THE RATE OF CONVERGENCE
OF THE BLOCK COUNTING PROCESS
OF EXCHANGEABLE COALESCENTS WITH DUST

Martin Möhle, University of Tübingen, Germany

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Exchangeable coalescents

- Exchangeable coalescents are Markov processes $\Pi = (\Pi_t)_{t \geq 0}$ with state space $\mathcal{P}$, the set of partitions (equivalence relations) on $\mathbb{N} := \{1, 2, \ldots\}$.

- During each transition, blocks (equivalence classes) merge together. Simultaneous multiple collisions of blocks are allowed.

- Schweinsberg (2000) characterizes exchangeable coalescents via a finite measure $\Xi$ on the infinite simplex

$$\Delta := \{u = (u_1, u_2, \ldots) : u_1 \geq u_2 \geq \cdots \geq 0, \ |u| := \sum_{r \in \mathbb{N}} u_r \leq 1\}.$$

- These processes are therefore also called $\Xi$-coalescents.

- The subclass of $\Lambda$-coalescents is obtained if $\Xi$ is concentrated on $\{u \in \Delta : u_2 = 0\}$. In this case $\Lambda(B) = \Xi(B \times \{0\} \times \{0\} \times \cdots)$ for all Borel sets $B \subseteq [0, 1]$. 
An urn model, Kingman’s paintbox

Fix $u = (u_1, u_2, \ldots) \in \Delta$. Note that $|u| := \sum_{r \in \mathbb{N}} u_r \leq 1$. Define $u_0 := 1 - |u|$. Imagine a countable infinite number of boxes having labels $r \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$.

<table>
<thead>
<tr>
<th>box 0</th>
<th>box 1</th>
<th>box 2</th>
<th>box 3</th>
<th>box 4</th>
<th>box 5</th>
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<tbody>
<tr>
<td>0</td>
<td>$u_0$</td>
<td>$u_1$</td>
<td>$u_2$</td>
<td>$u_3$</td>
<td>$u_4$</td>
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Balls are allocated successively to these boxes. It is assumed that every ball goes to box $r \in \mathbb{N}_0$ with probability $u_r$ independently of the other balls.

Let $X_r(i, u)$ be the number of balls in box $r$ after $i \in \mathbb{N}$ balls have been allocated.

Then $(X_0(i, u), X_1(i, u), \ldots)$ has an infinite multinomial distribution with parameters $i$ and $(u_0, u_1, u_2, \ldots)$. 
The block counting process

Let \( N_t^{(n)} \) denote the number of blocks of \( \Pi_t^{(n)} \), the restriction of \( \Pi_t \) to a sample of size \( n \).

The block counting process \( N^{(n)} := (N_t^{(n)})_{t \geq 0} \) moves from state \( i \) to state \( j < i \) at the rate

\[
q_{ij} = \Xi(\{0\}) \binom{i}{2} \delta_{j,i-1} + \int_{\Delta} \mathbb{P}(Y(i,u) = j) \nu(du)
\]

where \( \nu(du) := \Xi(du)/(u,u) \) with \( (u,u) := \sum_{r \in \mathbb{N}} u_r^2 \) and

\[
Y(i,u) := \text{number of balls in box 0 plus number of other non-empty boxes}
\]

\[
= X_0(i,u) + \sum_{r \in \mathbb{N}} 1\{X_r(i,u) > 0\}.
\]

Remark. Note that \( \mathbb{P}(Y(i,u) = j) = \sum_{k=1}^{j} f_{ijk}(u) \), where

\[
f_{ijk}(u) := \frac{u_0^{i-k}}{(j-k)!} \sum_{i_1,\ldots,i_k \geq 1 \atop i_1+\cdots+i_k = i-j+k} \frac{i!}{i_1! \cdots i_k!} \sum_{1 \leq r_1 < \cdots < r_k} u_{r_1}^{i_1} \cdots u_{r_k}^{i_k}.
\]
The fixation line (Hénard (2015), Gaiser and M. (2016))

For $n \in \mathbb{N}$ and $t \geq 0$ define $L_t^{(n)} := \sup \{k \in \mathbb{N} : N_t^{(k)} \leq n \}$. The fixation line $L_t^{(n)} := (L_t^{(n)})_{t \geq 0}$ moves from state $i$ to state $j > i$ at the rate

$$
\gamma_{ij} = \Xi(\{0\}) \binom{j}{2} \delta_{j,i+1} + \int_\Delta \mathbb{P}(Y(j,u) = i, Y(j + 1, u) = i + 1) \nu(du)
$$

with $Y(., u)$ and $\nu$ as before.

**Remark.** The probability below the integral can be provided explicitly as

$$
\mathbb{P}(Y(j,u) = i, Y(j + 1, u) = i + 1) = \sum_{k=1}^{i} g_{ijk}(u), \text{ where}
$$

$$
g_{ijk}(u) := \frac{u_0^{i-k}}{(i-k)!} \sum_{i_1, \ldots, i_k \geq 1 \atop i_1 + \cdots + i_k = j-i+k} \frac{j!}{i_1! \cdots i_k!} \sum_{1 \leq r_1 < \cdots < r_k} u_{r_1}^{i_1} \cdots u_{r_k}^{i_k} (1 - (u_{r_1} + \cdots + u_{r_k})).\]
Siegmund duality

Let $\Pi$ be a $\Xi$-coalescent and let $N^{(n)} = (N_t^{(n)})_{t \geq 0}$ and $L^{(n)} = (L_t^{(n)})_{t \geq 0}$ denote the block counting process and the fixation line of $\Pi$ respectively.

**Theorem 1.** (Gaiser and M., 2016)

The block counting process is Siegmund dual to the fixation line, that is

$$\mathbb{P}(N_t^{(i)} \leq j) = \mathbb{P}(L_t^{(j)} \geq i)$$

for all $i, j \geq 1$ and $t \geq 0$. 

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Exchangeable coalescents with dust

**Definition.** A $\Xi$-coalescent $\Pi = (\Pi_t)_{t \geq 0}$ has proper frequencies (no dust) if, for all times $t \geq 0$, the frequency $S_t$ of singletons of $\Pi_t$ satisfies $S_t = 0$ almost surely.

**Proposition 1.** (Schweinsberg, 2000)

A coalescent $\Pi$ has dust if and only if $\Xi(\{0\}) = 0$ and
\[
\int_\Delta |u| \nu(du) < \infty,
\]
where $|u| := \sum_{r \in \mathbb{N}} u_r$ and $\nu(du) := \Xi(du)/(u, u)$ with $(u, u) := \sum_{r \in \mathbb{N}} u_r^2$.

**Remark.** If $\Pi$ has dust then $Z = (Z_t)_{t \geq 0} := (-\log S_t)_{t \geq 0}$ is a drift-free subordinator (Lévy process with non-decreasing paths) with Laplace exponent
\[
\Phi(q) := \int_\Delta (1 - (1 - |u|)^q) \nu(du), \quad q \geq 0.
\]
Note that $\mathbb{E}(S_t^q) = \mathbb{E}(e^{-qZ_t}) = e^{-t\Phi(q)}, t, q \geq 0$. 
Asymptotics for large sample size

**Theorem 2.** (Gaiser and M., 2016)

Let \( \Pi \) be a \( \Xi \)-coalescent with dust, i.e. \( \Xi(\{0\}) = 0 \) and \( \int_{\Delta} |u| \nu(du) < \infty \). Then the following two assertions hold.

a) As \( n \to \infty \) the scaled block counting process \( (N_t^{(n)}/n)_{t \geq 0} \) converges in \( D_{[0,1]}[0, \infty) \) to the frequency of singleton process \( S = (S_t)_{t \geq 0} = (e^{-Z_t})_{t \geq 0} \).

b) As \( n \to \infty \) the scaled fixation line \( (L_t^{(n)}/n)_{t \geq 0} \) converges in \( D_{[1,\infty]}[0, \infty) \) to the reciprocal frequency of singleton process \( (1/S_t)_{t \geq 0} = (e^{Z_t})_{t \geq 0} \).
Remarks

- Proof of part a) based on the method of moments and some general weak convergence machinery. Proof of part b) uses duality.

- For $\Xi$-coalescents with dust, both processes $(\log n - \log N_t^{(n)})_{t \geq 0}$ and $(\log L_t^{(n)} - \log n)$ converge in $D_{[0,\infty]}[0,\infty)$ to the drift-free subordinator $Z$.

- Let $\Xi$ be concentrated on $\Delta^* := \{u \in \Delta : |u| = 1\}$. Then $\Pi$ has dust if and only if $\nu$ is finite. In this case $S_t \overset{d}{=} 1\{T_f > t\}$, where $T_f$ is exponentially distributed with parameter $\nu(\Delta^*)$. Examples are Dirichlet coalescents and Poisson–Dirichlet coalescents.
A Bernstein function

Let \( \Pi \) be a \( \Xi \)-coalescent with dust. For \( q \geq 0 \) define

\[
\tilde{\Phi}(q) := \int_\Delta \sum_{r \in \mathbb{N}} (1 - (1 - u_r)^q) \nu(du) .
\]

Properties:

- In general \( \tilde{\Phi} \) differs from \( \Phi \). For \( \Lambda \)-coalescents \( \tilde{\Phi} \) coincides with \( \Phi \).
- \( \tilde{\Phi}(0) = 0, \tilde{\Phi}(1) = \int_\Delta |u| \nu(du), \Phi(n) \leq \tilde{\Phi}(n) \) for all \( n \in \mathbb{N} \)
- \( \tilde{\Phi} \) is a Bernstein function (infinitely often differentiable on \( (0, \infty) \) with \( (-1)^{k-1} \tilde{\Phi}^{(k)} \geq 0 \) for all \( k \in \mathbb{N} \) and \( q > 0 \)).
- Lévy–Khintchine representation:

\[
\tilde{\Phi}(q) = \int_{(0,1]} (1 - (1 - x)^q) \tilde{\nu}(dx) ,
\]

where \( \tilde{\nu}(B) := \int_\Delta \sum_{r \in \mathbb{N}} 1_B(u_r) \nu(du) \) for all Borel sets \( B \subseteq (0,1] \).
Rate of convergence

Notation: \( E := [0, 1], E_n := \{ k/n : k \in \{1, \ldots, n\} \}. \)
\[
\pi_n : B(E) \to B(E_n), \pi_n f(x) := f(x) \text{ for } f \in B(E) \text{ and } x \in E_n.
\]

**Theorem 3.** (Rate of convergence of the block counting process, M., 2019)
Let \( \Pi = (\Pi_t)_{t \geq 0} \) be a \( \Xi \)-coalescent with dust and let \( A_n \) and \( A \) denote the generators of the scaled block counting process \( (N_t^{(n)}/n)_{t \geq 0} \) and the frequency of singleton process \( (S_t)_{t \geq 0} \) respectively. Then, for all \( n \in \mathbb{N} \) and \( f \in C^2([0, 1]) \),

\[
\|A_n \pi_n f - \pi_n A f\| := \sup_{x \in E_n} |A_n \pi_n f(x) - \pi_n A f(x)| \leq C_f r(n),
\]

where \( C_f := \|f'\| + 2\|f''\| \) and \( r(n) := \frac{\tilde{\Phi}(n)}{n} \).

We call \( r(n) \) the rate function.

Remark: The rate \( r(n) \) in Theorem 3 is optimal.
Sketch of proof

Define $u_0 := 1 - |u|$. Proof of Theorem 3 uses the paintbox representation

$$A_n^\pi_n f(x) - \pi_n Af(x) = \int_\Delta \left( \mathbb{E} \left( f \left( \frac{Y(nx, u)}{n} \right) \right) - f(xu_0) \right) \nu(du),$$

the Taylor expansion

$$f \left( \frac{Y(nx, u)}{n} \right) - f(xu_0) = f'(xu_0)x\tilde{Y} + f''(\xi)x^2\tilde{Y}^2,$$

for some random point $\xi$ taking values between $Y(nx, u)/n$ and $xu_0$, and the concentration inequality $\mathbb{E}(\tilde{Y}^2) \leq 2\mathbb{E}(\tilde{Y})$, where

$$\tilde{Y} := \tilde{Y}(nx, u) := \frac{Y(nx, u)}{nx} - u_0.$$

**Remark.** For $\Lambda$-coalescents the sharper concentration inequality $\mathbb{E}(\tilde{Y}^2) \leq \mathbb{E}(\tilde{Y})$ holds. We conjecture that this sharper inequality holds for all $\Xi$-coalescents.
Corollary 1. (Rate of convergence, semigroup version, M., 2019)

In the situation of Theorem 3, let \( (T_t^{(n)})_{t \geq 0} \) and \( (T_t)_{t \geq 0} \) denote the semigroups of the scaled block counting process and the frequency of singleton process \( (S_t)_{t \geq 0} \) respectively. Then, for all \( t \geq 0, n \in \mathbb{N}, \) and \( f \in C^2(E), \)

\[
\|T_t^{(n)} \pi_n f - \pi_n T_t f\| \leq t C_f r(n),
\]

where \( C_f \) is the constant from Theorem 3 and the rate \( r(n) \) is defined as before.
Rate of convergence of the fixation line

Notation: \( F := [1, \infty) \), \( F_n := \{k/n : k \in \{n, n+1, \ldots\}\} \cup \{\infty\} \).

\[ \tau_n : B(F') \to B(F_n), \tau_n g(y) := g(y) \text{ for } g \in B(F) \text{ and } y \in F_n. \]

**Conjecture.** (Rate of convergence of the fixation line; work in progress)

Let \( \Pi \) be a \( \Xi \)-coalescent with dust and let \( B_n \) and \( B \) denote the generators of the scaled fixation line \( (L_t^{(n)}/n)_{t \geq 0} \) and the reciprocal frequency of singleton process \( (1/S_t)_{t \geq 0} \) respectively. Then, for all \( n \in \mathbb{N} \) and \( g \in C^2_c([1, \infty]) \),

\[
\|B_n\tau_n g - \tau_n B g\| := \sup_{y \in F_n} |B_n\tau_n g(y) - \tau_n B g(y)| \leq D_g r(n)
\]

with rate \( r(n) \) as before and constant \( D_g := \|f'\| + 2\|f''\| \), where \( f(x) := g(1/x) \).

**Remark.** Conjecture holds for \( \Lambda \)-coalescents, even with improved constant \( D_g := \|f'\| + \|f''\| \). Some technical gaps in the proof for the \( \Xi \)-coalescent.
Example 1: Dirichlet coalescent

Let \((X_1, \ldots, X_N) \overset{d}{=} D_N(\alpha)\) be symmetric Dirichlet distributed with parameters \(N \in \mathbb{N}\) and \(\alpha > 0\).

Let \(X_{(1)} \geq \cdots \geq X_{(N)}\) denote the order statistics.

\(\nu := \text{distribution of} \ (X_{(1)}, \ldots, X_{(N)}, 0, 0, \ldots)\).

The associated exchangeable coalescent is called the Dirichlet coalescent.

This coalescent neither comes down from infinity nor stays infinite, since
\[
\mathbb{P}(N_t = \infty) = \mathbb{P}(T_f > t) = e^{-t} \text{ for all } t \geq 0.
\]
Dirichlet coalescent (continued)

Notation. $[x|y]_n := \prod_{k=0}^{n-1} (x + ky)$, $(x|y)_n := \prod_{k=0}^{n-1} (x - ky)$,

$[x]_n := [x|1]_n$, $(x)_n := (x|1)_n$

Rates of the block counting process:

\[ q_{ij} = \frac{(N\alpha|\alpha)_j}{[N\alpha]_i} S_\alpha(i, j), j < i \]

Rates of the fixation line:

\[ \gamma_{ij} = \frac{(N\alpha|\alpha)_{i+1}}{[N\alpha]_{j+1}} S_\alpha(j, i), i < j \]

$S_\alpha(i, j) := S(i, j; -1, \alpha, 0)$ is the generalized Stirling number as defined in Hsu and Shiue (1998).
Define $\Delta_N := \{u \in \Delta : u_1 + \cdots + u_N = 1\}$. Then

$$\tilde{\Phi}(q) = \int_{\Delta_N} \sum_{r=1}^{N} (1 - (1 - u_r)^q) \nu(du)$$

$$= \int_{\mathbb{R}^N} \sum_{r=1}^{N} (1 - (1 - u_r)^q) D_N(\alpha)(du_1, \ldots, du_N)$$

$$= N \mathbb{E}(1 - (1 - X_1)^q),$$

If $N = 1$ then $X_1 \equiv 1$ and $\tilde{\Phi} = \Phi$. If $N > 1$ then $X_1$ is beta distributed with parameters $\alpha$ and $N \alpha - \alpha$ and

$$\tilde{\Phi}(q) = N \left(1 - \frac{\Gamma(N\alpha)\Gamma(N\alpha - \alpha + q)}{\Gamma(N\alpha - \alpha)\Gamma(N\alpha + q)}\right) \sim N, \quad q \to \infty,$$

differs from $\Phi(q) = 1, q > 0$. 

Dirichlet coalescent (continued)
Example 2: Poisson–Dirichlet coalescent
(Sagitov (2003), M. (2010), Gaiser and M. (2016))

This is the coalescent where $\nu$ is the Poisson–Dirichlet distribution with parameters $0 \leq \alpha < 1$ and $\theta > -\alpha$.

Rates of the block counting process:

$$ q_{ij} = c_{j,\alpha,\theta} \frac{\Gamma(\theta + \alpha j)}{\Gamma(\theta + i)} s_\alpha(i, j), \ j < i $$

Normalizing constant: $c_{j,\alpha,\theta} := \prod_{k=1}^{j} \frac{\Gamma(\theta + 1 + (k - 1)\alpha)}{\Gamma(\theta + k\alpha)}$

Rates of the fixation line:

$$ \gamma_{ij} = c_{i,\alpha,\theta} \frac{\Gamma(\theta + \alpha i + 1)}{\Gamma(\theta + j + 1)} s_\alpha(j, i), \ i < j $$

$s_\alpha(i, j) := S(i, j; -1, -\alpha, 0)$ is the generalized absolute Stirling number of the first kind as defined in Hsu and Shiue (1998).
Poisson–Dirichlet coalescent (continued)

By a result of Handa (2009), applied with \( f(x) := 1 - (1 - x)^q \),

\[
\tilde{\Phi}(q) = \int_{\Delta} \sum_{r \in \mathbb{N}} f(u_r) \nu(du) = \int_{\mathbb{R}} (1 - (1 - x)^q) \mu_1(dx), \quad q \geq 0,
\]

where \( \mu_1 \) denotes the correlation measure associated with the Poisson–Dirichlet coalescent. The density of \( \mu_1 \) is explicitly known (see Handa, 2009), and it follows that

\[
\tilde{\Phi}(q) = c_{1,\alpha,\theta} \int_0^1 (1 - (1 - x)^q)x^{-\alpha-1}(1 - x)^{\theta+\alpha-1}dx, \quad q \geq 0,
\]

with normalizing constant \( c_{1,\alpha,\theta} := B(1 - \alpha, \theta + \alpha) \). For \( \alpha > 0 \) this leads to

\[
\tilde{\Phi}(q) = \frac{\theta + q \Gamma(\theta + \alpha + q)\Gamma(\theta + 1)}{\alpha \Gamma(\theta + 1 + q)\Gamma(\theta + \alpha)} - \frac{\theta}{\alpha} \sim \frac{\Gamma(\theta + 1) \; q^\alpha}{\Gamma(\theta + \alpha) \; \alpha}, \quad q \to \infty.
\]

For \( \alpha = 0 \) it follows that \( \tilde{\Phi}(q) = \theta(\Psi(q + \theta) - \Psi(\theta)) \sim \theta \log q \) as \( q \to \infty \), where \( \Psi := \Gamma'/\Gamma \). The associated subordinator is the \( \Psi \)-subordinator. In all cases \( \tilde{\Phi} \) differs from \( \Phi \).
Example 3: A symmetric coalescent

Let \((m_k)_{k \in \mathbb{N}}\) be a sequence of non-negative real numbers satisfying \(\sum_{k \in \mathbb{N}} m_k/k < \infty\). Suppose \(\nu\) assigns for each \(k \in \mathbb{N}\) mass \(m_k\) to 
\[u^{(k)} := (1/k, \ldots, 1/k, 0, 0, \ldots) \in \Delta^*\].
This coalescent occurs in González Casanova, Miró Pina and Siri-Jégousse (2019).

Rates of the block counting process:
\[q_{ij} = S(i, j) \sum_{k \in \mathbb{N}} \frac{(k)^j}{k^i} m_k, \; j < i\]

Rates of the fixation line:
\[\gamma_{ij} = S(j, i) \sum_{k \in \mathbb{N}} \frac{(k)_{i+1}}{k^{j+1}} m_k, \; i < j\]

\([x]^i := x(x - 1) \cdots (x - i + 1), \; S(., .) \) are the Stirling number of the second kind]
A symmetric coalescent (continued)

The symmetric coalescent has dust if and only if $\nu(\Delta) = \sum_{k \in \mathbb{N}} m_k < \infty$. In this case

$$\tilde{\Phi}(q) = \sum_{k \in \mathbb{N}} km_k \left(1 - \left(1 - \frac{1}{k}\right)^q\right), \quad q \geq 0.$$ 

For example, if $m_k = k^{-\alpha}$ with $\alpha > 1$, then, as $q \to \infty$,

$$\tilde{\Phi}(q) \sim \begin{cases} \zeta(\alpha - 1) & \text{if } \alpha > 2, \\ \log q & \text{if } \alpha = 2, \\ -\Gamma(\alpha - 2)q^{2-\alpha} & \text{if } \alpha \in (1, 2). \end{cases}$$

$$r(n) = O\left(\frac{1}{n}\right) \text{ for } \alpha > 2, \quad r(n) = O\left(\frac{\log n}{n}\right) \text{ for } \alpha = 2, \quad r(n) = O\left(\frac{1}{n^{\alpha-1}}\right) \text{ for } \alpha \in (1, 2).$$
Thank you very much for your attention!
References I


References II


