in $D_\mathbb{R}[0,\infty)$ is then established using a criterion of Aldous [3]. The following proof is relatively short and elegant.

**Proof.** (of Theorem 1) Let us compute for $s, t \geq 0$ the covariance of $Z_s$ and $Z_{s+t}$. For $k \in \mathbb{N}_0$,

$$
\mathbb{E}((Z_s - m(s))(Z_{s+t} - m(s + t)) \mid Z_s = k) = (k - m(s))\mathbb{E}(Z_t^{(k)} - m(s + t))
$$

$$
= (k - m(s))(km(t) - m(s)m(t)) = m(t)(k - m(s))^2.
$$

Thus, $\mathbb{E}((Z_s - m(s))(Z_{s+t} - m(s + t)) \mid Z_s) = m(t)(Z_s - m(s))^2$ almost surely. Taking expectation yields $\text{Cov}(Z_s, Z_{s+t}) = m(t)\text{Var}(Z_s) = m(t)\sigma^2(s)$.

In order to verify the convergence $X^{(n)} \overset{fd}{\to} X$ of the finite-dimensional distributions fix $k \in \mathbb{N}$ and $0 \leq t_1 < \cdots < t_k < \infty$, define the $\mathbb{R}^k$-valued random variable $Y := (Z_{t_1} - m(t_1), \ldots, Z_{t_k} - m(t_k))$ and let $Y_1, Y_2, \ldots$ be independent copies of $Y$. By the branching property, $(X_1^{(n)}, \ldots, X_k^{(n)}) = ((Z_{t_1}^{(n)} - nm(t_1))/\sqrt{n}, \ldots, (Z_{t_k}^{(n)} - nm(t_k))/\sqrt{n})$ has the same distribution as $(Y_1 + \cdots + Y_n)/\sqrt{n}$, which by the multivariate central limit theorem (see, for example, [48, p. 16, Example 2.18]) converges in distribution as $n \to \infty$ to a centered normal distribution $N(0, \Sigma)$ with covariance matrix $\Sigma = (\sigma_{ij})_{1 \leq i,j \leq k}$ having entries $\sigma_{ij} := \mathbb{E}((Z_{t_i} - m(t_i))(Z_{t_j} - m(t_j))) = \text{Cov}(Z_{t_i}, Z_{t_j}) = m(t_i - t_j)|\sigma^2(t_i - t_j)$. Thus the convergence $X^{(n)} \overset{fd}{\to} X$ holds.

The convergence $X^{(n)} \to X$ in $D_\mathbb{R}[0,\infty)$ is achieved as follows. Define the processes $M^{(n)} := (M_t^{(n)})_{t \geq 0}$, $n \in \mathbb{N}$, and $M := (M_t)_{t \geq 0}$ via

$$
M_t^{(n)} := \frac{X_t^{(n)}}{m(t)} = \sqrt{n}\left(\frac{Z_t^{(n)}}{nm(t)} - 1\right) \text{ and } M_t := \frac{X_t}{m(t)}, \quad n \in \mathbb{N}, t \geq 0.
$$

Then, $M, M^{(1)}, M^{(2)}, \ldots$ are martingales and $M$ is continuous, since the Gaussian process $X$ is continuous and $m(.)$ is continuous. Since $\mathbb{E}((M_t^{(n)})^2) = \text{Var}(M_t^{(n)}) = \text{Var}(Z_t^{(n)})/(nm(t))^2 = \sigma^2(t)/(m(t))^2 < \infty$ does not depend on $n \in \mathbb{N}$, we conclude that, for each $t \geq 0$, the family $\{M_t^{(n)} : n \in \mathbb{N}\}$ is uniformly integrable. The convergence $M^{(n)} \to M$ in $D_\mathbb{R}[0,\infty)$ therefore follows from Aldous’ criterion [3, Proposition 1.2]. Since the map $t \mapsto m(t)$ is continuous and deterministic it follows by multiplication with $m(t)$ that $X^{(n)} \to X$ in $D_\mathbb{R}[0,\infty)$.

4 Proofs concerning Theorem 2

This section contains the proofs of Lemma 1 and Theorem 2.

**Proof.** (of Lemma 1) The proof distinguishes the critical and non-critical case. Both cases are handled with different techniques. The representation in the critical case (for age-dependent branching processes) follows via an equivalence for the extinction probability from a combination of the results of Slack [45, Theorem 1] and Vatutin [49, Theorem 1]. The following more elementary proof (see Case 1) is based on the backward equation and does not use extinction probabilities.

**Case 1.** ($\lambda = 0$) Let $t \geq 0$. In the critical case Kolmogorov’s backward equation is

$$
at = \int_s^{F(s,t)} \frac{1}{f(x) - x} \, dx = \int_s^{F(s,t)} \frac{1}{(1-x)^{\alpha}L((1-x)^{-1})} \, dx, \quad s \in [0, 1].
$$

Since the map $x \mapsto f(x) - x$ is non-negative and non-increasing on $[0, 1]$ it follows that

$$
\frac{F(s,t) - s}{(1-s)^{\alpha}L((1-s)^{-1})} \leq at \leq \frac{F(s,t) - s}{(1 - F(s,t))^{\alpha}L((1-F(s,t))^{-1})}
$$

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and, hence,
\[
\limsup_{s \to 1} \frac{F(s, t) - s}{(1-s)^{\alpha}L((1-s)^{-1})} \leq \alpha t \leq \liminf_{s \to 1} \frac{F(s, t) - s}{(1-s)^{\alpha}L((1-s)^{-1})} = \liminf_{s \to 1} \frac{F(s, t) - s}{(1-s)^{\alpha}L((1-s)^{-1})},
\]
where the last equality holds since \(1 - F(s, t) \sim 1 - s\) as \(s \to 1\). Thus, \(\lim_{s \to 1}(F(s, t) - s)/(1-s)^{\alpha}L((1-s)^{-1}) = \alpha t\).

**Case 2.** (\(\lambda \neq 0\)) Fix \(t \geq 0\). Set \(h_1(s) := (1-s)m(t) - (1-F(s, t))\) and \(h_2(s) := (1-s)^{\alpha}L((1-s)^{-1})\) for \(s \in [0, 1]\). We have to verify that \(\lim_{s \to 1} h_1(s)/h_2(s) = c(t)\), where \(c(t)\) is defined via \([5]\).

By the Kolmogorov forward and backward equations, \(h_1'(s) = -m(t) + \frac{\partial}{\partial s}F(s, t) = -m(t) + (f(F(s, t)) - F(s, t))/(f(s) - s)\). Moreover, \(h_2'(s) = (1-s)^{\alpha-1}L((1-s)^{-1})/(1-s)^{-1}/L((1-s)^{-1}) - \alpha\). From Assumption \([3]\), the asymptotics \(1 - F(s, t) \sim m(t)(1-s)\) as \(s \to 1\) and \((m(t))^{\alpha} = m(\alpha)\) it follows that

\[
m(\alpha t) - m(t) = \lim_{s \to 1} \left( \frac{(1-s)m(t) - (1-F(s, t))}{(1-s)^{\alpha}L((1-s)^{-1})} - m(t) \right) \frac{(1-s)m(1-s) - 1-f(s)}{(1-s)^{\alpha}L((1-s)^{-1})}
\]

\[
= \lim_{s \to 1} \left( (1-s)m(t) - (1-F(s, t)) \right) \frac{1}{(1-s)^{\alpha}L((1-s)^{-1})} + \frac{m(t)(1-s) - 1-f(s)}{(1-s)^{\alpha}L((1-s)^{-1})}
\]

\[
= \lim_{s \to 1} \left( (1-s)m(t) - (1-F(s, t)) \right) \frac{1}{(1-s)^{\alpha}L((1-s)^{-1})} + \frac{m(t)(1-s) - 1-f(s)}{(1-s)^{\alpha}L((1-s)^{-1})}
\]

\[
= \lim_{s \to 1} \left( (1-s)m(t) - (1-F(s, t)) \right) \frac{1}{(1-s)^{\alpha}L((1-s)^{-1})} + \frac{m(t)(1-s) - 1-f(s)}{(1-s)^{\alpha}L((1-s)^{-1})}
\]

\[
= \lim_{s \to 1} \left( (1-s)m(t) - (1-F(s, t)) \right) \frac{1}{(1-s)^{\alpha}L((1-s)^{-1})} + \frac{m(t)(1-s) - 1-f(s)}{(1-s)^{\alpha}L((1-s)^{-1})}
\]

\[
= (m-1) \lim_{s \to 1} \left( \alpha \frac{h_1'(s)}{h_2'(s)} + R(s) \right) - \frac{h_1(s)}{h_2(s)}.
\]

Using

\[
\frac{(1-s)(1-s)}{f(s) - s} = \frac{1-m}{1-m + (1-s)^{\alpha-1}L((1-s)^{-1})}
\]

we see that \(R(s)\) is given by

\[
R(s) = -\alpha(1-s)^{\alpha-1}L((1-s)^{-1}) - \frac{1-m}{1-m + (1-s)^{\alpha-1}L((1-s)^{-1})}
\]

\[
- \frac{1-m}{1-m + (1-s)^{\alpha-1}L((1-s)^{-1})} \left( \frac{L'((1-s)^{-1})/(1-s)^{-1}}{L((1-s)^{-1})} - \alpha \right)
\]

\[
= \alpha(1-s)^{\alpha-1}L((1-s)^{-1}) \left( 1 - \frac{1-m}{1-m + (1-s)^{\alpha-1}L((1-s)^{-1})} \right) - \frac{L'((1-s)^{-1})/(1-s)^{-1}}{\alpha L((1-s)^{-1})}
\]

In order to see that \(\lim_{x \to \infty} xL'(x)/L(x) = 0\) we proceed as follows. Define \(U : [1, \infty) \to (0, \infty)\) via \(U(x) := m - x(1 - f(1 - 1/x)) = x^{-\alpha}L(x)\) for \(x \geq 1\), where the last equality holds by \([3]\). Note that \(U(x) = \int_{x}^{\infty} u(y) dy\), where \(u : [1, \infty) \to (0, \infty)\) is defined via \(u(x) := -U'(x) = 1 - f(1 - 1/x) - f'(1 - 1/x)/x\). The function \(u\) is non-increasing, since \(u'(x) = -f''(1 - 1/x)/x^3 \leq 0\) by the convexity of \(f\). By a variant of the monotone density theorem (see, for example, Bingham, Goldie and Teugels \([9]\), p. 39, Theorem 1.7.2) and the comments thereafter for integrals of the form \(U(x) = \int_{e}^{\infty} u(y) dy\) it follows that \(u(x) \sim (\alpha-1)x^{-\alpha}L(x)\) as \(x \to \infty\). Thus, \(\lim_{x \to \infty} xU'(x)/U(x) = 1 - \alpha\). Noting that \(xU'(x)/U(x) = 1 - \alpha + xL'(x)/L(x)\) we conclude that \(\lim_{x \to \infty} xL'(x)/L(x) = 0\). Applying this relation with \(x := (1-s)^{-1}\) yields

\[
\lim_{s \to 1} \frac{R(s)}{h_2'(s)} = \lim_{s \to 1} \left( \alpha \frac{1-m}{1-m + (1-s)^{\alpha-1}L((1-s)^{-1})} \right) - \frac{L'((1-s)^{-1})/(1-s)^{-1}}{\alpha L((1-s)^{-1})} = 0.
\]

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The three quantities \( h_1(s), h_2(s) \) and \( (m(\alpha t) - m(t))/(m - 1) \) are non-negative, so from (21) necessarily \( \liminf_{s \to 1} h_1'(s)/(h_2'(s) + R(s)) \geq 0 \), leading to the boundary \( h_1'(s)/(h_2'(s) + R(s)) \geq (1 - \delta)h_1'(s)/h_2'(s) \) for any \( 0 < \delta < (\alpha - 1)/\alpha \) and \( s \) sufficiently large. Then
\[
\frac{m(\alpha t) - m(t)}{m - 1} \geq \limsup_{s \to 1} \left( \alpha(1 - \delta) \frac{h_1'(s)}{h_2'(s)} - \frac{h_1(s)}{h_2(s)} \right),
\]
and the second part of Lemma 5 provides
\[
\limsup_{s \to 1} \frac{h_1(s)}{h_2(s)} \leq \frac{m(\alpha t) - m(t)}{m - 1}.
\]
Now (21), (22) and (23) yield
\[
\frac{m(\alpha t) - m(t)}{m - 1} = \lim_{s \to 1} \left( \frac{h_1'(s)}{h_2'(s)} - \frac{h_1(s)}{h_2(s)} \right) \left( \frac{h_1(s)}{h_2(s)} \frac{h_2(s)}{h_2'(s)} + R(s) - \frac{h_1(s)}{h_2(s)} \frac{R(s)}{h_2'(s)} \right) = \lim_{s \to 1} \left( \frac{h_1'(s)}{h_2'(s)} - \frac{h_1(s)}{h_2(s)} \right).
\]
The claim follows again from Lemma 5 in the appendix. Note that Lemma 5 is applicable in both cases due to Lemma 4.

**Proof.** (of Theorem 2) The proof is divided into four parts. The first part establishes the convergence of the one-dimensional distributions. The second and third part give two auxiliary results, one is about the normalizing sequence \((a_n)_{n \in \mathbb{N}}\) and the other is a kind of upper bound for the process, used in the final part to conclude the convergence in \( D_\mathbb{R}[0, \infty) \).

**Part 1.** (Convergence of the one-dimensional distributions) Fix \( t \in [0, \infty) \), define \( Y := Z_t \) for convenience and let \( Y_1, Y_2, \ldots \) be independent copies of \( Y \). By Lemma 1, Eq. (4) holds. Assume first that \( \alpha \in (1, 2) \). Then, by Bingham and Doney [8, Theorem A], applied with \( n = 1 \), Eq. (4) is equivalent to \( \mathbb{P}(Y > x) \sim c(t)(-\Gamma(1 - \alpha)^{-1}L(x)x^{-\alpha}, x \to \infty) \). In particular, the map \( x \mapsto \mathbb{P}(Y > x) \) is regularly varying (at infinity) with index \( -\alpha \). By Theorem 1 (ii) \( \Rightarrow \) (i) of Geluk and de Haan [18] (note that \( p = 1 \) since \( Y \geq 0 \)) it follows that the distribution function of \( Y \) is in the domain of attraction of an \( \alpha \)-stable distribution. The results on p. 174 in [18] on the choice of the normalizing sequences \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) furthermore show that, if we choose \((a_n)_{n \in \mathbb{N}}\) such that \( L(a_n) \sim a_n^\alpha/(\alpha n) \) as \( n \to \infty \) and \( b_n := n\mathbb{E}(Y)/a_n = nm(t)/a_n \), then \( (Z^n - nm(t))/a_n \overset{d}{=} (Y_1 + \cdots + Y_n)/a_n - b_n \to X_t \) in distribution as \( n \to \infty \), where \( X_t \) is \( \alpha \)-stable with characteristic function \( u \mapsto \exp(c(t)(-iu)^\alpha)/u \), \( u \in \mathbb{R} \). Thus, the convergence of the one-dimensional distributions holds.

The case \( \alpha = 2 \) is handled similarly by noting that [4] is then equivalent (see [8]) to \( \mathbb{E}(1_{Y \leq x}Y^2) \sim 2c(t)L(x) \) as \( x \to \infty \) such that we can apply Theorem 2 of [18].

**Part 2.** (Asymptotic relation for \((a_n)_{n \in \mathbb{N}}\)) Let \((\varepsilon_n)_{n \in \mathbb{N}}\) be an arbitrary sequence of positive real numbers converging to zero as \( n \to \infty \). For \( n \in \mathbb{N} \) and \( T > 0 \) define \( S_{n,T} := [-\varepsilon_n n/a_n, \varepsilon_n n/a_n] \times [0, T] \), where \((a_n)_{n \in \mathbb{N}}\) is the normalizing sequence satisfying \( a_n/(L(a_n))^{1/\alpha} \sim (\alpha n)^{1/\alpha} \) as \( n \to \infty \). Bojanić and Seneta [11, p. 308] provide the existence of another slowly varying function \( L^* \) such that \( a_n \sim (\alpha n)^{1/\alpha} L^*(n^{1/\alpha}) \) as \( n \to \infty \). Set \( h(n) := (\alpha n)^{1/\alpha} L^*(n^{1/\alpha})/a_n \) for \( n \in \mathbb{N} \) and \( h(r) := h([r]) \) for \( r \in \mathbb{R}, r \geq 1 \). Then the asymptotic relation simply means \( \lim_{r \to \infty} h(r) = 1 \). From
\[
\lim_{n \to \infty} \inf_{(x,s) \in S_{n,T}} (nm(s) + xa_n) = \infty
\]
and it follows that \( \sup_{(x,s) \in S_{n,T}} |h(nm(s) + xa_n) - 1| \to 0 \) as \( n \) tends to infinity. Furthermore, \( \lim_{n \to \infty} \sup_{(x,s) \in S_{n,T}} |x\varepsilon_n n/a_n| \leq \lim_{n \to \infty} \varepsilon_n = 0 \) implies \( \lim_{n \to \infty} \sup_{(x,s) \in S_{n,T}} |(m(s) + xa_n/n)^{1/\alpha} - (m(s))^{1/\alpha}| = 0 \) as well as, using the uniform convergence theorem for slowly varying functions (see, for example, Bingham, Goldie and Teugels [9, Theorem 1.2.1] or Bojanić and Seneta [11])
\[
\lim_{n \to \infty} \sup_{(x,s) \in S_{n,T}} \left| \frac{L^*(n^{1/\alpha}(m(s) + xa_n/n)^{1/\alpha})}{L^*(n^{1/\alpha})} - 1 \right| = 0.
\]
Having bounded limits, the listed uniformly convergent sequences are uniformly bounded and thus their product converges again uniformly, yielding

\[
\lim_{n \to \infty} \sup_{(x,s) \in S_n,T} \left| \frac{a_n m(s) + xa_n}{n} - (m(s))^{1/\alpha} \right| = \lim_{n \to \infty} \sup_{(x,s) \in S_n,T} \left| \frac{h(n)}{h(n m(s) + xa_n)} \frac{L^*((nm(s) + xa_n)^{1/\alpha})}{L^*(n^{1/\alpha})} \left( m(s) + \frac{x a_n}{n} \right)^{1/\alpha} - (m(s))^{1/\alpha} \right| = 0. \tag{25}
\]

**Part 3.** (Kind of upper bound for \(X_t^{(n)}\)) In this part it is shown that for each \(T > 0\) there exists a sequence \((\varepsilon_n)_{n \in \mathbb{N}}\) of positive real numbers with \(\lim_{n \to \infty} \varepsilon_n = 0\) such that

\[
\lim_{n \to \infty} \mathbb{P} \left( \sup_{t \in [0,T]} |X_t^{(n)}| \geq \frac{\varepsilon_n n}{a_n} \right) = 0. \tag{26}
\]

Let \(\delta := 0\) if \(m < 1\) and \(\delta := T\) if \(m \geq 1\). Then, for any sequence \((\varepsilon_n)_{n \in \mathbb{N}}\) of positive real numbers,

\[
\mathbb{P} \left( \sup_{t \in [0,T]} |X_t^{(n)}| \geq \frac{\varepsilon_n n}{m(t)} \right) \leq \mathbb{P} \left( \sup_{t \in [0,T]} \left| a_n X_t^{(n)} \right| \geq \frac{\varepsilon_n n}{m(\delta)} \right).
\]

Applying Doob’s submartingale inequality to the martingale \((a_n X_t^{(n)}/m(t))_{t \geq 0} = (Z_t^{(n)}/m(t) - n)_{t \geq 0}\) yields

\[
\mathbb{P} \left( \sup_{t \in [0,T]} \left| a_n X_t^{(n)} \right| / m(t) \geq \frac{\varepsilon_n n}{m(\delta)} \right) \leq \frac{m(\delta)}{\varepsilon_n n} \mathbb{E} \left( \left| Z_T^{(n)} / m(T) - n \right| \right) = \frac{m(\delta)}{m(T)} \frac{1}{\varepsilon_n} \mathbb{E} \left( \left| Z_T^{(n)} / n - m(T) \right| \right).
\]

By the law of large numbers the latter expectation converges to 0 as \(n \to \infty\). Thus the sequence \((\varepsilon_n)_{n \in \mathbb{N}}\) can be chosen such that \(\lim_{n \to \infty} \varepsilon_n = 0\) and such that the right-hand side still converges to 0, which implies that (26) holds for the particular sequence \((\varepsilon_n)_{n \in \mathbb{N}}\).

**Part 4.** (Convergence in \(D_S([0,\infty))\)) In general, the processes \(X^{(n)}\) and \(X\) are time-inhomogeneous. Let \(Y^{(n)} := (X_t^{(n)},t)_{t \geq 0}\) and \(Y := (X_t,t)_{t \geq 0}\) denote the space-time processes of \(X^{(n)}\) and \(X\) respectively. According to Revuz and Yor [40] p. 85, Exercise (1.10) the processes \(Y^{(n)}\) and \(Y\) are time-homogeneous Markov processes with state space \(S := \mathbb{R} \times [0,\infty)\). Recall that \(S_n,T = [-\varepsilon_n n/a_n, \varepsilon_n n/a_n] \times [0,T]\), where \((\varepsilon_n)_{n \in \mathbb{N}}\) is the sequence defined in Part 2. In terms of \(Y^{(n)}, (26)\) is simply

\[
\lim_{n \to \infty} \mathbb{P} \left( Y_t^{(n)} \in S_n,T, 0 \leq t \leq T \right) = 1. \tag{27}
\]

Corollary 8.7 on p. 232 of Ethier and Kurtz [14] states that (27) jointly with the uniform convergence of the semigroups on the restricted area \(S_n,T\) implies the convergence of \(Y^{(n)}\) to \(Y\) in \(D_S([0,\infty))\), hence the desired convergence of \(X^{(n)}\) to \(X\) in \(D_R([0,\infty))\). Thus it remains to show that for each \(f \in \hat{C}(S)\), the space of real valued continuous functions on \(S\) vanishing at infinity, and \(t \in [0,T]\)

\[
\lim_{n \to \infty} \sup_{(x,s) \in S_n,T} \left| \hat{T}_t^{(n)} f(x,s) - \hat{T}_t f(x,s) \right| = 0, \tag{28}
\]

where \((\hat{T}_t^{(n)})_{t \geq 0}\) and \((\hat{T}_t)_{t \geq 0}\) denote the semigroups of \(Y^{(n)}\) and \(Y\) respectively, that is \(\hat{T}_t^{(n)} f(x,s) = \mathbb{E}(f(X_{s+t}^{(n)}, s+t) | X_s^{(n)} = x)\) and \(\hat{T}_t f(x,s) = \mathbb{E}(f(X_{s+t}, s+t) | X_s = x)\) for all \(f \in \hat{C}(S)\) and \((x,s) \in S\). By Lemma 7 the space of all maps of the form \((x,s) \mapsto \sum_{l=1}^{\infty} g_l(x) h_l(s)\) with \(l \in \mathbb{N}\), \(g_l \in \hat{C}((\mathbb{R})\) and \(h_l \in \hat{C}([0,\infty))\) is dense in \(\hat{C}(S)\). Hence it suffices to show (28) for \(f = gh\) with \(g \in \hat{C}((\mathbb{R})\) and \(h \in \hat{C}([0,\infty))\), in which case

\[
\hat{T}_t^{(n)} f(x,s) = h(s+t) \mathbb{E}(g(X_{s+t}^{(n)}^{(n)} | X_s^{(n)} = x) = \hat{T}_t f(x,s) = h(s+t) \mathbb{E}\left( g \left( \frac{a_n}{n} X_t^{(k)} + xa_n \right) \right), \quad (x,s) \in S,
\]
where \( k := k(n, s, x) := nm(s) + xa_n \), and

\[
\bar{T}_t f(x, s) = h(s + t) \mathbb{E}(g(X_{s+t}), X_s = x) = h(s + t) \mathbb{E}(g(m(s)^{1/\alpha}X_t + xm(t))), \quad (x, s) \in S.
\]

Let \( \varepsilon > 0. \) Choose \( C > 0 \) such that \( \sup_{n \in \mathbb{N}} \mathbb{P}(|X_t^{(n)}| > C) < \varepsilon. \) Splitting the mean along the event \( A_k := \{|X_t^{(k)}| \leq C\} \) yields

\[
\sup_{(x, s) \in S, t} |\bar{T}_t^{(n)} f(x, s) - \bar{T}_t f(x, s)|
\]

\[
= \sup_{(x, s) \in S, t} h(s + t) \left| \mathbb{E}\left(g\left(\frac{a_k}{a_n} X_t^{(k)} + xm(t)\right)\right) - \mathbb{E}(g((m(s))^{1/\alpha}X_t + xm(t)))\right|
\]

\[
\leq \|h\| \left( \sup_{(x, s) \in S, t} \left| \mathbb{E}(g((m(s))^{1/\alpha}X_t^{(k)} + xm(t))) - \mathbb{E}(g((m(s))^{1/\alpha}X_t + xm(t)))\right| + 2\|g\| \varepsilon + \sup_{(x, s) \in S, t} \mathbb{E}\left(1_{A_k} \left|g\left(\frac{a_k}{a_n} X_t^{(k)} + xm(t)\right) - g((m(s))^{1/\alpha}X_t^{(k)} + xm(t))\right| \right)\right).
\]

The second last supremum converges to 0 as \( n \to \infty \) by Lemma 5 and since \( k \to \infty \) as \( n \to \infty \) by 24. The last supremum converges as well to 0 by 25, together with the uniform continuity of \( g. \) Since \( \varepsilon > 0 \) can be chosen arbitrarily, 26 holds, which completes the proof. \( \square \)

5 Proofs concerning Theorem 3

This section contains the proofs of Lemma 2, Lemma 3 and Theorem 3.

**Proof.** (of Lemma 2) Fix \( t \geq 0. \) By Theorem 2 or Corollary 2.2 of Lamperti 28, applied with \( x := 1 - s \) to the function \( x \mapsto 1 - F(1-x, t) \), (11) holds if and only if

\[
\lim_{s \to 1} \alpha(s, t) = \alpha(t), \quad (29)
\]

where

\[
\alpha(s, t) := \frac{(1-s) \frac{\partial}{\partial s} F(s, t) - f(s, t)}{1 - F(s, t)} = \frac{f(F(s, t)) - F(s, t)}{1 - F(s, t)} \frac{1 - s}{f(s) - s} = \frac{L((1-F(s, t))^{-1}) - 1}{L((1-s)^{-1}) - 1}
\]

for all \( s \in (0, 1). \) Thus (i) and (ii) are equivalent. By Kolmogorov’s backward equation,

\[
at = \int_s^{F(s, t)} \frac{1}{f(u) - u} du = \int_{(1-F(s, t))^{-1}}^{(1-s)^{-1}} \frac{1}{x(L(x) - 1)} dx. \quad (30)
\]

Also, note that

\[
\log \frac{1}{\alpha(s, t)} = \log(L((1-s)^{-1}) - 1) - \log(L((1-F(s, t))^{-1}) - 1) = \int_{(1-F(s, t))^{-1}}^{(1-s)^{-1}} \frac{L'(x)}{L(x) - 1} dx.
\]

(iii) \( \Rightarrow \) (ii): Applying integration by parts to (30) yields

\[
at = \frac{\log x}{L(x) - 1} \bigg|_{x = (1-s)^{-1}}^{x = (1-F(s, t))^{-1}} + \int_{(1-F(s, t))^{-1}}^{(1-s)^{-1}} \frac{\log x}{L(x) - 1} \frac{L'(x)}{L(x) - 1} dx. \quad (31)
\]

Let \( \varepsilon > 0 \) be arbitrary. Since \( L(x)/\log x \to A > 0 \) as \( x \to \infty \) there exists \( K > 0 \) such that \( 1 - \varepsilon \leq A \log x/(L(x) - 1) \leq 1 + \varepsilon \) for all \( x \geq K. \) But, if \( s \) is sufficiently close to 1, both inequalities hold on the interval where it is integrated above in (31), implying that \( Aat = \lim_{s \to 1} \log(\alpha(s, t))^{-1}, \) which is exactly (29).

(i) \( \Rightarrow \) (iii): Assume that (11) holds for all \( t \geq 0. \) By (13),

\[
\alpha(t) = \lim_{s \to 1} \frac{\log(1 - F(s, t))}{\log(1 - s)}.
\]
As already seen before Lemma \[3\] there exists $C \geq 0$ such that $\alpha(t) = e^{-Ct}$. Thus,

$$Ct = - \lim_{s \to 1} \log \frac{\log(1 - F(s, 1))}{\log(1 - s)} = \lim_{s \to 1} \int_{(1-F(s, t))^{-1}}^{1} \frac{1}{x \log x} \, dx. \quad (32)$$

Division of (32) by (30) leads to

$$A := \frac{C}{a} = \lim_{s \to 1} \int_{(1-F(s, t))^{-1}}^{1} \frac{1}{x \log x} \, dx \, \frac{1}{a}. \quad \frac{1}{x} \log x$$

Now exploit the monotonicity of $\log x$ and $L(x)$ to conclude that

$$A \leq \lim_{s \to 1} \inf \frac{1}{\log((1-F(s, t))^{-1})} \int_{(1-F(s, t))^{-1}}^{1} \frac{1}{x \log x} \, dx \leq \frac{1}{\alpha(t)} \lim_{s \to 1} \sup \frac{1}{\log((1-F(s, t))^{-1})} \int_{(1-F(s, t))^{-1}}^{1} \frac{1}{x \log x} \, dx \leq \frac{1}{\alpha(t)} \lim_{s \to 1} \frac{L(1-F(s, t))^{-1}}{s}. \quad (33)$$

Similarly, $A \geq \alpha(t) \lim_{s \to 1} \frac{L(1-F(s, t))^{-1}}{\log((1-F(s, t))^{-1})}$. Letting $t \to 0$ yields $A = \lim_{s \to 1} \frac{L(1-F(s, t))^{-1}}{\log((1-F(s, t))^{-1})}$, which is (iii) and completes the proof. \(\square\)

**Proof.** (of Lemma \[3\]) By assumption, $H(x) := L(x) - 1 - A \log x, x \geq 1$, satisfies $\lim_{x \to \infty} H(x) = B - 1$. Moreover, $\beta(t)$, defined via (16), satisfies

$$\log \beta(t) = a \int_{0}^{t} (B - 1 - A \log \beta(s)) \, ds, \quad t \geq 0. \quad (34)$$

Computing the derivative of $L_t(x)$ with respect to $t$ provides a representation for $L_t(x)$ similar to (33), namely

$$\begin{align*}
\frac{\partial}{\partial t} L_t(x) &= \frac{\partial}{\partial t} (x^{\alpha(t)}(1 - F(1 - x^{-1}, t))) \\
&= x^{\alpha(t)} \alpha'(t) \log(x) (1 - F(1 - x^{-1}, t)) - x^{\alpha(t)} a(1 - F(1 - x^{-1}, t)) - F(1 - x^{-1}, t) \\
&= aL_t(x) \left( \frac{1 - F(1 - x^{-1}, t)}{1 - F(1 - x^{-1}, t)} - 1 - A \log x^{\alpha(t)} \right) \\
&= aL_t(x) \left( \frac{1 - F(1 - x^{-1}, t)}{1 - F(1 - x^{-1}, t)} - 1 - A \log x^{\alpha(t)} \right) \\
&= aL_t(x) \left( H(x^{\alpha(t)} L_t(x)) - A \log L_t(x) \right), \quad t \geq 0, x \geq 1.
\end{align*}$$

Therefore

$$\log L_t(x) = \int_{0}^{t} \frac{\partial}{\partial t} L_s(x) \, ds = a \int_{0}^{t} (H(x^{\alpha(s)} L_s^{-1}(x)) - A \log L_s(x)) \, ds, \quad t \geq 0. \quad (34)$$

Let $t > 0$ be fixed and $\varepsilon > 0$ be arbitrary. If $1 - x^{-1} > q$, where $q$ denotes the extinction probability, then the map $s \to x^{\alpha(s)} L_s^{-1}(x) = (1 - F(1 - x^{-1}, s))^{-1}$ is non-increasing. Hence $|H(x^{\alpha(s)} L_s^{-1}(x)) - (B - 1)| < \varepsilon$ for all $s \in [0, t]$ and all sufficiently large $x$. From (33) and (34),

$$|\log L_t(x) - \log \beta(t)| \leq a\varepsilon t + aA \int_{0}^{t} |\log L_s(x) - \log \beta(s)| \, ds.$$

By Gronwall’s inequality,

$$|\log L_t(x) - \log \beta(t)| \leq a\varepsilon t + aA \int_{0}^{t} a\varepsilon s \exp \left( \int_{s}^{t} aA \, d\sigma \right) \, ds \leq a\varepsilon \left( 1 + \int_{0}^{t} aA \exp(aA(t - s)) \, ds \right) = a\varepsilon \exp(aAt).$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, the result $\lim_{x \to \infty} L_t(x) = \beta(t)$ follows. \(\square\)
Proof. (of Theorem 3) The proof is divided into two steps. First the assumption (15) is used to establish the convergence of the one-dimensional distributions. Afterwards it is shown with some general weak convergence machinery for Markov processes that the convergence of the one-dimensional distributions is already sufficient for convergence in $D(E[0, \infty])$, where $E := [0, \infty)$.

**Step 1.** (Convergence of the one-dimensional distributions) Fix $\lambda, t \geq 0$. Define $s_n := \exp(-\lambda n^{-1/\alpha(t)}), n \in \mathbb{N}$. Note that $s_n \to 1$ as $n \to \infty$. We have $E(\exp(-\lambda X_t^n(\cdot))) = E(\exp(-\lambda n^{-1/\alpha(t)} Z_t^n)) = (E(\exp(-\lambda n^{-1/\alpha(t)} Z_t^n)))^n = (F(s_n, t))^n$. Taking the logarithm yields

$\log E(\exp(-\lambda X_t^n)) = n \log(1 - (1 - F(s_n, t))) \sim n(1 - F(s_n, t)) \sim -n\beta(t)(1 - s_n)^{\alpha(t)}$

as $n \to \infty$ by (15). Since $1 - s_n = 1 - \exp(-\lambda n^{-1/\alpha(t)}) \to \lambda n^{-1/\alpha(t)}$ as $n \to \infty$ it follows that the latter expression is asymptotically equal to $-n\beta(t)(\lambda n^{-1/\alpha(t)})^{\alpha(t)} = -\beta(t)\lambda^{\alpha(t)}$. Therefore

$\lim_{n \to \infty} E(\exp(-\lambda X_t^n)) = \exp(-\beta(t)\lambda^{\alpha(t)}) = E(\exp(-\lambda X_t))$. This pointwise convergence of the Laplace transforms implies the convergence $X_t^n \to X_t$ in distribution as $n \to \infty$.

**Step 2.** (Convergence in $D(E[0, \infty])$) We proceed as in the proof of [27, Theorem 2.1]. For $n \in \mathbb{N}$ and $t \geq 0$ define $E_n, t := \{j/n^{1/\alpha(t)} : j \in \mathbb{N}\}$. In general the process $X_t^n$ is time-inhomogeneous. Let $Y(t) := (X_t^n, t)_{t \geq 0}$ and $Y := (X_t, t)_{t \geq 0}$ denote the space-time processes of $X_t^n$ and $X_t$, respectively. Note that $Y(n)$ has state space $N_n := \{(j/n^{1/\alpha(t)}, t) : j \in \mathbb{N}, t \geq 0) = \bigcup_{n \geq 0}(E_n, t \times \{t\})$ and $Y$ has state space $S := [0, \infty)^2$. According to Revuz and Yor [40, p. 85, Exercise (1.10)] the process $Y(n)$ is time-homogeneous. Define $\pi_n : B(S) \to B(S)$ via $\pi_n g(x, s) := g(x, s)$ for $g \in B(S)$ and $(x, s) \in S_n$. In the following it is shown that $Y(n)$ converges in $D(E[0, \infty])$ to $Y$ as $n \to \infty$. Note that this convergence implies the desired convergence of $X_t^n$ in $D(E[0, \infty])$ to $X_t$ as $n \to \infty$. For $\lambda, \mu > 0$ denote the test function $g_{\lambda, \mu}$ via $g_{\lambda, \mu}(x, s) := e^{-\lambda x - \mu s}, (x, s) \in S$. By [27, Proposition 5.4] it suffices to verify that for every $t \geq 0$ and $\lambda, \mu > 0$,

$$\lim_{n \to \infty} \sup_{s \geq 0} s \in E_{n,s} |U^n_t \pi_{n, \lambda, \mu}(x, s) - \pi_{n, \lambda, \mu}(x, s)| = 0,$$

where $U^n_t : B(S_n) \to B(S_n)$ is defined via $U^n_t g(x, s) := E(g(X_{s+t}^n) \mid X_s^n) = x), g \in B(S_n)$, $s \geq 0, x \in E_{n,s}$. Note that $(U^n_t)_{t \geq 0}$ is the semigroup of $Y(n)$.

Fix $t \geq 0$ and $\lambda, \mu > 0$. For all $n \in \mathbb{N}$, $s \geq 0$ and $x \in E_{n,s}$,

$$U^n_t \pi_{n, \lambda, \mu}(x, s) = E(\pi_{n, \lambda, \mu}(X_{s+t}^n) \mid X_s^n = x) = E(\exp(-\lambda X_{s+t}^n - \mu(s+t)) \mid X_s^n = x) = e^{s+t}E(\exp(-\lambda n^{-1/\alpha(t)} Z_{s+t}^n) \mid Z_s^n = x n^{1/(\alpha(s))}) = e^{s+t}E(\exp(-\lambda n^{-1/\alpha(t)} Z_{s+t}^n))$$

and

$$\pi_{n, \lambda, \mu}(x, s) = U_t g_{\lambda, \mu}(x, s) = E(\exp(-\lambda X_{s+t} - \mu(s+t)) \mid X_s = x) = e^{s+t}E(\exp(-\lambda X_{s+t}) \mid X_s = x) = e^{s+t}E(\exp(-\lambda n^{-1/\alpha(t)} X_t)).$$

Thus, one has to verify that

$$\lim_{n \to \infty} \sup_{s \geq 0} s \in E_{n,s} e^{s+t}E(\exp(-\lambda n^{-1/\alpha(t)} Z_{s+t}^n)) - E(\exp(-\lambda n^{-1/\alpha(t)} X_t)) = 0.$$ 

We will even verify that

$$\lim_{n \to \infty} \sup_{s \geq 0} s \in E_{n,s} e^{s+t}E(\exp(-\lambda n^{-1/\alpha(t)} Z_{s+t}^n)) - E(\exp(-\lambda n^{-1/\alpha(t)} X_t)) = 0.$$ 

Since $\alpha(s+t) = \alpha(s)\alpha(t)$, the quantity inside the absolute values depends on $n$ and $s$ only via $n^{1/(\alpha(s))}$. Since $n^{1/(\alpha(s))}$ is non-decreasing in $s$ it follows that the convergence for fixed $s \geq 0$ is
Let \( \aleph > 1 \) \( \leq \) \( \cdots \) almost surely it follows by Pólya’s theorem [39, Satz I] that it suffices to verify the convergence of the one-dimensional distributions \( X_{\lambda k} = k^{-1/\alpha(t)} Z_{\lambda}^{(k)} \rightarrow X_{\lambda} \) in distribution as \( k \rightarrow \infty \), \( t \geq 0 \). But the convergence of the one-dimensional distributions holds by Step 1. The proof is complete.

6 Appendix

In this appendix five auxiliary results are provided. Lemma 4 and Lemma 5 below are used in the proof of Lemma 4. Lemma 4 provides an asymptotic statement for Laplace transforms and generating functions respectively. Lemma 5 is a version of L’Hospital’s rule, which is stated for completeness.

**Lemma 4** Let \( \xi \) be a nonnegative real valued random variable with \( m := \mathbb{E}(\xi) < \infty \). Suppose that the distribution function \( F \) of \( \xi \) satisfies \( 1 - F(x) \leq Cx^{-\alpha} \) for all \( x \geq 0 \) for some \( C < \infty \) and \( \alpha > 1 \). Then, for every \( \varepsilon \in [0, \min(\alpha - 1, 1)) \),

\[
\lim_{\lambda \to 0} \frac{1 - \varphi(\lambda) + \lambda m}{\lambda^{1+\varepsilon}} = 0, \tag{36}
\]

where \( \varphi \) denotes the Laplace transform of \( \xi \). If \( \xi \) takes only values in \( \mathbb{N}_0 \), then, for the same range of values of \( \varepsilon \) as above,

\[
\lim_{s \to 1} \frac{(1 - s)m - (1 - f(s))}{(1 - s)^{1+\varepsilon}} = 0, \tag{37}
\]

where \( f \) denotes the pgf of \( \xi \).

**Remark.** The tail condition holds if \( \mathbb{E}(\xi^\alpha) < \infty \), since, by Markov’s inequality, \( 1 - F(x) = \mathbb{P}(\xi^\alpha > x) \leq x^{-\alpha} \mathbb{E}(\xi^\alpha) \).

**Proof.** (of Lemma 4) Applying the well known formula \( \mathbb{E}(g(\xi)) = g(0) + \int_0^\infty g'(x)(1 - F(x)) \, dx \), \( g \in C^1([0, \infty)) \), to the function \( g(x) := e^{-\lambda x} - 1 + \lambda x \)

\[
\frac{\varphi(\lambda) - 1 + \lambda m}{\lambda^{1+\varepsilon}} = \frac{1}{\lambda^\varepsilon} \int_0^\infty (1 - F(x))(1 - e^{-\lambda x}) \, dx \leq \int_0^1 \frac{1 - e^{-\lambda x}}{\lambda^\varepsilon} \, dx + C \int_1^\infty \frac{1 - e^{-\lambda x}}{(\lambda x)^{\alpha - \varepsilon}} \, dx.
\]

Since \( \varepsilon < 1 \), \( \lim_{\lambda \to 0}(1 - e^{-\lambda x})/\lambda^\varepsilon = 0 \) and the first integral converges to 0 by the dominated convergence theorem. Since \( (1 - e^{-\lambda x})/(\lambda x)^\varepsilon \) is bounded uniformly in \( \lambda \) and \( x \), and \( \alpha - \varepsilon > 1 \), the dominated convergence theorem is again applicable and the second integral converges to 0. If \( \xi \) takes only values in \( \mathbb{N}_0 \) then (37) follows from (36) via the substitution \( \lambda := -\log s \), \( s \in (0, 1) \), and the fact that \( -\log s = (1 - s) + O((1 - s)^2) \) as \( s \to 1 \).

The situation in the following lemma is the one of L’Hospital’s rule.

**Lemma 5** Let \( c, x_0 \in [-\infty, \infty] \). Let \( f, g : I \to \mathbb{R} \) be continuously differentiable on an open interval \( I \) containing \( x_0 \) or having \( x_0 \) as a limit point if the limit is one-sided. Assume further that \( g'(x) \neq 0 \) for all \( x \in I \setminus \{x_0\} \). Let \( \alpha \in \mathbb{R} \setminus \{1\} \). If either

\[
\lim_{x \to x_0} g^{-1/\alpha}(x) = \lim_{x \to x_0} f(x) / g^{1/\alpha}(x) = 0
\]

or

\[
\lim_{x \to x_0} g^{-1/\alpha}(x) = \lim_{x \to x_0} f(x) / g^{1/\alpha}(x) = \infty,
\]

slower as \( s \) is smaller. So the slowest convergence holds for \( s = 0 \) \( \Rightarrow \alpha(s) = 1 \). Thus it suffices to verify that for every \( t \geq 0 \) and \( \lambda > 0 \)

\[
\lim_{n \to \infty} \sup_{x > 0} |\mathbb{E}(\exp(-\lambda n^{-1/\alpha(t)} Z_{\lambda}^{(\lfloor xn \rfloor)})) - \mathbb{E}(\exp(-\lambda x^{1/\alpha(t)} X_{\lambda}))| = 0.
\]

The map \( x \mapsto \mathbb{E}(\exp(-\lambda x^{1/\alpha(t)} X_{\lambda})) \) is bounded, continuous and non-increasing. Since \( Z_{\lambda}^{(1)} \leq Z_{\lambda}^{(2)} \leq \cdots \) almost surely it follows by Pólya’s theorem [39, Satz I] that it suffices to verify the above convergence pointwise for every \( x > 0 \). Defining \( k := \lfloor xn \rfloor \) it is readily seen that this is equivalent to the convergence of the one-dimensional distributions \( X_{\lambda k} = k^{-1/\alpha(t)} Z_{\lambda}^{(k)} \rightarrow X_{\lambda} \) in distribution as \( k \rightarrow \infty \), \( t \geq 0 \). But the convergence of the one-dimensional distributions holds by Step 1. The proof is complete.
and
\[
\lim_{x \to x_0} (\alpha f'(x)/g'(x) - f(x)/g(x)) = c,
\]
then \(\lim_{x \to x_0} f(x)/g(x) = c(\alpha - 1)^{-1}\). If the limit (38) does not exist it still holds that
\[
\lim \inf_{x \to x_0} \left( \frac{\alpha f'(x)}{g'(x)} - \frac{f(x)}{g(x)} \right) \leq \lim \inf_{x \to x_0} \left( \frac{f(x)}{g(x)} \right) = \lim \sup_{x \to x_0} \left( \frac{f(x)}{g(x)} \right) \leq \lim \sup_{x \to x_0} \left( \frac{\alpha f'(x)}{g'(x)} - \frac{f(x)}{g(x)} \right).
\]

**Proof.** A straightforward computation shows that
\[
\frac{1}{\alpha - 1} \left( \frac{f'(x)}{g'(x)} - \frac{f(x)}{g(x)} \right) = \frac{f'(x)g^{-1/\alpha}(x) - (1/\alpha)g^{-1-1/\alpha}(x)g'(x)f(x)}{(1-1/\alpha)g^{-1/\alpha}g'(x)},
\]
The following two results are needed in the proof of Theorem 2. Lemma 6 contains a statement on uniform weak convergence. The last result (Lemma 7) provides a certain dense subset of \(\hat{C}(\mathbb{R} \times [0, \infty))\).

**Lemma 6** Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of real-valued random variables converging weakly to a real-valued random variable \(X\). Then, for every bounded and continuous function \(f : \mathbb{R} \to \mathbb{R}\) and \(A, B > 0\),
\[
\lim_{n \to \infty} \sup_{|a| \leq A, |b| \leq B} |\mathbb{E}(f(aX_n + b)) - \mathbb{E}(f(aX + b))| = 0.
\]

If \(f \in \hat{C}(\mathbb{R})\), then (39) even holds if the supremum is taken over \([-A, A] \times \mathbb{R}\) instead of \([-A, A] \times [-B, B]\).

**Proof.** For \(n \in \mathbb{N}\) define \(g_n : \mathbb{R}^2 \to \mathbb{R}\) via \(g_n(a, b) := \mathbb{E}(f(aX_n + b))\), \(a, b \in \mathbb{R}\), and \(g\) similarly with \(X_n\) replaced by \(X\). Fix \(A, B > 0\). Obtaining pointwise convergence of \(g_n\) to \(g\) from weak convergence, (39) follows, in view of the Arzelà–Ascoli theorem, from the uniform equicontinuity of \(\{g_n : n \in \mathbb{N}\}\) on \(K := [-A, A] \times [-B, B]\), that is, for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that \(\max(|a-a'|, |b-b'|) < \delta\) implies \(|g_n(a, b) - g_n(a', b')| < \varepsilon\) for all \(n \in \mathbb{N}\) and all \((a, b), (a', b') \in K\). Let \(\varepsilon > 0\). By Prohorov’s theorem the family of distributions of the weakly convergent sequence \((X_n)_{n \in \mathbb{N}}\) is tight. Thus, there exists \(C \in (0, \infty)\) such that \(\sup_{n \in \mathbb{N}} \mathbb{P}(|X_n| > C) < \varepsilon\) and \(\mathbb{P}(|X| > C) < \varepsilon\). Using the uniform continuity of \(f\) on \(K\), choose \(\delta > 0\) such that \(|x-y| < \delta (C+1)\) implies \(|f(x) - f(y)| < \varepsilon\). Consequently,
\[
|g_n(a, b) - g_n(a', b')| = |\mathbb{E}(f(aX_n + b)) - \mathbb{E}(f(a'X_n + b'))| \\
\leq \varepsilon \|f\| + \mathbb{E}(1_{|X_n| \leq C}|f(aX_n + b) - f(a'X_n + b'))| \\
\leq 2\varepsilon \|f\| + \varepsilon
\]
for \((a, b), (a', b') \in K\) with \(\max(|a-a'|, |b-b'|) < \delta\), proving the first statement.

If \(f \in \hat{C}(\mathbb{R})\) then there exists \(L > 0\) such that \(|f(x)| < \varepsilon\) for all \(|x| > L\). In particular (39) holds for \(B := AC + L\). On the remaining area \([-A, A] \times (\mathbb{R} \setminus [-B, B])\) all the functions \(g_n\) and \(g\) are sufficiently small. More precisely, if \(|a| \leq A\) and \(|b| > B\), then \(|aX_n + b| > L\) on the event \(|X_n| \leq C\), hence
\[
|g_n(a, b)| = |\mathbb{E}(f(aX_n + b))| \leq \varepsilon \|f\| + \mathbb{E}(1_{|X_n| \leq C}|f(aX_n + b)|) \leq \varepsilon \|f\| + \varepsilon
\]
for all \(n \in \mathbb{N}\), and similarly \(|g(a, b)| \leq \varepsilon \|f\| + \varepsilon\), which proves the additional statement. \(\square\)
Lemma 7 Let $S := \mathbb{R} \times [0, \infty)$. The space of functions $f : S \to \mathbb{R}$ of the form $f(x,y) = \sum_{i=1}^{l} g_i(x) h_i(y)$ with $l \in \mathbb{N}$, $g_i, \ldots, g_l \in C(\mathbb{R})$ and $h_1, \ldots, h_l \in \dot{C}([0, \infty))$ is dense in $\dot{C}(S)$.

Proof. Two proofs are provided. The first proof is elementary and constructive. The second proof exploits the Stone–Weierstrass theorem for locally compact spaces.

Proof 1. (elementary) Each $f \in \dot{C}(S)$ can be transformed (with the additional definition $f(\pm \infty, y) := 0$ for all $y \in [0, \infty)$ and $f(x, \infty) := 0$ for all $x \in \mathbb{R}$) into a map $\bar{f} \in C([0,1]^2)$ satisfying $\bar{f}(0, y) = \bar{f}(1, y) = \bar{f}(x, 1) = 0$ for all $x, y \in [0,1]$ via

$$\bar{f}(x, y) := f \left( \frac{1}{1-x} - 1, \frac{y}{1-y} \right), \quad x, y \in [0,1]^2.$$ 

Thus, it suffices to verify that the space $D$ of functions $f : [0,1]^2 \to \mathbb{R}$ of the form $f(x,y) = \sum_{i=1}^{l} g_i(x) h_i(y)$ with $l \in \mathbb{N}$, $g_i, \ldots, g_l \in D_1 := \{ g \in C([0,1]) : g(0) = g(1) = 0 \}$ and $h_1, \ldots, h_l \in D_2 := \{ h \in C([0,1]) : h(1) = 0 \}$ is dense in $\{ f \in C([0,1]^2) : f(0, y) = f(1, y) = f(x, 1) = 0 \}$ for all $x, y \in [0,1]$. This is seen as follows. Let $m \in \mathbb{N}$. For $i \in \{ 0, \ldots, m \}$ define $x_i := i/m$ and $g_i : [0,1] \to [0,1]$ via

$$g_i(x) := (1 - m|x - x_i|) 1_{\{ |x - x_i| \leq 1/m \}}, \quad x \in [0,1].$$

Note that $g_0, \ldots, g_m$ form a partition of unity, i.e. $\sum_{i=0}^{m} g_i(x) = 1$ for all $x \in [0,1]$. Moreover, $g_1, \ldots, g_m \in D_1$. In the same manner define $y_i := j/m$ and $h_j : [0,1] \to [0,1]$ via $h_j(y) := (1 - m|y - y_j|) 1_{\{ |y - y_j| \leq 1/m \}}$ for all $j \in \{ 0, \ldots, m \}$. Again, $h_0, \ldots, h_m$ form a partition of unity, i.e. $\sum_{j=0}^{m} h_j(y) = 1$ for all $y \in [0,1]$. Moreover, $h_0, \ldots, h_{m-1} \in D_2$. Now define $f_m : [0,1]^2 \to \mathbb{R}$ via

$$f_m(x,y) := \sum_{i,j=0}^{m} f(x_i, y_j) g_i(x) h_j(y) = \sum_{i=1}^{m-1} \sum_{j=0}^{m-1} f(x_i, y_j) g_i(x) h_j(y), \quad x, y \in [0,1],$$

where the last equality holds since $f(0,y) = f(1,y) = f(x,1) = 0$ for all $x, y \in [0,1]$. From $g_1, \ldots, g_m \in D_1$ and $h_0, \ldots, h_{m-1} \in D_2$ it follows that $f_m \in D$. It remains to verify that $\lim_{m \to \infty} \| f_m - f \| = 0$. Let $\varepsilon > 0$. Since $f$ is uniformly continuous on $[0,1]^2$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|f(x', y') - f(x, y)| < \varepsilon$ for all $x, y, x', y' \in [0,1]$ with $|x - x'| < \delta$ and $|y - y'| < \delta$. For all $x, y \in [0,1]$ it follows from $\sum_{i,j=0}^{m} g_i(x) h_j(y) = 1$ that

$$|f_m(x,y) - f(x,y)| = \left| \sum_{i,j=0}^{m} \left( f(x_i, y_j) - f(x, y) \right) g_i(x) h_j(y) \right| \leq \sum_{i,j=0}^{m} \left| f(x_i, y_j) - f(x, y) \right| g_i(x) h_j(y).$$

Now for each $(x, y) \in [0,1]^2$ there exist $i_0, j_0 \in \{ 0, \ldots, m \} - \{ 0, i_0 \}$ (depending on $x$ and $y$) such that $x_{i_0} \leq x \leq x_{i_0+1}$ and $y_{j_0} \leq y \leq y_{j_0+1}$. Since $g_i(x) = 0$ for all $i \in \{ 0, \ldots, m \} - \{ i_0, i_0+1 \}$ and $h_j(y) = 0$ for all $j \in \{ 0, \ldots, m \} - \{ j_0, j_0 + 1 \}$ we conclude that

$$|f_m(x,y) - f(x,y)| \leq \left| f(x_{i_0}, y_{j_0}) - f(x, y) \right| + \left| f(x_{i_0+1}, y_{j_0+1}) - f(x, y) \right| + \left| f(x_{i_0+1}, y_{j_0}) - f(x, y) \right| + \left| f(x_{i_0}, y_{j_0+1}) - f(x, y) \right| \leq 4\varepsilon$$

for all $m \in \mathbb{N}$ with $m > 1/\delta$. Thus, $\lim_{m \to \infty} \| f_m - f \| = 0$. \qed

Proof 2. (using the Stone–Weierstrass theorem) The space of functions $f : S \to \mathbb{R}$ of the form $f(x,y) = \sum_{i=1}^{l} g_i(x) h_i(y)$ with $l \in \mathbb{N}$, $g_i, \ldots, g_l \in \dot{C}(\mathbb{R})$ and $h_1, \ldots, h_l \in \dot{C}([0, \infty))$ is a subalgebra of $\dot{C}(S)$, which separates points and vanishes nowhere, whence is dense in $\dot{C}(S)$ by the Stone–Weierstrass theorem (see, for example, [13]). In [13] the theorem is stated for complex-valued functions, but it remains true for real-valued functions. To see this, let $f \in \dot{C}(S) \subseteq C(S, \mathbb{C})$ be arbitrary. By the theorem there exist $g_1, g_2, \ldots \in C(S, \mathbb{C})$ such that $\lim_{n \to \infty} \| g_n - f \| = 0$. Then $f_n := \text{Re}(g_n) \in \dot{C}(S)$, $n \in \mathbb{N}$, and $\| f_n - f \| \leq \| g_n - f \| \to 0$ as $n \to \infty$. \qed
The last result (Lemma 8) states that a particular function is absolutely monotone. This result is needed in Example 7 but is as well of its own interest. For general information on absolutely and completely monotonic functions we refer the reader to (Appendix A, § 4 of) Steutel and van Harn [16] and Chapter IV of Widder [50].

**Lemma 8** The function \( g(z) := (-\log(1 - z)) \log(1 - \log(1 - z)) \) is absolutely monotone on \([0, 1)\).

**Remark.** The following proof of Lemma 8 is based on a particular integral representation (see (41)) for the coefficient \( g_n := g^{(n)}(0)/n! \) in front of \( z^n \) in the Taylor expansion of \( g(z) \) around 0. Concrete calculations of the coefficients \( g_n \) via the summation formula (40) or the integral representation (41) yield
\[
g(z) = z^2 + \frac{1}{2} z^3 + \frac{1}{2} z^4 + \frac{3}{8} z^5 + \frac{247}{720} z^6 + \frac{7}{24} z^7 + \frac{535}{2016} z^8 + \frac{2051}{8640} z^9 + O(z^{10}).
\]

**Proof.** (of Lemma 8) We have to verify that \( g_n \geq 0 \) for all \( n \in \mathbb{N}_0 \). Clearly, \( g_0 = g(0) = 0 \) and \( g = u \circ v \), where \( u(x) := x \log(x + 1) \) for \( x \geq 0 \) and \( v(z) := -\log(1 - z) \) for \( 0 \leq z < 1 \). The functions \( u \) and \( v \) have derivatives \( u'(x) = \log(x + 1) + x/(x + 1), x \geq 0 \),
\[
u^{(k)}(x) = (-1)^k \left( \frac{(k-2)!}{(x+1)^{k-1}} + \frac{(k-1)!}{(x+1)^k} \right), \quad k \in \mathbb{N} \setminus \{1\}, x \geq 0,
\]
and \( v^{(m)}(z) = (m-1)!/(1-z)^m \), \( m \in \mathbb{N}, 0 \leq z < 1 \). Note that \( v \) is absolutely monotone, but \( u \) is not absolutely monotone, which explains why the statement of Lemma 8 is less simple as it seems at a first glance. By Faà di Bruno’s formula,
\[
g^{(n)}(z) = \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{n!}{k_1! \cdots k_n!} u^{(k_1+\cdots+k_n)}(v(z)) \prod_{m=1}^{n} \left( \frac{v^{(m)}(z)}{m!} \right)^{k_m}
\]
\[
= \frac{1}{(1-z)^n} \sum_{k=1}^{n} u^{(k)}(v(z)) \sum_{1k_1+2k_2+\cdots+nk_n=k} \frac{n!}{k_1! \cdots k_n! k_1 \cdots nk_n}
\]
\[
= \frac{1}{(1-z)^n} \sum_{k=1}^{n} u^{(k)}(v(z)) s(n,k), \quad n \in \mathbb{N}, 0 \leq z < 1,
\]
where \( s(\_, \_) \) denote the Stirling numbers of the first kind. Thus,
\[
g_n := \frac{g^{(n)}(0)}{n!} = \frac{1}{n!} \sum_{k=1}^{n} u^{(k)}(0)s(n, k) = \frac{1}{n!} \sum_{k=2}^{n} (-1)^k k! \frac{k!}{k-1} |s(n, k)|, \quad n \in \mathbb{N}. \tag{40}
\]
Plugging in \( k!/(k-1) = (k-2)! + (k-1)! = \int_{t=0}^{\infty} t^k (1/t + 1/t^2) e^{-t} dt \) and interchanging the sum with the integral yields
\[
g_n = \frac{1}{n!} \int_{t=0}^{\infty} \sum_{k=2}^{n} (-t)^k |s(n, k)| \left( \frac{1}{t} + \frac{1}{t^2} \right) e^{-t} dt, \quad n \in \mathbb{N}.
\]
Applying the relation \( \sum_{k=0}^{n} x^k |s(n, k)| = [x]^n := x(x+1) \cdots (x+n-1) \) to the point \( x := -t \) and noting that \( s(n, 0) = 0 \) and \( s(n, 1) = (n-1)! \) for \( n \in \mathbb{N} \) yields
\[
g_n = \frac{1}{n!} \int_{0}^{\infty} [t]^n + t(n-1)! \left( \frac{1}{t} + \frac{1}{t^2} \right) e^{-t} dt, \quad n \in \mathbb{N}.
\]
Noting that \([t]^n/n! = (-1)^n \binom{t}{n} \) shows that \( g_n \) has the integral representation
\[
g_n = \int_{0}^{\infty} \left( (-1)^n \binom{t}{n} + \frac{t}{n} \right) \left( \frac{1}{t} + \frac{1}{t^2} \right) e^{-t} dt, \quad n \in \mathbb{N}. \tag{41}
\]
In particular, \( g_1 = 0 \). For \( n \in \{2, 4, \ldots\} \) the map \( t \mapsto (-1)^n \left( \frac{t}{n} + \frac{1}{n} \right) \) is non-decreasing and hence nonnegative on \([0, \infty)\) implying that \( g_n \geq 0 \) for even \( n \). Assume now that \( n \in \{3, 5, \ldots\} \).

Then the map \( t \mapsto (-1)^n \left( \frac{t}{n} + \frac{1}{n} \right) \), \( t > 0 \), has a single root at \( t = n \) and is positive for \( t \in (0, n) \) and negative for \( t \in (n, \infty) \). Decompose \( g_n = I_1 - I_2 \), where

\[
I_1 := \int_0^n \left( \frac{t}{n} - \frac{1}{n^2} \right) e^{-t} \, dt \quad \text{and} \quad I_2 := \int_n^\infty \left( \frac{t}{n} - \frac{1}{n^2} \right) e^{-t} \, dt.
\]

The map \( t \mapsto \left( \frac{t}{n} - \frac{1}{n^2} \right) \) takes on the interval \((0, n-1)\) its minimum value \( 1/(n-1) \) at the right most point \( n-1 \). For \( n \in \{3, 5, \ldots\} \) we hence obtain for \( I_1 \) the lower bound

\[
I_1 \geq \int_0^{n-1} \left( \frac{t}{n} - \frac{1}{n^2} \right) e^{-t} \, dt \geq \frac{1}{n-1} \int_0^{n-1} e^{-t} \, dt = \frac{1-e^{-(n-1)}}{n-1} \geq \frac{1}{n}.
\]

For \( I_2 \) we obtain the upper bound

\[
I_2 \leq \int_n^\infty \left( \frac{t}{n} - \frac{1}{n^2} \right) e^{-t} \, dt \leq \int_n^\infty \frac{2}{n^2} \, dt = \frac{2}{n} \int_n^\infty \mathbb{P}(N_t = n-1) \, dt,
\]

where \( N_t \) is Poisson distributed with parameter \( t \). Applying the formula \( \int_n^\infty \mathbb{P}(N_t = k) \, dt = \mathbb{P}(N_n \leq k) \) with \( k = n-1 \) leads to \( I_2 \leq (2/n) \mathbb{P}(N_n \leq n-1) = 1/n \), since \( \mathbb{P}(N_n \leq n-1) \) is increasing in \( n \) with \( \lim_{n \to \infty} \mathbb{P}(N_n \leq n-1) = 1/2 \); see, for example, Teicher [17]. From \( I_1 \geq 1/n \) and \( I_2 \leq 1/n \) it follows that \( g_n = I_1 - I_2 \geq 0 \) for \( n \in \{3, 5, \ldots\} \). In summary, \( g_n \geq 0 \) for all \( n \in \mathbb{N}_0 \).

\[ \square \]

**Remark.** (Asymptotics of \( g_n \)) It is easily seen that \( g'(z) \sim (1-z)^{-1}L((1-z)^{-1}) \) as \( z \to 1 \) in \( \Delta \setminus \{1\} \), where \( L(u) := \log \log u \) and \( \Delta \) is defined as in Flajolet and Odlyzko [16, Eq. (2.5)]. By Theorem 5 of [16], applied with \( \alpha = -1 \) and \( f \) replaced by \( g' \), \( \lim_n [z^n] g'(z) \sim L(n) \) as \( n \to \infty \). From \( [z^n] g'(z) = (n+1)g_{n+1} \) it follows that \( g_n \sim n^{-1} \log \log n \) as \( n \to \infty \).

**References**


