THE RATE OF CONVERGENCE
OF THE BLOCK COUNTING PROCESS
OF EXCHANGEABLE COALESCENTS WITH DUST

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Exchangeable coalescents

- **Exchangeable coalescents** are Markov processes \( \Pi = (\Pi_t)_{t \geq 0} \) with state space \( \mathcal{P} \), the set of partitions (equivalence relations) on \( \mathbb{N} := \{1, 2, \ldots\} \).

- During each transition, blocks (equivalence classes) merge together. **Simultaneous multiple collisions** of blocks are allowed.

- Schweinsberg (2000) characterizes exchangeable coalescents via a finite measure \( \Xi \) on the infinite simplex

\[
\Delta := \{ u = (u_1, u_2, \ldots) : u_1 \geq u_2 \geq \cdots \geq 0, |u| := \sum_{r \in \mathbb{N}} u_r \leq 1 \}.
\]

- These processes are therefore also called \( \Xi \)-coalescents.

- The subclass of \( \Lambda \)-coalescents is obtained if \( \Xi \) is concentrated on \( \{ u \in \Delta : u_2 = 0 \} \). In this case \( \Lambda(B) = \Xi(B \times \{0\} \times \{0\} \times \cdots) \) for all Borel sets \( B \subseteq [0, 1] \).
An urn model, Kingman’s paintbox

Fix \( u = (u_1, u_2, \ldots) \in \Delta \). Note that \(|u| := \sum_{r \in \mathbb{N}} u_r \leq 1\). Define \( u_0 := 1 - |u| \).

Imagine a countable infinite number of boxes having labels \( r \in \mathbb{N}_0 := \{0, 1, 2, \ldots\} \).

\[
\begin{array}{ccccccc}
\text{box 0} & \text{box 1} & \text{box 2} & \text{box 3} & \text{box 4} & \text{box 5} \\
0 & u_0 & u_1 & u_2 & u_3 & u_4 & u_5 & 1
\end{array}
\]

Balls are allocated successively to these boxes. It is assumed that every ball goes to box \( r \in \mathbb{N}_0 \) with probability \( u_r \) independently of the other balls.

Let \( X_r(i, u) \) be the number of balls in box \( r \) after \( i \in \mathbb{N} \) balls have been allocated.

Then \( (X_0(i, u), X_1(i, u), \ldots) \) has an infinite multinomial distribution with parameters \( i \) and \((u_0, u_1, u_2, \ldots)\).
The block counting process

Let $N_t^{(n)}$ denote the number of blocks of $\Pi_t^{(n)}$, the restriction of $\Pi_t$ to a sample of size $n$. The block counting process $N^{(n)} := (N_t^{(n)})_{t \geq 0}$ moves from state $i$ to state $j < i$ at the rate

$$q_{ij} = \Xi(\{0\}) \left( \begin{array}{c} i \\ 2 \end{array} \right) \delta_{j,i-1} + \int_\Delta \mathbb{P}(Y(i,u) = j) \nu(du)$$

where $\nu(du) := \Xi(du)/(u,u)$ with $(u,u) := \sum_{r \in \mathbb{N}} u_r^2$ and

$$Y(i,u) := \text{number of balls in box 0 plus number of other non-empty boxes}$$

$$= X_0(i,u) + \sum_{r \in \mathbb{N}} 1\{X_r(i,u) > 0\}.$$ 

**Remark.** Note that $\mathbb{P}(Y(i,u) = j) = \sum_{k=1}^j f_{ijk}(u)$, where

$$f_{ijk}(u) := \frac{u_0^{i-k}}{(j-k)!} \sum_{i_1,\ldots,i_k \geq 1 \atop i_1 + \cdots + i_k = i-j+k} \frac{i!}{i_1! \cdots i_k!} \sum_{1 \leq r_1 < \cdots < r_k} u_{r_1}^{i_1} \cdots u_{r_k}^{i_k}.$$
The fixation line (Hénard (2015), Gaiser and M. (2016))

For \( n \in \mathbb{N} \) and \( t \geq 0 \) define \( L^{(n)}_t := \sup\{k \in \mathbb{N} : N^{(k)}_t \leq n\} \).

The fixation line \( L^{(n)} := (L^{(n)}_t)_{t \geq 0} \) moves from state \( i \) to state \( j > i \) at the rate

\[
\gamma_{ij} = \Xi(\{0\}) \binom{j}{2} \delta_{j,i+1} + \int_{\Delta} \mathbb{P}(Y(j,u) = i, Y(j+1,u) = i+1) \nu(du)
\]

with \( Y(\cdot,u) \) and \( \nu \) as before.

**Remark.** The probability below the integral can be provided explicitly as

\[
\mathbb{P}(Y(j,u) = i, Y(j+1,u) = i+1) = \sum_{k=1}^{i} g_{ijk}(u), \text{ where}
\]

\[
g_{ijk}(u) := \frac{u^{i-k}}{(i-k)!} \sum_{i_1,\ldots,i_k \geq 1 \atop i_1+\cdots+i_k = j-i+k} \frac{j!}{i_1! \cdots i_k!} \sum_{1 \leq r_1 < \cdots < r_k} u^{i_1}_{r_1} \cdots u^{i_k}_{r_k} (1-(u_{r_1} + \cdots + u_{r_k})).
\]
Siegmund duality

Let $\Pi$ be a $\Xi$-coalescent and let $N^{(n)} = (N_{t}^{(n)})_{t \geq 0}$ and $L^{(n)} = (L_{t}^{(n)})_{t \geq 0}$ denote the block counting process and the fixation line of $\Pi$ respectively.

**Theorem 1.** (Gaiser and M., 2016)

The block counting process is Siegmund dual to the fixation line, that is

$$\mathbb{P}(N_{t}^{(i)} \leq j) = \mathbb{P}(L_{t}^{(j)} \geq i)$$

for all $i, j \geq 1$ and $t \geq 0$. 

Exchangeable coalescents with dust

**Definition.** A $\Xi$-coalescent $\Pi = (\Pi_t)_{t \geq 0}$ has proper frequencies (no dust) if, for all times $t \geq 0$, the frequency $S_t$ of singletons of $\Pi_t$ satisfies $S_t = 0$ almost surely.

**Proposition 1.** (Schweinsberg, 2000)
A coalescent $\Pi$ has dust if and only if $\Xi(\{0\}) = 0$ and $\int_{\Delta} |u| \nu(du) < \infty$, where $|u| := \sum_{r \in \mathbb{N}} u_r$ and $\nu(du) := \Xi(du)/(u,u)$ with $(u,u) := \sum_{r \in \mathbb{N}} u_r^2$.

**Remark.** If $\Pi$ has dust then $Z = (Z_t)_{t \geq 0} := (-\log S_t)_{t \geq 0}$ is a drift-free subordinator (Lévy process with non-decreasing paths) with Laplace exponent

$$\Phi(q) := \int_{\Delta} (1 - (1 - |u|)^q) \nu(du), \quad q \geq 0.$$ 

Note that $\mathbb{E}(S_t^q) = \mathbb{E}(e^{-qZ_t}) = e^{-t\Phi(q)}$ for $t, q \geq 0$. 
Asymptotics for large sample size

Theorem 2. (Gaiser and M., 2016)
Let $\Pi$ be a $\Xi$-coalescent with dust, i.e. $\Xi(\{0\}) = 0$ and $\int_{\Delta} |u| \nu(du) < \infty$. Then the following two assertions hold.

a) As $n \to \infty$ the scaled block counting process $(N_t^{(n)}/n)_{t \geq 0}$ converges in $D_{[0,1]} [0, \infty)$ to the frequency of singleton process $S = (S_t)_{t \geq 0} = (e^{-Z_t})_{t \geq 0}$.

b) As $n \to \infty$ the scaled fixation line $(L_t^{(n)}/n)_{t \geq 0}$ converges in $D_{[1,\infty]} [0, \infty)$ to the reciprocal frequency of singleton process $(1/S_t)_{t \geq 0} = (e^{Z_t})_{t \geq 0}$.
Remarks

○ Proof of part a) based on the method of moments and some general weak convergence machinery. Proof of part b) uses duality.

○ For $\Xi$-coalescents with dust, both processes $(\log n - \log N_t^{(n)})_{t \geq 0}$ and $(\log L_t^{(n)} - \log n)$ converge in $D_{[0, \infty]}[0, \infty)$ to the drift-free subordinator $Z$.

○ Let $\Xi$ be concentrated on $\Delta^* := \{u \in \Delta : |u| = 1\}$.

  Then $\Pi$ has dust if and only if $\nu$ is finite.

  In this case $S_t \overset{d}{=} 1\{T_f > t\}$, where $T_f$ is exponentially distributed with parameter $\nu(\Delta^*)$.

  Examples are Dirichlet coalescents and Poisson–Dirichlet coalescents.
A Bernstein function

Let $\Pi$ be a $\Xi$-coalescent with dust. For $q \geq 0$ define

$$\tilde{\Phi}(q) := \int_{\Delta} \sum_{r \in \mathbb{N}} (1 - (1 - u_r)^q) \nu(du).$$

Properties:

- In general $\tilde{\Phi}$ differs from $\Phi$. For $\Lambda$-coalescents $\tilde{\Phi}$ coincides with $\Phi$.
- $\tilde{\Phi}(0) = 0$, $\tilde{\Phi}(1) = \int_{\Delta} |u| \nu(du)$, $\Phi(n) \leq \tilde{\Phi}(n)$ for all $n \in \mathbb{N}$
- $\tilde{\Phi}$ is a Bernstein function (infinitely often differentiable on $(0, \infty)$ with $(-1)^{k-1}\tilde{\Phi}^{(k)} \geq 0$ for all $k \in \mathbb{N}$ and $q > 0$).
- Lévy–Khintchine representation: 

$$\tilde{\Phi}(q) = \int_{(0,1]} (1 - (1 - x)^q) \tilde{\nu}(dx),$$

where $\tilde{\nu}(B) := \int_{\Delta} \sum_{r \in \mathbb{N}} 1_B(u_r) \nu(du)$ for all Borel sets $B \subseteq (0, 1]$. 
Rate of convergence

Notation: \( E := [0, 1], E_n := \{ k/n : k \in \{1, \ldots, n\} \}. \)

\[ \pi_n : B(E) \to B(E_n), \pi_n f(x) := f(x) \text{ for } f \in B(E) \text{ and } x \in E_n. \]

**Theorem 3.** (Rate of convergence of the block counting process, M., 2019)

Let \( \Pi = (\Pi_t)_{t \geq 0} \) be a \( \Xi \)-coalescent with dust and let \( A_n \) and \( A \) denote the generators of the scaled block counting process \( (N_t^{(n)}/n)_{t \geq 0} \) and the frequency of singleton process \( (S_t)_{t \geq 0} \) respectively. Then, for all \( n \in \mathbb{N} \) and \( f \in C^2([0, 1]) \),

\[
\| A_n \pi_n f - \pi_n Af \| := \sup_{x \in E_n} | A_n \pi_n f(x) - \pi_n Af(x) | \leq C_f r(n),
\]

where \( C_f := \| f' \| + 2\| f'' \| \) and \( r(n) := \frac{\Phi(n)}{n} \).

We call \( r(n) \) the rate function.

Remark: The rate \( r(n) \) in Theorem 3 is optimal.
Sketch of proof

Define \( u_0 := 1 - |u| \). Proof of Theorem 3 uses the paintbox representation

\[
A_n \pi_n f(x) - \pi_n Af(x) = \int_\Delta \left( \mathbb{E} \left( f \left( \frac{Y(nx, u)}{n} \right) \right) - f(xu_0) \right) \nu(du),
\]

the Taylor expansion

\[
f \left( \frac{Y(nx, u)}{n} \right) - f(xu_0) = f'(xu_0)x\tilde{Y} + f''(\xi)x^2\tilde{Y}^2,
\]

for some random point \( \xi \) taking values between \( Y(nx, u)/n \) and \( xu_0 \), and the concentration inequality \( \mathbb{E}(\tilde{Y}^2) \leq 2\mathbb{E}(\tilde{Y}) \), where

\[
\tilde{Y} := \tilde{Y}(nx, u) := \frac{Y(nx, u)}{nx} - u_0.
\]

Remark. For \( \Lambda \)-coalescents the sharper concentration inequality \( \mathbb{E}(\tilde{Y}^2) \leq \mathbb{E}(\tilde{Y}) \) holds. We conjecture that this sharper inequality holds for all \( \Xi \)-coalescents.
Rate of convergence (continued)

**Corollary 1.** (Rate of convergence, semigroup version, M., 2019)

In the situation of Theorem 3, let \((T_t^{(n)})_{t \geq 0}\) and \((T_t)_{t \geq 0}\) denote the semigroups of the scaled block counting process and the frequency of singleton process \((S_t)_{t \geq 0}\) respectively. Then, for all \(t \geq 0\), \(n \in \mathbb{N}\), and \(f \in C^2(E)\),

\[
\|T_t^{(n)} \pi_n f - \pi_n T_t f\| \leq t C_f r(n),
\]

where \(C_f\) is the constant from Theorem 3 and the rate \(r(n)\) is defined as before.
Rate of convergence of the fixation line

Notation: \( F := [1, \infty], F_n := \{ k/n : k \in \{n, n + 1, \ldots\} \} \cup \{\infty\} \).

\[ \tau_n : B(F) \to B(F_n), \tau_n g(y) := g(y) \text{ for } g \in B(F) \text{ and } y \in F_n. \]

**Conjecture.** (Rate of convergence of the fixation line; work in progress)

Let \( \Pi \) be a \( \Xi \)-coalescent with dust and let \( B_n \) and \( B \) denote the generators of the scaled fixation line \( (L_t^{(n)}/n)_{t \geq 0} \) and the reciprocal frequency of singleton process \( (1/S_t)_{t \geq 0} \) respectively. Then, for all \( n \in \mathbb{N} \) and \( g \in C^2([1, \infty]) \),

\[
\| B_n \tau_n g - \tau_n B g \| := \sup_{y \in F_n} |B_n \tau_n g(y) - \tau_n B g(y)| \leq D_g r(n)
\]

with rate \( r(n) \) as before and constant \( D_g := \|f'| + 2\|f''| \), where \( f(x) := g(1/x) \).

**Remark.** Conjecture holds for \( \Lambda \)-coalescents, even with improved constant \( D_g := \|f'| + \|f''| \). Some technical gaps in the proof for the \( \Xi \)-coalescent.
Example 1: Dirac coalescent

Let $\nu = \delta_a$ be the Dirac measure at $a = (a_1, a_2, \ldots) \in \Delta \setminus \{0\}$. Then

$$\tilde{\Phi}(q) = \sum_{i \in \mathbb{N}} (1 - (1 - a_i)^q) = q \int_0^\infty e^{-qx} u(x) \, dx, \quad q > 0,$$

where $u(x) := \mu([1 - e^{-x}, 1])$ for $x \geq 0$ with $\mu := \sum_{i \geq 1} \delta_{a_i}$.

Thus, $\frac{\tilde{\Phi}(q)}{q}$ coincides with the Laplace transform of $u$.

Asymptotics of $\tilde{\Phi}(q)$ difficult to describe in general.

Set $U(x) := \int_0^x u(t) \, dt$. If $U(x) \sim x^{1-\alpha} \ell(1/x)$, $x \to 0$, for some $\alpha \in [0, 1]$ and some function $\ell$ slowly varying at infinity, then, by a Tauberian argument,

$$\tilde{\Phi}(q) \sim \Gamma(2 - \alpha) q^\alpha \ell(q), \quad q \to \infty.$$
### Dirac coalescent (continued)

Examples with $\alpha = 0$ (slow variation), $0 < \alpha < 1$ and $\alpha = 1$ (rapid variation):

<table>
<thead>
<tr>
<th>$a_i$</th>
<th>Parameter</th>
<th>Constant $\alpha$</th>
<th>Asymptotics of $\tilde{\Phi}(q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$cp^i$</td>
<td>$0 &lt; p &lt; 1$</td>
<td>0</td>
<td>$\mu \log q$ with $\mu := -1 / \log p$</td>
</tr>
<tr>
<td>$ci^{-\beta}$</td>
<td>$\beta &gt; 1$</td>
<td>$\beta^{-1} \in (0, 1)$</td>
<td>$\Gamma(1 - \alpha)c^\alpha q^\alpha$</td>
</tr>
<tr>
<td>$\frac{c}{i(\log i)^\beta}$</td>
<td>$\beta &gt; 1$</td>
<td>1</td>
<td>$\frac{c}{\beta - 1} \frac{q}{(\log q)^{\beta-1}}$</td>
</tr>
</tbody>
</table>

Constant $c > 0$ chosen sufficiently small such that $|a| \leq 1$
Example 2: Dirichlet coalescent

Let \((X_1, \ldots, X_N) \overset{d}{=} D_N(\alpha)\) be symmetric Dirichlet distributed with parameters \(N \in \mathbb{N}\) and \(\alpha > 0\).

Let \(X_{(1)} \geq \cdots \geq X_{(N)}\) denote the order statistics.

\[\nu := \text{distribution of } (X_{(1)}, \ldots, X_{(N)}, 0, 0, \ldots).\]

The associated exchangeable coalescent is called the Dirichlet coalescent.

This coalescent neither comes down from infinity nor stays infinite, since

\[\mathbb{P}(N_t = \infty) = \mathbb{P}(T_f > t) = e^{-t} \text{ for all } t \geq 0.\]
Dirichlet coalescent (continued)

**Notation.** 
\[
[x|y]_n := \prod_{k=0}^{n-1} (x + ky), \quad (x|y)_n := \prod_{k=0}^{n-1} (x - ky),
\]
\[
[x]_n := [x|1]_n, \quad (x)_n := (x|1)_n
\]

Rates of the block counting process:
\[
q_{ij} = \frac{(N\alpha|\alpha)_j}{[N\alpha]_i} S_\alpha(i, j), \ j < i
\]

Rates of the fixation line:
\[
\gamma_{ij} = \frac{(N\alpha|\alpha)_{i+1}}{[N\alpha]_{j+1}} S_\alpha(j, i), \ i < j
\]

\(S_\alpha(i, j) := S(i, j; -1, \alpha, 0)\) is the generalized Stirling number as defined in Hsu and Shiue (1998).
Dirichlet coalescent (continued)

Define $\Delta_N := \{ u \in \Delta : u_1 + \cdots + u_N = 1 \}$. Then

$$
\tilde{\Phi}(q) = \int_{\Delta_N} \sum_{r=1}^{N} (1 - (1 - u_r)^q) \nu(du) \\
= \int_{\mathbb{R}^N} \sum_{r=1}^{N} (1 - (1 - u_r)^q) D_N(\alpha)(du_1, \ldots, du_N) \\
= N \mathbb{E}(1 - (1 - X_1)^q),
$$

If $N = 1$ then $X_1 \equiv 1$ and $\tilde{\Phi} = \Phi$. If $N > 1$ then $X_1$ is beta distributed with parameters $\alpha$ and $N\alpha - \alpha$ and

$$
\tilde{\Phi}(q) = N \left( 1 - \frac{\Gamma(N\alpha)\Gamma(N\alpha - \alpha + q)}{\Gamma(N\alpha - \alpha)\Gamma(N\alpha + q)} \right) \sim N, \quad q \to \infty,
$$

differs from $\Phi(q) = 1, q > 0$. 
Example 3: Poisson–Dirichlet coalescent
(Sagitov (2003), M. (2010), Gaiser and M. (2016))

This is the coalescent where $\nu$ is the **Poisson–Dirichlet distribution** with parameters $0 \leq \alpha < 1$ and $\theta > -\alpha$.

Rates of the block counting process:

$$q_{ij} = c_{j,\alpha,\theta} \frac{\Gamma(\theta + \alpha j)}{\Gamma(\theta + i)} s_{\alpha}(i, j), j < i$$

Normalizing constant: $c_{j,\alpha,\theta} := \prod_{k=1}^{j} \frac{\Gamma(\theta + 1 + (k - 1)\alpha)}{\Gamma(\theta + k\alpha)}$

Rates of the fixation line:

$$\gamma_{ij} = c_{i,\alpha,\theta} \frac{\Gamma(\theta + \alpha i + 1)}{\Gamma(\theta + j + 1)} s_{\alpha}(j, i), i < j$$

$s_{\alpha}(i, j) := S(i, j; -1, -\alpha, 0)$ is the generalized absolute Stirling number of the first kind as defined in Hsu and Shiue (1998).
Poisson–Dirichlet coalescent (continued)

By a result of Handa (2009), applied with \( f(x) := 1 - (1 - x)^q \),

\[
\tilde{\Phi}(q) = \int \sum_{r \in \mathbb{N}} f(u_r) \nu(du) = \int (1 - (1 - x)^q) \mu_1(dx), \quad q \geq 0,
\]

where \( \mu_1 \) denotes the correlation measure associated with the Poisson–Dirichlet coalescent. The density of \( \mu_1 \) is explicitly known (see Handa, 2009), and it follows that

\[
\tilde{\Phi}(q) = c_{1, \alpha, \theta} \int_0^1 (1 - (1 - x)^q)x^{-\alpha - 1}(1 - x)^{\theta + \alpha - 1} dx, \quad q \geq 0,
\]

with normalizing constant \( c_{1, \alpha, \theta} := B(1 - \alpha, \theta + \alpha) \). For \( \alpha > 0 \) this leads to

\[
\tilde{\Phi}(q) = \frac{\theta + q \Gamma(\theta + \alpha + q)\Gamma(\theta + 1)}{\alpha \Gamma(\theta + 1 + q)\Gamma(\theta + \alpha)} - \frac{\theta}{\alpha} \sim \frac{\Gamma(\theta + 1) q^\alpha}{\Gamma(\theta + \alpha) \alpha}, \quad q \to \infty.
\]

For \( \alpha = 0 \) it follows that

\[
\tilde{\Phi}(q) = \theta(\Psi(q + \theta) - \Psi(\theta)) \sim \theta \log q \quad \text{as} \quad q \to \infty,
\]

where \( \Psi := \Gamma'/\Gamma \). The associated subordinator is the \( \Psi \)-subordinator. In all cases \( \tilde{\Phi} \) differs from \( \Phi \).
Example 4: A symmetric coalescent

Let \( (m_k)_{k \in \mathbb{N}} \) be a sequence of non-negative real numbers satisfying \( \sum_{k \in \mathbb{N}} m_k/k < \infty \).

Suppose \( \nu \) assigns for each \( k \in \mathbb{N} \) mass \( m_k \) to \( u^{(k)} := (1/k, \ldots, 1/k, 0, 0, \ldots) \in \Delta^* \).

This coalescent occurs in González Casanova, Miró Pina and Siri-Jégousse (2019).

Rates of the block counting process:
\[
q_{ij} = S(i, j) \sum_{k \in \mathbb{N}} \frac{(k)_j}{k^i} m_k, \ j < i
\]

Rates of the fixation line:
\[
\gamma_{ij} = S(j, i) \sum_{k \in \mathbb{N}} \frac{(k)_{i+1}}{k^{j+1}} m_k, \ i < j
\]

\([ (x)_i := x(x - 1) \cdots (x - i + 1), S(., .) \text{ are the Stirling number of the second kind} ]\)
A symmetric coalescent (continued)

The symmetric coalescent has dust if and only if \( \nu(\Delta) = \sum_{k \in \mathbb{N}} m_k < \infty \). In this case

\[
\tilde{\Phi}(q) = \sum_{k \in \mathbb{N}} k m_k \left( 1 - \left( 1 - \frac{1}{k} \right)^q \right), \quad q \geq 0.
\]

For example, if \( m_k = k^{-\alpha} \) with \( \alpha > 1 \), then, as \( q \to \infty \),

\[
\tilde{\Phi}(q) \sim \begin{cases} 
\zeta(\alpha - 1) & \text{if } \alpha > 2, \\
\log q & \text{if } \alpha = 2, \\
-\Gamma(\alpha - 2)q^{2-\alpha} & \text{if } \alpha \in (1, 2).
\end{cases}
\]

\( r(n) = O\left( \frac{1}{n} \right) \) for \( \alpha > 2 \), \( r(n) = O\left( \frac{\log n}{n} \right) \) for \( \alpha = 2 \), \( r(n) = O\left( \frac{1}{n^{\alpha-1}} \right) \) for \( \alpha \in (1, 2) \).
Thank you very much for your attention!
References I


References II


