2 Branching processes

Notation: $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$

2.1 Definition and branching property

2.1.1 Def. (GWP)
Let Y_{nj}, $n \in \mathbb{N}_0$, $j \in \mathbb{N}$, be independent and identically distributed (iid) r.v. taking values in $\mathbb{N}_0$. The process $(Z_n)_{n\in\mathbb{N}_0}$, defined via $Z_0 := 1$ and

$$Z_{n+1} := \sum_{j=1}^{Z_n} Y_{nj}, \qquad n \in \mathbb{N}_0,$$

is called Bienaymé–Galton–Watson branching process (GWP). Z_n is interpreted as the size of a population at generation n.

Define $p_k := P(Z_1 = k) = P(Y_{01} = k)$, $k \in \mathbb{N}_0$.

We call $(p_k)_{k\in\mathbb{N}_0}$ the reproduction distribution or offspring distribution.

For $i, j, i_{n-1}, \ldots, i_0 \in \mathbb{N}_0$ we have (as long as the conditional probability is defined)

$$\begin{aligned} \pi_{ij} &:= P(Z_{n+1} = j \,|\, Z_n = i, Z_{n-1} = i_{n-1}, \ldots, Z_0 = i_0) \\ &= P(Y_{n1} + \cdots + Y_{ni} = j) = \sum_{\substack{j_1,\ldots,j_i\in\mathbb{N}_0 \\ j_1+\cdots+j_i=j}} p_{j_1} \cdots p_{j_i} =: p_j^{*i}, \end{aligned}$$

where $p_j^{*i} = P(Y_{01} + \cdots + Y_{0i} = j)$ denotes the i-fold convolution of the offspring distribution at j. Thus, $(Z_n)_{n\in\mathbb{N}_0}$ is a HMC with transition probabilities

$$\pi_{ij} := P(Z_{n+1} = j \,|\, Z_n = i) = p_j^{*i}, \qquad i, j \in \mathbb{N}_0.$$

2.1.2 Example (Poisson GWP)
Let $\alpha > 0$, $p_k := e^{-\alpha}\alpha^k/k!$, $k \in \mathbb{N}_0$. In this case, $\pi_{ij} = e^{-\alpha i}(\alpha i)^j/j!$, $i, j \in \mathbb{N}_0$.

2.1.3 Example (Binary GWP)
Let $p \in [0, 1]$, $p_0 := 1 - p$ and $p_2 := p$. In this case,

$$\pi_{ij} = \begin{cases} 0 & \text{if } j \text{ is odd,} \\ \binom{i}{j/2} p^{j/2}(1-p)^{i-j/2} & \text{if } j \text{ is even.} \end{cases}$$

Thus,

$$P(Z_1 = 2i_1, \ldots, Z_n = 2i_n) = \prod_{k=1}^{n} \left(\binom{2i_{k-1}}{i_k} p^{i_k}(1-p)^{2i_{k-1}-i_k} \right)$$

with the convention $2i_0 = 1$. For $p = 1$ we have $Z_n = 2^n$ a.s. for all $n \in \mathbb{N}_0$.

2.1.4 Lemma
Let $(\Omega, \mathcal{F}, P)$ be a probability space, $A, B_i \in \mathcal{F}$ ($i \in I$, I at most countable) with $B_i \cap B_j$ $\forall\, i, j \in I$ with $i \neq j$ and assume that $B := \cup_{i\in I} B_i$ satisfies $P(B) > 0$. If for every $i \in I$ with $P(B_i) > 0$ the probability $P(A|B_i)$ does not depend on i, i.e. $P(A|B_i) =: c$ for all $i \in I$ with $P(B_i) > 0$, then $P(A|B) = c$.

Proof.
We have

$$\begin{aligned} P(A\cap B) &= P(A \cap \bigcup_{i\in I} B_i) = \sum_{i\in I} P(A\cap B_i) \\ &= \sum_{\substack{i\in I \\ P(B_i)>0}} \underbrace{P(A|B_i)}_{=c} P(B_i) = c \sum_{\substack{i\in I \\ P(B_i)>0}} P(B_i) = c\,P(B). \end{aligned}$$

Now divide by $P(B) > 0$. $\square$

2.1.5 Lemma
Let $(Z_n^{(l)})_{n\in\mathbb{N}_0}$, $l \in \mathbb{N}$, be independent GWPs, each distributed as $Z := (Z_n)_{n\in\mathbb{N}_0}$. Then, for any $k \in \mathbb{N}$, the process $\widetilde{Z} := (\widetilde{Z}_n)_{n\in\mathbb{N}_0}$, defined via $\widetilde{Z}_n := \sum_{l=1}^k Z_n^{(l)}$ for all $n \in \mathbb{N}_0$, is a HMC with the same transition probabilities as Z.

Proof.
Let $j, i_1, \ldots, i_k \in \mathbb{N}_0$ with $P(Z_n^{(1)} = i_1, \ldots, Z_n^{(k)} = i_k) > 0$. Define $i := i_1 + \cdots + i_k$. The conditional probability

$$\begin{aligned} &P(\widetilde{Z}_{n+1} = j \,|\, Z_n^{(1)} = i_1, \ldots, Z_n^{(k)} = i_k) \\ &\quad= \sum_{\substack{j_1,\ldots,j_k\in\mathbb{N}_0 \\ j_1+\cdots+j_k=j}} P(Z_{n+1}^{(1)} = j_1, \ldots, Z_{n+1}^{(k)} = j_k \,|\, Z_n^{(1)} = i_1, \ldots, Z_n^{(k)} = i_k) \\ &\quad= \sum_{\substack{j_1,\ldots,j_k\in\mathbb{N}_0 \\ j_1+\cdots+j_k=j}} \prod_{l=1}^k P(Z_{n+1}^{(l)} = j_l \,|\, Z_n^{(l)} = i_l) = \sum_{\substack{j_1,\ldots,j_k\in\mathbb{N}_0 \\ j_1+\cdots+j_k=j}} \prod_{l=1}^k p_{j_l}^{*i_l} = p_j^{*i} \end{aligned}$$

only depends via $i = i_1 + \cdots + i_k$ on $i_1, \ldots, i_k$. Lemma 2.1.4. $\Rightarrow$

$$\begin{aligned} P(\widetilde{Z}_{n+1} = j \,|\, \widetilde{Z}_n = i) &= P(\widetilde{Z}_{n+1} = j \,|\, \bigcup_{\substack{i_1,\ldots,i_k\in\mathbb{N}_0 \\ i_1+\cdots+i_k=i}} \{Z_n^{(1)} = i_1, \ldots Z_n^{(k)} = i_k\}) \\ &= p_j^{*i}, \qquad i, j \in \mathbb{N}_0. \end{aligned}$$

The calculation does not change, if the condition $\widetilde{Z}_n = i$ is replaced by $\widetilde{Z}_n = i, \widetilde{Z}_{n-1} = i_{n-1}, \ldots, \widetilde{Z}_0 = i_0$. $\Rightarrow$ $\widetilde{Z}$ is a HMC with the same transition probabilities as Z. $\square$

In the following let $\mathbb{N}_0^\infty := \times_{i\in\mathbb{N}}\mathbb{N}_0$. For $i \in \mathbb{N}$ let $\pi_i : \mathbb{N}_0^\infty \to \mathbb{N}_0$ be the projection to the i-th component, i.e. $\pi_i(k) = k_i$ for all $k = (k_i)_{i\in\mathbb{N}} \in \mathbb{N}_0^\infty$. Furthermore, let $\mathcal{G}$ denote the smallest σ-algebra in $\mathbb{N}_0^\infty$ such that all projections π_i, $i \in \mathbb{N}$, are measurable. $\mathcal{G}$ is called the product-σ-algebra of $\mathbb{N}_0^\infty$. It is easily seen that $\mathcal{G} = \mathcal{F}(\pi_i, i \in \mathbb{N}) = \mathcal{F}(\{\pi_i^{-1}(A_i) : i \in \mathbb{N}, A_i \subseteq \mathbb{N}_0\})$.

2.1.6 Theorem (Branching Property)
For $r \in \mathbb{N}_0$, $k \in \mathbb{N}$ and $A \in \mathcal{G}$,

$$P((Z_n)_{n>r} \in A \,|\, Z_r = k) \;=\; P((\widetilde{Z}_n)_{n\in\mathbb{N}} \in A), \qquad (*)$$

where $\widetilde{Z}_n := \sum_{j=1}^k Z_n^{(j)}$ and $(Z_n^{(j)})_{n\in\mathbb{N}_0}$, $j \in \{1,\ldots,k\}$, are independent GWPs, all distributed as $Z := (Z_n)_{n\in\mathbb{N}_0}$.

Proof.
For fixed $r \in \mathbb{N}_0$ and $k \in \mathbb{N}$ let $\mathcal{D}$ be the set of all $A \in \mathcal{G}$ satisfying $(*)$. It is easily seen that $\mathcal{D}$ is a Dynkin system in $\mathbb{N}_0^\infty$. Consider the system $\mathcal{E}$ of all A of the form $A = A_1 \times \cdots \times A_m \times \mathbb{N}_0 \times \mathbb{N}_0 \times \cdots$ with $m \in \mathbb{N}$ and $A_1,\ldots,A_m \subseteq \mathbb{N}_0$. Obviously, $\mathcal{E}$ is a $\cap$-stable generator of $\mathcal{G}$, i.e. $\mathcal{F}(\mathcal{E}) = \mathcal{G}$. If we can verify that $\mathcal{E} \subseteq \mathcal{D}$, then the statement follows, since then $\mathcal{G} = \mathcal{F}(\mathcal{E}) = \mathcal{D}(\mathcal{E}) \subseteq \mathcal{D}(\mathcal{D}) = \mathcal{D}$. It remains to verify that $(*)$ holds for $A \in \mathcal{E}$. Each such A is a at most countable union of sets of the form

$$\{k_1\} \times \cdots \times \{k_m\} \times \mathbb{N}_0 \times \mathbb{N}_0 \times \cdots \qquad (**)$$

with $m \in \mathbb{N}$ and $k_1,\ldots,k_m \in \mathbb{N}_0$. Because of the σ-additivity of the two probability measures on the left-hand and right-hand side in $(*)$ it suffices to verify $(*)$ for sets of the form $(**)$. In this case the left-hand side in $(*)$ is equal to

$$\begin{aligned} &P(Z_{r+1} = k_1,\ldots,Z_{r+m} = k_m \,|\, Z_r = k)\\ &\quad=\; \prod_{n=1}^m P(Z_{r+n} = k_n \,|\, Z_{r+n-1} = k_{n-1}) \;=\; \prod_{n=1}^m \pi_{k_{n-1},k_n}, \end{aligned}$$

where $k_0 := k$. The right-hand side in $(*)$ is as well equal to

$$P(\widetilde{Z}_1 = k_1,\ldots,\widetilde{Z}_m = k_m) \;=\; \prod_{n=1}^m P(\widetilde{Z}_n = k_n \,|\, \widetilde{Z}_{n-1} = k_{n-1}) \;=\; \prod_{n=1}^m \pi_{k_{n-1},k_n},$$

since, by Lemma 2.1.5, $\widetilde{Z}$ has the same transition probabilities as Z. □

2.1.7 Corollary
For $k, n \in \mathbb{N}_0$ and any function $h : \mathbb{N}_0 \to [0,\infty)$,

$$\mathrm{E}(h(Z_{n+1}) \,|\, Z_1 = k) \;=\; \mathrm{E}\left(h\left(\sum_{j=1}^k Z_n^{(j)}\right)\right),$$

where $(Z_n^{(j)})_{n\in\mathbb{N}_0}$, $j \in \mathbb{N}$, are independent copies of Z.

Proof.
For $k = 0$ both sides are equal to $h(0)$. Assume now that $k \in \mathbb{N}$. For $h = 1_B$ with $B \subseteq \mathbb{N}_0$ the statement follows from Theorem 2.1.6 (branching property) with the choice $A := \pi_n^{-1}(B) \in \mathcal{G}$. Thus, the statement holds for elementary functions. If $h : \mathbb{N}_0 \to [0, \infty)$ is arbitrary, then there exist elementary functions $0 \le h_1 \le h_2 \le \cdots$ with $\lim_{m\to\infty} h_m = h$. The statement then follows by two-times applying the theorem of monotone convergence. □

In order to compute the mean and the variance of Z_n, the following lemma will be useful.

2.1.8 Lemma
Let $X_1, X_2, \dots$ be iid $\mathbb{N}_0$-valued r.v. and let Y be a further $\mathbb{N}_0$-valued r.v. being independent of $(X_n)_{n\in\mathbb{N}}$. If g denotes the probability generation function (pgf) of X_1 and h the pgf of Y, then $S := \sum_{j=1}^{Y} X_j$ has the pgf $h \circ g$ and $\mathrm{E}(S) = \mathrm{E}(Y)\mathrm{E}(X_1) \in [0, \infty]$. If $\mathrm{E}(S) < \infty$ then $\mathrm{Var}(S) = \mathrm{Var}(Y)(\mathrm{E}(X_1))^2 + \mathrm{E}(Y)\mathrm{Var}(X_1)$.

Proof.
Let $s \in [0, 1]$. For $k \in \mathbb{N}_0$, $\mathrm{E}(s^S \,|\, Y = k) = \mathrm{E}(s^{X_1+\cdots+X_k} \,|\, Y = k) = \mathrm{E}(s^{X_1} \cdots s^{X_k}) = (\mathrm{E}(s^{X_1}))^k = (g(s))^k$. Multiplication with $P(Y = k)$ and summation over all $k \in \mathbb{N}_0$ yields

$$\mathrm{E}(s^S) \;=\; \sum_{k=0}^{\infty} \mathrm{E}(s^S \,|\, Y = k)P(Y = k) \;=\; \sum_{k=0}^{\infty} (g(s))^k P(Y = k) \;=\; h(g(s)).$$

Thus, S has the pgf $h \circ g$. It follows that $\mathrm{E}(S) = (h \circ g)'(1) = h'(g(1))g'(1) = h'(1)g'(1) = \mathrm{E}(Y)\mathrm{E}(X_1)$ and

$$\begin{aligned}
\mathrm{E}(S(S-1)) &= (h \circ g)''(1) \;=\; h''(g(1))(g'(1))^2 + h'(g(1))g''(1) \\
&= h''(1)(g'(1))^2 + h'(1)g''(1) \\
&= \mathrm{E}(Y(Y-1))(\mathrm{E}(X_1))^2 + \mathrm{E}(Y)\mathrm{E}(X_1(X_1-1)).
\end{aligned}$$

Assume now that $\mathrm{E}(S) < \infty$. Summation of $\mathrm{E}(S) - (\mathrm{E}(S))^2 = \mathrm{E}(Y)\mathrm{E}(X_1) - (\mathrm{E}(Y))^2(\mathrm{E}(X_1))^2$ yields

$$\begin{aligned}
\mathrm{Var}(S) &= \mathrm{E}(Y(Y-1))(\mathrm{E}(X_1))^2 + \mathrm{E}(Y)\mathrm{E}(X_1^2) - (\mathrm{E}(Y))^2(\mathrm{E}(X_1))^2 \\
&= \mathrm{E}(Y^2)(\mathrm{E}(X_1))^2 + \mathrm{E}(Y)(\mathrm{E}(X_1^2) - (\mathrm{E}(X_1))^2) - (\mathrm{E}(Y))^2(\mathrm{E}(X_1))^2 \\
&= \mathrm{Var}(Y)(\mathrm{E}(X_1))^2 + \mathrm{E}(Y)\mathrm{Var}(X_1).
\end{aligned}$$ □

Now let f_n denote the pgf of Z_n, i.e.

$$f_n(s) \;:=\; \mathrm{E}(s^{Z_n}) \;=\; \sum_{k=0}^{\infty} P(Z_n = k)s^k, \qquad s \in [0, 1].$$

Define $f := f_1$, i.e. $f(s) = \sum_{k=0}^{\infty} p_k s^k$, $s \in [0,1]$. Lemma 2.1.8 (applied with $Y := Z_{n-1}$ and $X_j := Y_{n-1,j}$) yields

$$f_n = f_{n-1} \circ f, \qquad n \in \mathbb{N},$$

and

$$\mathrm{E}(Z_n) = m\mathrm{E}(Z_{n-1}), \qquad n \in \mathbb{N},$$

where $m := f'(1) = \sum_{k=1}^{\infty} k p_k = \mathrm{E}(Z_1)$ is the expected number of offspring of any individual. Moreover, for $m < \infty$, Lemma 2.1.8 yields

$$\mathrm{Var}(Z_n) = \sigma^2 \mathrm{E}(Z_{n-1}) + m^2 \mathrm{Var}(Z_{n-1}), \qquad n \in \mathbb{N},$$

where $\sigma^2 := \mathrm{Var}(Z_1) = \sum_{k=1}^{\infty} k^2 p_k - m^2 = f''(1) + f'(1) - (f'(1))^2$ is the reproductive variance. In particular,

$$f_n = \underbrace{f \circ \cdots \circ f}_{n-\text{times}}$$

is the n-fold convolution of f and the mean of Z_n is $\boxed{\mathrm{E}(Z_n) = m^n}$, $n \in \mathbb{N}_0$. Moreover, if $m < \infty$, an induction on n shows that the variance of Z_n is

$$\mathrm{Var}(Z_n) = \begin{cases} \dfrac{\sigma^2 m^{n-1}(m^n - 1)}{m-1} & \text{if } m \neq 1, \\ n\sigma^2 & \text{if } m = 1. \end{cases}$$

2.2 Extinction probability

Given: GWP $Z = (Z_n)_{n \in \mathbb{N}_0}$ with $Z_0 = 1$ and offspring distribution $(p_k)_{k \in \mathbb{N}_0}$.

Notation: $f_n :=$ pgf of Z_n.

Known: $f_n = \underbrace{f \circ \cdots \circ f}_{n-\text{times}}$, where $f(s) := \displaystyle\sum_{k=0}^{\infty} p_k s^k$, $s \in [0,1]$.

$$m := \mathrm{E}(Z_1) = f'(1-) = \sum_{k=1}^{\infty} k p_k \in [0, \infty]$$

2.2.1 Def. (Extinction Probability)
The event

$$Q := \{Z_n = 0 \text{ eventually}\} := \liminf_{n \to \infty} \{Z_n = 0\} := \bigcup_{n \in \mathbb{N}} \bigcap_{m=n}^{\infty} \{Z_m = 0\}$$

is called the *extinction event* *and*

$$q := P(Q) = P(Z_n = 0 \text{ eventually}) = \lim_{n \to \infty} P(Z_n = 0) = \lim_{n \to \infty} f_n(0)$$

the *extinction probability* *of* Z.

2.2.2 Theorem (Fixed Point Theorem)
The fixed point equation $f(s) = s$ has exactly one solution in $[0, 1)$ if $m > 1$ and no solution in $[0, 1)$ if $m \le 1$ and $p_1 < 1$. The extinction probability q is the smallest fixed point of f in the interval $[0, 1]$.

Proof.
We exclude the trivial case $p_1 = 1$. From $f_n(0) \to q$ and the continuity of f it follows that

$$f(q) \leftarrow f(f_n(0)) \;=\; f_{n+1}(0) \to q.$$

Thus, $f(q) = q$. Now let $a \in [0, 1]$ be arbitrary with $f(a) = a$. By induction on $n \in \mathbb{N}$ it follows that $f_n(0) \le a$: For $n = 1$ this is clear, since f is nondecreasing and hence $f_1(0) = f(0) \le f(a)$. The induction step from n to $n + 1$ reads $f_{n+1}(0) = f(f_n(0)) \le f(a) = a$. Letting $n \to \infty$ yields $q \le a$, i.e. q is the smallest solution of the equation $f(s) = s$ in $[0, 1]$.

Define $\varphi(s) := f(s) - s$, $s \in [0, 1]$.

Assume first that $m \le 1$ and $p_1 < 1$. Then, for all $s \in [0, 1)$,

$$\varphi'(s) \;=\; f'(s) - 1 \;<\; f'(1) - 1 \;\le\; 0,$$

i.e. φ is strictly decreasing. In particular, $\varphi(s) > \varphi(1) = 0$, so $f(s) > s$ for all $s \in [0, 1)$. Therefore, the equation $f(s) = s$ has no solution in $[0, 1)$.

Assume now that $m > 1$. Then,

$$\frac{1 - f(s)}{1 - s} \;\to\; f'(1) \;=\; m \;>\; 1, \qquad s \to 1.$$

Thus, $1 - f(s) > 1 - s$, so $\varphi(s) < 0$ for all s in a left neighborhood of 1. On the other hand, $\varphi(0) = f(0) \ge 0$. Thus, by the intermediate value theorem, there exists $s \in [0, 1)$ with $\varphi(s) = 0$, i.e. $f(s) = s$. To see that there is only one such s, assume that there exist $0 \le s_1 < s_2 < 1$ with $f(s_1) = s_1$ and $f(s_2) = s_2$. Then, $\varphi(s_1) = \varphi(s_2) = 0 = \varphi(1)$. Thus, by the theorem of Rolle, there exist a, b with $s_1 < a < s_2 < b < 1$ and $\varphi'(a) = \varphi'(b) = 0$, i.e. $f'(a) = f'(b)$, in contradiction to the fact that f' is strictly increasing[1] if $m > 1$. Thus, the assumption is wrong, so there exists exactly one $s \in [0, 1)$ with $f(s) = s$. □

2.2.3 Theorem
If $p_1 < 1$ then $P(Z_n \to 0) + P(Z_n \to \infty) = 1$.

[1] Since $m > 1$ there exists $k_0 \in \{2, 3, \ldots\}$ with $p_{k_0} > 0$. For all $s \in (0, 1)$ it follows that $f''(s) = \sum_{k=2}^{\infty} k(k-1)p_k s^{k-2} \ge k_0(k_0 - 1)p_{k_0} s^{k_0 - 1} > 0$. Hence, f' is strictly increasing.

Proof.
For $q = 1$ there is nothing to show. Thus, let $q < 1$. Then, by Theorem 2.2.2, $m > 1$ and f' is strictly increasing.

Assume that $f'(q) \geq 1$. Then, for all $s \in (q, 1)$, $f'(s) > f'(q) \geq 1$, and hence $f(1) - q = f(1) - f(q) = \int_q^1 f'(s)\,\mathrm{d}s > \int_q^1 1\,\mathrm{d}s = 1 - q$, so $f(1) > 1$, an obvious contradiction. Thus, $f'(q) < 1$.

Induction yields

$$f_n'(q) \;=\; (f'(q))^n, \qquad n \in \mathbb{N}.$$

For $n = 1$ this is clear, since $f_1 = f$. The step from n to $n+1$ reads $f_{n+1}'(q) = (f_n \circ f)'(q) = f_n'(f(q))f'(q) = f_n'(q)f'(q) \overset{IV}{=} (f'(q))^n f'(q) = (f'(q))^{n+1}$.

<u>Case 1:</u> Let $q \in (0, 1)$. Then, for all $k, n \in \mathbb{N}$,

$$P(1 \leq Z_n \leq k) = \sum_{j=1}^{k} P(Z_n = j) \leq \sum_{j=1}^{k} P(Z_n = j)\frac{jq^{j-1}}{q^k} \leq \frac{f_n'(q)}{q^k} = \frac{(f'(q))^n}{q^k}.$$

$$\Rightarrow \sum_{n=1}^{\infty} P(1 \leq Z_n \leq k) \;<\; \infty.$$

Borel–Cantelli lemma. $\Rightarrow$ $P(1 \leq Z_n \leq k\ \infty\text{-often}) = 0$, and the assertion follows for $q > 0$.

<u>Case 2:</u> Assume now that $q = 0$. Then, $p_0 = f(0) = f(q) = q = 0$ and hence $Z_1 \leq Z_2 \leq \cdots$ almost surely. For each $n \in \mathbb{N}$ it follows that

$$\begin{aligned}
P\Big(\bigcap_{m=n}^{\infty}\{Z_m = Z_{m+1}\}\Big) \;&=\; \lim_{N\to\infty} P\Big(\bigcap_{m=n}^{n+N-1}\{Z_m = Z_{m+1}\}\Big)\\
&= \lim_{N\to\infty}\sum_{k=0}^{\infty} P(k = Z_n = Z_{n+1} = \cdots = Z_{n+N})\\
&= \lim_{N\to\infty}\sum_{k=0}^{\infty} P(Z_n = k)\underbrace{p_1^k \cdots p_1^k}_{N-\text{times}} \quad (\text{since } Z_1 \leq Z_2 \leq \cdots \text{ a.s.})\\
&= \lim_{N\to\infty}\sum_{k=0}^{\infty} P(Z_n = k)(p_1^N)^k \;=\; \lim_{N\to\infty} f_n(p_1^N) \;\overset{p_1<1}{=}\; f_n(0).
\end{aligned}$$

Thus,

$$\begin{aligned}
P(Z_n = Z_{n+1} \text{ eventually}) \;&=\; P\Big(\bigcup_{n\in\mathbb{N}}\bigcap_{m=n}^{\infty}\{Z_{m+1} = Z_m\}\Big)\\
&= \lim_{n\to\infty} P\Big(\bigcap_{m=n}^{\infty}\{Z_m = Z_{m+1}\}\Big) \;=\; \lim_{n\to\infty} f_n(0) \;=\; q \;=\; 0,
\end{aligned}$$

i.e. $P(Z_n \to \infty) = P(Z_n < Z_{n+1}\ \infty\text{-often}) = 1 - P(Z_n = Z_{n+1} \text{ eventually}) = 1$. □

2.2.4 Def.
A GWP Z is called subcritical if $m < 1$, critical if $m = 1$ and supercritical if $m > 1$.

2.3 Critical case

Given: GWP $Z = (Z_n)_{n \in \mathbb{N}_0}$ with $Z_0 = 1$

It is assumed that $m = 1$ and $p_1 < 1$.

Known:

- $q := P(Z_n \to 0) = 1$
- $\mathrm{E}(Z_n) = m^n = 1$ for all $n \in \mathbb{N}_0$
- $\mathrm{Var}(Z_n) = n\sigma^2 \to \infty$

  Note that $p_1 < 1$ is equivalent to $\sigma^2 := \mathrm{Var}(Z_1) > 0$.

2.3.1 Lemma (Basic Lemma)
If $m = 1$ and $\sigma^2 \in (0, \infty)$ then

$$\lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{1 - f_n(s)} - \frac{1}{1-s}\right) = \frac{\sigma^2}{2}$$

uniformly for $s \in [0, 1)$.

2.3.2 Theorem (Yaglom Limit)
Let $m = 1$ and $\sigma^2 \in (0, \infty)$. Then

(a) $\displaystyle\lim_{n\to\infty} nP(Z_n > 0) = \frac{2}{\sigma^2},$ *(Kolmogorov, 1938)*

(b) $\displaystyle\lim_{n\to\infty} \mathrm{E}\left(\frac{Z_n}{n} \,\middle|\, Z_n > 0\right) = \frac{\sigma^2}{2},$ *and*

(c) Exponential limit law:

$$\lim_{n\to\infty} P\left(\frac{Z_n}{n} \le u \,\middle|\, Z_n > 0\right) = 1 - e^{-2u/\sigma^2},\ u \ge 0.$$ *(Yaglom, 1947)*

Rem.
Conditional on $Z_n > 0$, the r.v. Z_n/n converges in distribution to an exponential distribution with parameter $2/\sigma^2$.

Proof.
(of Lemma 2.3.1) Let $s \in [0,1)$.

Taylor expansion in 1: $f(s) = s + \frac{\sigma^2}{2}(1-s)^2 + r(s)(1-s)^2$ for a continuous function r with $\lim_{s\uparrow 1} r(s) = 0$. $\Rightarrow$

$$\begin{aligned}\frac{1}{1-f(s)} - \frac{1}{1-s} &= \frac{f(s)-s}{(1-f(s))(1-s)} = \frac{\frac{\sigma^2}{2}(1-s)^2 + r(s)(1-s)^2}{(1-f(s))(1-s)} \\ &= \frac{1-s}{1-f(s)}\left(\frac{\sigma^2}{2} + r(s)\right) = \frac{\sigma^2}{2} + \rho(s),\end{aligned}$$

where again ρ is continuous with $\lim_{s\uparrow 1} \rho(s) = 0$. Iteration yields

$$\begin{aligned}\frac{1}{n}\left(\frac{1}{1-f_n(s)} - \frac{1}{1-s}\right) &= \frac{1}{n}\sum_{j=0}^{n-1}\left(\frac{1}{1-f(f_j(s))} - \frac{1}{1-f_j(s)}\right) \\ &= \frac{\sigma^2}{2} + \frac{1}{n}\sum_{j=0}^{n-1}\rho(f_j(s)).\end{aligned}$$

The convergence $f_n(s) \to 1$ is uniform in $s \in [0,1)$, since $f_n(0) \le f_n(s) \le 1$ and $f_n(0) \to 1$. The result follows since ρ is bounded. □

Proof.
(of Theorem 2.3.2)

(a) $nP(Z_n > 0) = n(1-f_n(0)) = \left(\frac{1}{n}\left(\frac{1}{1-f_n(0)} - 1\right) + \frac{1}{n}\right)^{-1} \to \frac{2}{\sigma^2}$ by Lemma 2.3.1 (applied with $s = 0$).

(b) $\mathrm{E}\left(\frac{Z_n}{n} \,\middle|\, Z_n > 0\right) = \frac{\mathrm{E}(Z_n)}{n(1-f_n(0))} = \frac{1}{nP(Z_n > 0)} \overset{(a)}{\to} \frac{\sigma^2}{2}$.

(c) Let $u > 0$. Define $\beta := 2/\sigma^2$.

$$\begin{aligned}\mathrm{E}(e^{-uZ_n/n} \mid Z_n > 0) &= \frac{f_n(e^{-u/n}) - f_n(0)}{1-f_n(0)} = 1 - \frac{1-f_n(e^{-u/n})}{1-f_n(0)} \\ = 1 - \frac{1}{nP(Z_n>0)}&\left(\frac{1}{n}\left(\frac{1}{1-f_n(e^{-u/n})} - \frac{1}{1-e^{-u/n}}\right) + \frac{1}{n(1-e^{-u/n})}\right)^{-1}\end{aligned}$$

converges by (a) and Lemma 2.3.1 to

$$1 - \frac{1}{\beta}\left(\frac{1}{\beta} + \frac{1}{u}\right)^{-1} = 1 - \frac{1}{\beta}\frac{\beta u}{\beta + u} = \frac{\beta}{\beta + u},$$

where the uniform convergence in Lemma 2.3.1 is essential here.

The map $u \mapsto \frac{\beta}{\beta+u}$ is the Laplace transform (LT) of $\mathrm{Exp}(\beta)$. By the continuity theorem for LT the pointwise convergence of the LTs implies the convergence in distribution. □

2.4 Subcritical case

Given: GWP $Z = (Z_n)_{n\in\mathbb{N}_0}$ with $Z_0 = 1$ and $m := \mathrm{E}(Z_1) < \infty$

Taylor expansion of the pgf f of Z_1 at 1:

$$f(s) \;=\; 1 - m(1-s) + r(s)(1-s), \qquad s \in [0,1].$$

2.4.1 Lemma (Comparison Lemma)
For all $\delta \in (0,1)$,

$$\sum_{k=1}^{\infty} r(1-\delta^k) \;<\; \infty. \quad \iff \quad \sum_{k=1}^{\infty} p_k k \log k \;<\; \infty.$$

Rem.
The condition on the right-hand side is equivalent to $\mathrm{E}(Z_1 \log Z_1) < \infty$.

2.4.2 Theorem (Kolmogorov, 1938)
If $p_0 < 1$ and $m < 1$ then the limit

$$\varphi(0) \;:=\; \lim_{n\to\infty} \frac{P(Z_n > 0)}{m^n}$$

exists with $\varphi(0) = 0$ if $\mathrm{E}(Z_1 \log Z_1) = \infty$ and $\varphi(0) > 0$ otherwise.

Rem.
The theorem thus states that $P(Z_n > 0)$ behaves (up to a multiplicative constant) as m^n provided that $\mathrm{E}(Z_1 \log Z_1) < \infty$.

The following result shows convergence conditional on non-extinction.

2.4.3 Theorem (Convergence for subcritical GWPs, Yaglom, 1947)
If $p_0 < 1$ and $m < 1$ then for each $k \in \mathbb{N}$ the limit

$$b_k \;:=\; \lim_{n\to\infty} P(Z_n = k \,|\, Z_n > 0)$$

exists and $\sum_{k=1}^{\infty} b_k = 1$, i.e. $(b_k)_{k\in\mathbb{N}}$ defines a distribution on $\mathbb{N}$. The mean of this distribution is finite if and only if $\mathrm{E}(Z_1 \log Z_1) < \infty$ and in this case

$$\sum_{k=1}^{\infty} k b_k \;=\; \frac{1}{\varphi(0)}.$$

The pgf $g(s) := \sum_{k=1}^{\infty} b_k s^k$, $s \in [0,1]$, is a solution to the equation

$$g(f(s)) \;=\; 1 - m(1-g(s)), \qquad s \in [0,1].$$

Proof.
(of Lemma 2.4.1) For all $s \in [0,1)$,

$$\begin{aligned} r(s) &= m - \frac{1-f(s)}{1-s} = m - \sum_{j\geq 0} s^j \Big(1 - \sum_{k\geq 0} p_k s^k\Big) \\ &= m - \sum_{j\geq 0} s^j + \sum_{k\geq 0} p_k \sum_{j\geq 0} s^{j+k} = m - \sum_{n\geq 0} s^n + \sum_{k\geq 0} p_k \sum_{n\geq k} s^n \\ &= m - \sum_{n\geq 0} s^n + \sum_{n\geq 0} \Big(\sum_{k=0}^{n} p_k\Big) s^n = m - \sum_{n\geq 0} a_n s^n, \end{aligned}$$

where $a_n := 1 - \sum_{k=0}^{n} p_k = \sum_{k>n} p_k$, $n \in \mathbb{N}_0$. Note that $r(1) = 0$ and, hence, $\sum_{n\geq 0} a_n = m$. In particular, r is a nonnegative nonincreasing function on $[0,1]$. Define $\alpha := -\log\delta$. r nonincreasing. $\Rightarrow$ For $j \in \mathbb{N}$

$$\begin{aligned} r(1-\delta) + \int_1^j r(1-e^{-\alpha x})\,\mathrm{d}x &\geq \sum_{k=1}^{j} r(1-\delta^k) \\ &\geq \int_1^j r(1-e^{-\alpha x})\,\mathrm{d}x = \frac{1}{\alpha}\int_{1-\delta}^{1-\delta^j} \frac{r(s)}{1-s}\,\mathrm{d}s, \end{aligned}$$

where the last equality follows from the substitution $s := 1 - e^{-\alpha x}$. Thus,

$$\sum_{k\geq 1} r(1-\delta^k) < \infty \quad \Longleftrightarrow \quad \int_0^1 \frac{r(s)}{1-s}\,\mathrm{d}s < \infty.$$

But, for all $s \in [0,1)$,

$$\begin{aligned} \frac{r(s)}{1-s} &= \sum_{j\geq 0} s^j \Big(m - \sum_{n\geq 0} a_n s^n\Big) = m\sum_{j\geq 0} s^j - \sum_{n\geq 0} a_n \sum_{j\geq 0} s^{j+n} \\ &= m\sum_{k\geq 0} s^k - \sum_{n\geq 0} a_n \sum_{k\geq n} s^k = m\sum_{k\geq 0} s^k - \sum_{k\geq 0}\Big(\sum_{n=0}^{k} a_n\Big) s^k \\ &= \sum_{k\geq 0}\Big(m - \sum_{n=0}^{k} a_n\Big) s^k = \sum_{k\geq 0}\Big(\sum_{n>k} a_n\Big) s^k. \end{aligned}$$

Integration yields

$$\int_0^1 \frac{r(s)}{1-s}\,\mathrm{d}s = \int_0^1 \sum_{k\geq 0}\Big(\sum_{n>k} a_n\Big) s^k\,\mathrm{d}s = \sum_{k\geq 0}\sum_{n>k} \frac{a_n}{k+1} = \sum_{n\geq 1} a_n \sum_{k=0}^{n-1} \frac{1}{k+1}.$$

Since $\sum_{k=0}^{n-1} \frac{1}{k+1} \sim \log n$ as $n \to \infty$, this series converges if and only if the series $\sum_{n\geq 1} a_n \log n$ converges. Now,

$$\sum_{n\geq 1} a_n \log n = \sum_{n\geq 1}\Big(\sum_{k>n} p_k\Big)\log n = \sum_{k\geq 2} p_k \sum_{n=1}^{k-1} \log n.$$

Since $\sum_{n=1}^{k-1} \log n \sim \int_1^k \log x \, \mathrm{d}x = [x \log x - x]_1^k \sim k \log k$ as $k \to \infty$, this series converges if and only if $\sum_{k \geq 1} p_k k \log k < \infty$. $\square$

Proof.
(of Theorem 2.4.2) We have $\dfrac{1 - f(s)}{1 - s} = m - r(s)$. Replacing s by $f_k(s)$ yields

$$\frac{1 - f_{k+1}(s)}{1 - f_k(s)} = m\left(1 - \frac{r(f_k(s))}{m}\right)$$

and hence (taking products) $\dfrac{1 - f_n(s)}{1 - s} = m^n \prod_{k=0}^{n-1} \left(1 - \dfrac{r(f_k(s))}{m}\right)$.

$0 \leq r/m \leq 1$. $\Rightarrow$ $m^{-n}(1 - f_n(s))/(1 - s)$ is nonincreasing in n and hence converges to a limit $\varphi(s) \geq 0$. In particular (choose $s = 0$) $P(Z_n > 0) = 1 - f_n(0) \sim m^n \varphi(0)$. The well-known relation between convergence of sums and products shows that $\varphi(0) > 0$ if and only if $\sum_{k \geq 1} r(f_k(0)) < \infty$. Now, $1 - f(s) \leq m(1 - s)$ and, by induction, $1 - f_k(s) \leq m^k(1 - s)$ for all $k \in \mathbb{N}$. Similarly it follows that $1 - f_k(s) \geq (f'(s_0))^k(1 - s)$ for $s \geq s_0$ and with $s_0 = p_0 > 0$ it follows with the notation $a := f'(p_0) > 0$ that

$$1 - m^k \leq f_k(0) = f_{k-1}(p_0) \leq 1 - a^{k-1}(1 - p_0) \leq 1 - b^k,$$

where $b := a \wedge (1 - p_0)$. From Lemma 2.4.1 it follows that

$$\sum_{k \geq 1} r(f_k(0)) < \infty \iff \sum_{k \geq 1} p_k k \log k < \infty. \qquad \square$$

Proof.
(of Theorem 2.4.3) Define

$$\begin{aligned} g_n(s) &:= \mathrm{E}(s^{Z_n} \mid Z_n > 0) = \frac{f_n(s) - f_n(0)}{1 - f_n(0)} = 1 - \frac{1 - f_n(s)}{1 - f_n(0)} \\ &= 1 - (1 - s) \prod_{k=0}^{n-1} \frac{m - r(f_k(s))}{m - r(f_k(0))}. \end{aligned}$$

We have $f_k(s) \geq f_k(0)$ and r is nonincreasing. $\Rightarrow$ The fraction in the product is greater than or equal to 1. Thus, $g_n(s)$ is nonincreasing in n and, hence, converges to some $g(s)$. Obviously, $g_n(0) = 0$ and $g_n(1) = 1$ and, hence, $g(0) = 0$ and $g(1) = 1$. In order to verify that g is continuous at 1, it suffices (by the monotonicity of g and since $f_k(0) \to q = 1$) to verify that $\lim_{k \to \infty} g(f_k(0)) = 1$. We have

$$g_n(f_k(0)) = 1 - \frac{1 - f_n(f_k(0))}{1 - f_n(0)} = 1 - \frac{1 - f_k(f_n(0))}{1 - f_n(0)} \to 1 - m^k, \qquad n \to \infty.$$

Thus, $g(f_k(0)) = 1 - m^k$ and hence $\lim_{k\to\infty} g(f_k(0)) = \lim_{k\to\infty}(1-m^k) = 1$. Therefore, $g(1-) = 1$, so g is continuous at 1. By the continuity theorem for pgf's, all the limits

$$b_k \;:=\; \lim_{n\to\infty} P(Z_n = k \,|\, Z_n > 0), \qquad k \in \mathbb{N},$$

exist and g is as well a pgf of the form $g(s) = \sum_{k\geq 1} b_k s^k$. It follows that

$$\sum_{k\geq 1} kb_k \;=\; g'(1-) \;=\; \lim_{k\to\infty} \frac{1-g(f_k(0))}{1-f_k(0)} \;=\; \lim_{k\to\infty} \frac{m^k}{1-f_k(0)} \;\overset{\text{Thm. 2.4.2}}{=}\; \frac{1}{\varphi(0)}$$

and $g_n \circ f \;=\; 1 - \dfrac{1-f_{n+1}}{1-f_n(0)} \;=\; 1 - \dfrac{1-f_{n+1}}{1-f_{n+1}(0)}\dfrac{1-f(f_n(0))}{1-f_n(0)}$.

Letting $n \to \infty$ yields $g \circ f = 1 - (1-g)m$. □

2.5 Supercritical case

Given. $(Z_n)_{n\in\mathbb{N}_0}$ GWP with reproduction r.v. Z_1, where $m := \mathrm{E}(Z_1) \in (1,\infty)$ and $\sigma^2 := \mathrm{Var}(Z_1) \in (0,\infty]$. $q := P(Z_n \to 0)$ (extinction probability)

2.5.1 Theorem (Convergence Theorem for Supercritical GWPs)
Under the above assumptions there exist positive numbers $k_1, k_2, \ldots$ such that $W_n := k_n Z_n$ converges as $n \to \infty$ almost surely to a non-degenerate nonnegative real r.v. W. Moreover, $P(W = 0) = q$.

If $a \in [0, 1/m)$ then $a^n Z_n \to 0$ a.s..

If $a \in (1/m, \infty)$ then $a^n Z_n \to Z_\infty$ a.s., where $Z_\infty(\omega) := 0$ for $\omega \in \{Z_n \to 0\}$ and $Z_\infty(\omega) := \infty$ for $\omega \in \{Z_n \to \infty\}$.

If $\mathrm{E}(Z_1 \log Z_1) < \infty$ then one can choose $k_n := m^{-n}$.

If $\mathrm{E}(Z_1 \log Z_1) = \infty$ then $m^{-n} Z_n \to 0$ a.s. as $n \to \infty$.

Rem.
The numbers k_n, $n \in \mathbb{N}$, are called Seneta constants (Seneta, 1968). In particular, for $\mathrm{E}(Z_1 \log Z_1) < \infty$ and $k_n = m^{-n}$, one speaks of the Theorem of Kesten and Stigum (1966). Heyde (1970) has also provided important contributions to the convergence properties of supercritical GWPs.

2.5.2 Theorem (Characterization of the limit W)
The LT ψ of W is a solution to the equation

$$\psi(mu) = (f \circ \psi)(u), \quad u \geq 0.$$

The mean $\mathrm{E}(W) = -\psi'(0)$ *is finite if and only if* $\mathrm{E}(Z_1 \log Z_1) < \infty$ *and in this case there exists exactly one solution* ψ *of the above equation, which satisfies* $\psi(0) = 1$ *and whose derivative at* 0 *exists and is equal to a given value.*

Recapitulation.

$p_k := P(Z_1 = k)$, $k \in \mathbb{N}_0$.

pgf f of Z_1, $f(s) := \mathrm{E}(s^{Z_1}) = \sum_{k=0}^{\infty} p_k s^k$, $s \in [0,1]$.

Extinction probability $q := \lim_{n\to\infty} P(Z_n = 0) < 1$.

$q =$ smallest fixed point of f in the interval $[0,1]$.

Proof.
(of Theorem 2.5.1) f strictly increasing. $\Rightarrow g := f^{-1}$ exists.

Define $g_0 := \mathrm{id}$, $g_n := \underbrace{g \circ \cdots \circ g}_{n-\text{times}}$, $n \in \mathbb{N}$.

g is non-decreasing, concave, differentiable and maps $[q,1]$ to $[q,1]$.

Define $X_n(s) := (g_n(s))^{Z_n}$ and $\mathcal{F}_n := \mathcal{F}(Z_1, \ldots, Z_n)$, $s \in [q,1]$, $n \in \mathbb{N}_0$.

$X_n(s)$ is $\mathcal{F}_n$-measurable. Moreover, for $k \in \mathbb{N}_0$,

$$\begin{aligned}
\mathrm{E}\left(X_{n+1}(s) \,|\, Z_n = k\right) &= \mathrm{E}\left((g_{n+1}(s))^{Z_{n+1}} \,|\, Z_n = k\right) \\
&= \mathrm{E}\left((g_{n+1}(s))^{Y_{n1}+\cdots+Y_{nk}} \,|\, Z_n = k\right) \\
&= \mathrm{E}\left((g_{n+1}(s))^{Y_{n1}+\cdots+Y_{nk}}\right) \\
&= \mathrm{E}\left((g_{n+1}(s))^{Y_{n1}}\right) \cdots \mathrm{E}\left((g_{n+1}(s))^{Y_{nk}}\right) \\
&= \left(\mathrm{E}\left((g_{n+1}(s))^{Z_1}\right)\right)^k \\
&= (f(g_{n+1}(s)))^k \;=\; (g_n(s))^k.
\end{aligned}$$

$\Rightarrow \mathrm{E}(X_{n+1}(s) \,|\, Z_n) = (g_n(s))^{Z_n} = X_n(s)$ a.s.

$\Rightarrow \mathrm{E}(X_{n+1}(s) \,|\, \mathcal{F}_n) = X_n(s)$ a.s.

$\Rightarrow (X_n(s))_{n\in\mathbb{N}_0}$ is a nonnegative martingale w.r.t. $F := (\mathcal{F}_n)_{n\in\mathbb{N}_0}$.

$\Rightarrow X_\infty(s) := \lim\limits_{n\to\infty} X_n(s)$ exists a.s. (martingale convergence theorem)

Clear: $0 \le X_\infty(s) \le 1$, since $0 \le X_n(s) \le 1 \;\forall\, n \in \mathbb{N}_0$.

dominated convergence. $\Rightarrow \mathrm{E}(X_\infty(s)) = \mathrm{E}(X_1(s)) = \mathrm{E}(X_0(s)) = s$ a.s.

$\mathrm{E}(X_{n+1}^2(s) \,|\, \mathcal{F}_n) \ge (\mathrm{E}(X_{n+1}(s) \,|\, \mathcal{F}_n))^2 = X_n^2(s)$ a.s.

$\Rightarrow (X_n^2(s))_{n\in\mathbb{N}_0}$ submartingale w.r.t. F (again with values in $[0,1]$).

$\Rightarrow \mathrm{E}(X_\infty^2(s)) \ge \mathrm{E}(X_1^2(s)) > (\mathrm{E}(X_1(s)))^2$, since Z_1 is non-degenerate.

$\Rightarrow \mathrm{Var}(X_\infty(s)) \ge \mathrm{Var}(X_1(s)) > 0$.

Define $c_n(s) := -\log g_n(s)$, $Y(s) := -\log X_\infty(s)$.

$\Rightarrow$ $\boxed{c_n(s)Z_n \stackrel{a.s.}{\to} Y(s)}$ and $Y(s)$ is non-degenerate, which proves the first part of Theorem 2.5.1, except that it remains to verify that $Y(s)$ is a.s. finite.

$f(s) \le s$ for $s \in [q,1]$. $\Rightarrow$ $g(s) \ge s$ for $s \in [q,1]$. $\Rightarrow$ $g_n \nearrow g_\infty$ for some g_∞.

$s = f_n(g_n(s)) \le f_n(g_\infty(s)) \to q$, if $g_\infty(s) < 1$. $\Rightarrow$ $g_\infty(s) = 1$ for $s > q$.

Taylor expansion of f around 1 (as in the critical case). $\Rightarrow$ $1 - f(s) = (m - r(s))(1-s)$.

Replacing $s \in (q,1)$ by $g(s)$ yields

$$\frac{1-g(s)}{1-s} = \frac{1}{m - r(g(s))} = \frac{1}{m}\frac{1}{1 - \frac{r(g(s))}{m}}.$$

Repeating this and taking products. $\Rightarrow$

$$m^n(1 - g_n(s)) = \frac{1-s}{\prod_{k=1}^{n}\left(1 - \frac{r(g_k(s))}{m}\right)}. \qquad (*)$$

This tells us something on $c_n(s)$, since $-\log x \sim 1-x$ for $x \to 1$. In particular

$$\frac{c_n(s)}{c_{n-1}(s)} \sim \frac{1-g_n(s)}{1-g_{n-1}(s)} = \frac{1}{m}\frac{m^n(1-g_n(s))}{m^{n-1}(1-g_{n-1}(s))} = \frac{1}{m}\frac{1}{1-\frac{r(g_n(s))}{m}} \sim \frac{1}{m},$$

since $g_n(s) \to g_\infty(s) = 1$ and $r(1-) = 0$. Now use this to verify that $Y(s) := \lim_{n\to\infty} c_n(s)Z_n$ is a.s. finite.

Again, we have, with the notation $P(A \,|\, Z_1) := \mathrm{E}(1_A \,|\, Z_1) := \mathrm{E}(1_A \,|\, \mathcal{F}(Z_1))$

$$\begin{aligned}
P(Y(s) < \infty) &= \mathrm{E}(P(Y(s) < \infty \,|\, Z_1)) \\
&= \mathrm{E}\left(P\left(\lim_{n\to\infty} c_n(s)Z_n < \infty \,\middle|\, Z_1\right)\right) \\
&\stackrel{(+)}{=} \mathrm{E}\left(\left(P\left(\lim_{n\to\infty} c_n(s)Z_{n-1} < \infty\right)\right)^{Z_1}\right) \quad ((+) \text{ follows from} \\
&\qquad \text{the branching property, see Appendix 1)} \\
&= \mathrm{E}\left(\left(P\left(\lim_{n\to\infty} \frac{c_n(s)}{c_{n-1}(s)} c_{n-1}(s)Z_{n-1} < \infty\right)\right)^{Z_1}\right) \\
&= \mathrm{E}\left(\left(P\left(\frac{Y(s)}{m} < \infty\right)\right)^{Z_1}\right) \\
&= \mathrm{E}((P(Y(s) < \infty))^{Z_1}) = f(P(Y(s) < \infty)).
\end{aligned}$$

Analogous: $P(Y(s) = 0) = f(P(Y(s) = 0))$.

$\Rightarrow$ The probabilities $P(Y(s) < \infty)$ and $P(Y(s) = 0)$ are both fixed points of f and can hence only be equal to q or to 1. $Y(s)$ is non-degenerate. $\Rightarrow$ $P(Y(s) = 0) = q$.

$$s \;=\; \mathrm{E}(X_\infty(s)) \;=\; \mathrm{E}(e^{-Y(s)}) \;\leq\; P(Y(s) < \infty)$$

$\Rightarrow$ $\boxed{P(Y(s) < \infty) = 1}$ for $s > q$. We have

$$m^n c_n(s) \not\to \infty \;\overset{(*)}{\Longleftrightarrow}\; \prod_{n=1}^{\infty} \left(1 - \frac{r(g_n(s))}{m}\right) > 0 \;\Longleftrightarrow\; \sum_{n=1}^{\infty} r(g_n(s)) < \infty.$$

Choose $s_0 \in (q, 1)$ such that $m_0 := f'(s_0) > 1$ and k such that $g_k(s) \geq s_0$. Since

$$m_0^n (1-s) \leq 1 - f_n(s) \leq m^n (1-s) \;\forall\; s \in [s_0, 1],$$

it follows that

$$1 - m_0^{-(n-k)}(1 - g_k(s)) \leq g_n(s) \leq 1 - m^{-n}(1-s).$$

By the comparison lemma,

$$\lim_{n\to\infty} m^n c_n(s) < \infty. \;\Longleftrightarrow\; \mathrm{E}(Z_1 \log Z_1) < \infty. \qquad (**)$$

If $\mathrm{E}(Z_1 \log Z_1) < \infty$, then we can hence choose $k_n := m^{-n}$, since then $k_n Z_n = \frac{c_n(s) Z_n}{m^n c_n(s)}$ converges a.s..

If $\mathrm{E}(Z_1 \log Z_1) = \infty$, then $m^{-n} Z_n = \frac{c_n(s) Z_n}{m^n c_n(s)} \to 0$ a.s..

If $a < 1/m$, then always $a^n Z_n = \frac{(am)^n}{m^n c_n(s)} c_n(s) Z_n \to 0$ a.s..

Assume now that $a \in (1/m, \infty)$. Then,

$$\frac{c_n(s)}{a^n} \sim \frac{1 - g_n(s)}{a^n} \overset{(*)}{=} \frac{1-s}{\prod_{k=1}^{n} a(m - r(g_k(s)))}$$

and this expression has to converge to 0 for $s \in (q, 1)$, since

$$1 \;=\; m - r(q) \;\leq\; m - r(g_k(s)) \;\to\; m.$$

Therefore,

$$a^n Z_n = \frac{a^n}{c_n(s)} c_n(s) Z_n \;\to\; \begin{cases} 0 & \text{if } Z_n \to 0, \\ \infty & \text{if } Z_n \to \infty. \end{cases}$$

$\square$

Appendix 1 to (+): We have

$$\{\lim_{n\to\infty} c_n(s)Z_n < \infty\} \;=\; \{(Z_n)_{n>1} \in A_s\},$$

where

$$A_s \;:=\; \{a = (a_n)_{n>1} \in \mathbb{N}_0^\infty \,|\, \text{The sequence } (c_n(s)a_n)_{n>1} \text{ converges in } \mathbb{R}\}.$$

For $n > 1$ let $\pi_n : \mathbb{N}_0^\infty \to \mathbb{N}_0$ denote the projection to the n-th component, i.e. $\pi_n(a) = a_n$ for all $n > 1$ and $a = (a_n)_{n>1} \in \mathbb{N}_0^\infty$. Further, let $\mathcal{G} := \mathcal{F}(\pi_n, n > 1)$ denote the product-σ-algebra. Then,

$$\begin{aligned} A_s &= \{a = (a_n)_{n>1} \in \mathbb{N}_0^\infty \,|\, (c_n(s)a_n)_{n>1} \text{ is a Cauchy-sequence in } \mathbb{R}\} \\ &= \bigcap_{N\in\mathbb{N}} \bigcup_{n_0\in\mathbb{N}} \bigcap_{i,j>n_0} \{a = (a_n)_{n>1} \in \mathbb{N}_0^\infty \,|\, |c_i(s)\pi_i(a) - c_j(s)\pi_j(a)| < \tfrac{1}{N}\} \\ &= \bigcap_{N\in\mathbb{N}} \bigcup_{n_0\in\mathbb{N}} \bigcap_{i,j>n_0} (c_i(s)\pi_i - c_j(s)\pi_j)^{-1}((-\tfrac{1}{N}, \tfrac{1}{N})) \;\in\; \mathcal{G}, \end{aligned}$$

since, with π_i and π_j, also the map $c_i(s)\pi_i - c_j(s)\pi_j : \mathbb{N}_0^\infty \to \mathbb{R}$ is $\mathcal{G}$-$\mathcal{B}$-measurable.

Now, let $k \in \mathbb{N}_0$ and let $(Z_n^{(j)})_{n\in\mathbb{N}_0}$, $j \in \{1, \dots, k\}$, be independent GWPs, each distributed as $(Z_n)_{n\in\mathbb{N}_0}$. Define $\widetilde{Z}_n := \sum_{j=1}^k Z_n^{(j)}$. Then,

$$\begin{aligned} P(\lim_{n\to\infty} c_n(s)Z_n < \infty \,|\, Z_1 = k) &= P((Z_n)_{n>1} \in A_s \,|\, Z_1 = k) \\ &= P((\widetilde{Z}_n)_{n\in\mathbb{N}} \in A_s) \quad \text{(branching property, see before)} \\ &= P(\lim_{n\to\infty} c_n(s)\widetilde{Z}_{n-1} < \infty) \;=\; P(\sum_{j=1}^k \lim_{n\to\infty} c_n(s)Z_{n-1}^{(j)} < \infty) \\ &= P(\bigcap_{j=1}^k \{\lim_{n\to\infty} c_n(s)Z_{n-1}^{(j)} < \infty\}) \;=\; (P(\lim_{n\to\infty} c_n(s)Z_{n-1} < \infty))^k. \end{aligned}$$

Therefore,

$$P(\lim_{n\to\infty} c_n(s)Z_n < \infty \,|\, Z_1) \;=\; (P(\lim_{n\to\infty} c_n(s)Z_{n-1} < \infty))^{Z_1} \quad \text{a.s.}$$

and taking the mean yields (+).

Proof.
(of Theorem 2.5.2) Let $s \in (q, 1)$. Known (from the previous proof):

$\frac{c_{n+1}(s)}{c_n(s)} \to \frac{1}{m}$. With $n \to \infty$ it follows for $u \geq 0$

$$\begin{aligned} \psi(mu) &= \mathrm{E}(e^{-muY(s)}) \\ &\leftarrow \mathrm{E}(e^{-muc_{n+1}(s)Z_{n+1}}) \\ &= \mathrm{E}(\mathrm{E}(e^{-muc_{n+1}(s)Z_{n+1}} \mid Z_1)) \\ &= \mathrm{E}\left(\left(\mathrm{E}\left(e^{-um(\frac{c_{n+1}(s)}{c_n(s)}c_n(s)Z_n}\right)\right)^{Z_1}\right) \\ &\qquad \text{(follows from the branching property,} \\ &\qquad \text{see Theorem 2.1.6 and Corollary 2.1.7)} \\ &\to f(\psi(u)), \end{aligned}$$

where the theorem of dominated convergence was used several times.

The substitution $u \mapsto u/m$ and an application of $g := f^{-1}$ yields $\psi(u/m) = g(\psi(u))$. Since $\psi(u) \geq \lim_{u\to\infty} \psi(u) = \lim_{u\to\infty} \mathrm{E}(e^{-uY(s)}) = P(Y(s) = 0) = q$ one can iterate this to

$$1 - \psi(u/m^n) \; = \; 1 - g_n(\psi(u)) \; = \; O(m^{-n}) \stackrel{(**)}{\Longleftrightarrow} \mathrm{E}(Z_1 \log Z_1) < \infty.$$

ψ convex. $\Rightarrow$ The map $h \mapsto \frac{1-\psi(h)}{h}$ is non-increasing on $(0, \infty)$. $\Rightarrow$ The left-hand side above is equivalent to $\frac{1-\psi(h)}{h} = O(1)$, i.e. equivalent to the existence of the limit $\lim_{h\to 0} \frac{1-\psi(h)}{h} < \infty$, i.e. equivalent to the property, that $-\psi'(0)$ exists and is finite. This is well-known (see Appendix 2) to be equivalent to $\mathrm{E}(Y(s)) < \infty$ and in this case the equality $\mathrm{E}(Y(s)) = -\psi'(0)$ holds. Therefore, the second assertion follows.

To prove the uniqueness statement let ψ and ϕ be two solutions with $\psi(0) = \phi(0)$ finite and $\psi'(0) = \phi'(0)$ finite. Then, for any $u > 0$

$$\begin{aligned} |\psi(u) - \phi(u)| &= |f(\psi(u/m)) - f(\phi(u/m)| \; \leq \; m|\psi(u/m) - \phi(u/m)| \\ &\leq \cdots \leq m^n|\psi(u/m^n) - \phi(u/m^n)| \\ &= u\left|\frac{\psi(u/m^n) - \psi(0) - (\phi(u/m^n) - \phi(0))}{u/m^n}\right| \\ &\to u|\psi'(0) - \phi'(0)| \; = \; 0. \end{aligned}$$

□

Appendix 2

2.5.3 Lemma

Let X be a nonnegative real r.v. and $\psi : [0, \infty) \to (0, 1]$ the LT of X. Then, the mean $\mathrm{E}(X)$ is finite if and only if the derivative $\psi'(0)$ of ψ at 0 (in $\mathbb{R}$) exists. In this case, $\mathrm{E}(X) = -\psi'(0)$.

Proof.
'⇒': Let $\mathrm{E}(X) < \infty$. Define $f : [0,\infty) \to [0,\infty)$ via $f(u) := \mathrm{E}(Xe^{-uX})$ for all $u \in [0,\infty)$. Then, obviously, $-\psi$ is a antiderivative of f. By the mean value theorem there exists for each $h > 0$ a mean-value $\xi \in [0,h]$ with $1-\psi(h) = -\psi(h) - (-\psi(0)) = \int_0^h f(u)\,\mathrm{d}u = f(\xi)h$. Thus,

$$\frac{1-\psi(h)}{h} \;=\; f(\xi) \;=\; \mathrm{E}(Xe^{-\xi X}).$$

Letting $h \to 0$ (and hence also $\xi \to 0$) yields (on the left-hand side by the def. of the derivative of ψ in 0 and on the right-hand side by dominated convergence) $-\psi'(0) = \mathrm{E}(X) < \infty$.

'⇐': Conversely, assume that $-\psi'(0) < \infty$. Then, ψ' is defined on the full interval $[0,\infty)$ and

$$-\psi'(u) \;=\; \mathrm{E}(Xe^{-uX}), \qquad u \in [0,\infty).$$

Letting $u \to 0$ yields (on the left-hand side by the def. of the right-sided limit and on the right-hand side by dominated convergence)

$$-\psi'(0+) = \mathrm{E}(X).$$

On the other hand, the map $u \mapsto -\psi'(u) = \mathrm{E}(Xe^{-uX})$ is non-increasing on $[0,\infty)$. Thus,

$$-\psi'(u) \;\le\; -\psi'(0).$$

Letting $u \to 0$ yields $\mathrm{E}(X) = -\psi'(0+) \le -\psi'(0) < \infty$. □