1 Introduction

Main parts of probability theory deal with stochastic processes.

1.1 Stochastic processes

1.1.1 Def. (Stochastic Process)
Let T be some index set and $(S, \mathcal{S})$ a measure space. A family $(X_t)_{t \in T}$ is called a stochastic process on a probability space $(\Omega, \mathcal{F}, P)$, if each $X_t : \Omega \to S$ is a random variable (r.v.), i.e. $\{X_t \in B\} := X^{-1}(B) := \{\omega \in \Omega : X_t(\omega) \in B\} \in \mathcal{F}$ for all $B \in \mathcal{S}$ and $t \in T$.

The set S is called the state space of the process. In many cases X_t is interpreted as the random state of a stochastic system at 'time' t.

For any selection of finitely many time points $t_1, \ldots, t_k \in T$, the joint distribution of $(X_{t_1}, \ldots, X_{t_k})$ is called a finite-dimensional distribution of the process $(X_t)_{t \in T}$.

We are often dealing with stochastic processes with $T = \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ and $S \subseteq \mathbb{N}_0$.

The two most popular classes of stochastic processes are Markov chains and martingales.

1.2 Markov chains

1.2.1 Def. (Markov Chain)
A stochastic process $(X_n)_{n \in \mathbb{N}_0}$ with discrete state space S is called a Markov chain (MC) if it has Markov property (MP)

$$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) \;=\; P(X_{n+1} = j \mid X_n = i)$$

for all $n \in \mathbb{N}_0$, $i_0, \ldots, i_{n-1}, i, j \in S$ (as long as the conditional probabilities are defined).

The MP states that, if the current state is exactly known ($X_n = i$), then the additional knowledge of the past $(X_0, \ldots, X_{n-1})$ does not influence the probabilistic structure of the future (X_{n+1}).

The probabilities

$$\pi_{ij}(n) \;:=\; P(X_{n+1} = j \mid X_n = i), \quad n \in \mathbb{N}_0, i, j \in S,$$

are called the transition probabilities of the MC. If all the $\pi_{ij}(n) = \pi_{ij}$ do not depend on n, then $(X_n)_{n \in \mathbb{N}_0}$ is called homogeneous. We mainly deal with homogeneous Markov chains (HMC).

A state $i \in S$ is called absorbing if $\pi_{ii} = 1$.

$\Pi := (\pi_{ij})_{i,j\in S}$, the transition matrix of the MC, is a stochastic matrix, i.e. $\pi_{ij} \geq 0$ for all $i, j \in S$ and $\sum_{j\in S} \pi_{ij} = 1$ for all $i \in S$.

The n-step transition probabilities $\pi_{ij}^{(n)} := P(X_{m+n} = j \,|\, X_m = i)$, $n \in \mathbb{N}_0$, $i, j \in S$, satisfy the Chapman–Kolmogorov equations $\pi_{ij}^{(n)} = \sum_{k\in S} \pi_{ik}^{(r)} \pi_{kj}^{(s)}$ for all $n, r, s \in \mathbb{N}_0$ with $r + s = n$. It follows that the matrix $(\pi_{ij}^{(n)})_{i,j\in S}$ of the n-step transition probabilities coincides with the nth power Π^n of Π.

It can be shown that, for each $i \in S$,

$$P(\text{The chain visits } i \text{ infinitely often} \,|\, X_0 = i) \;\in\; \{0, 1\}.$$

If this probability is 0 then the state i is called transient, otherwise recurrent.

In this lecture we will meet several examples of MCs.

1.3 Martingales

1.3.1 Def. (Martingale)
A family $(X_n)_{n\in\mathbb{N}_0}$ of real-valued r.v. is called a martingale, if the following three conditions hold.

(i) *Each X_n is integrable, i.e.* $\mathrm{E}(|X_n|) < \infty$ *for all* $n \in \mathbb{N}_0$.

(ii) *Each X_n is measurable w.r.t. the σ-algebra* $\mathcal{F}_n := \mathcal{F}(X_0, \ldots, X_n)$.

(iii) $\mathrm{E}(1_D X_{n+1}) = \mathrm{E}(1_D X_n)$ *for all* $n \in \mathbb{N}_0$ *and* $D \in \mathcal{F}_n$.

In particular, $\mathrm{E}(X_n)$ does not depend on n. In this sense $(X_n)_{n\in\mathbb{N}_0}$ can be interpreted as a fair game (in the mean).

1.3.2 Exercise
If $(X_n)_{n\in\mathbb{N}_0}$ is a MC with state space $S \subseteq \mathbb{R}$, then (iii) can be replaced by

(iv) $\mathrm{E}(X_{n+1} \,|\, X_n = i) = i$ *for all* $n \in \mathbb{N}_0$ *and* $i \in S$ *with* $P(X_n = i) > 0$.

The following convergence theorem is one of the most important results in probability theory.

1.3.3 Theorem
Let $(X_n)_{n\in\mathbb{N}_0}$ be a martingale. If $\sup_{n\in\mathbb{N}_0} \mathrm{E}(|X_n|) < \infty$, *then X_n converges almost surely as $n \to \infty$ to an integrable real-valued limiting r.v. X_∞.*

Proof.
See, for example, Bauer, Probability Theory. □