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Rotational symmetric Willmore surfaces with umbilic lines

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Abstract: In this article, we give examples of rotational symmetric not totally umbilic Willmore immersions of a connected surface with countably many umbilic circles. Further we prove that any rotational symmetric Willmore immersion of a connected surface in codimension one with an umbilic line is after applying an appropriate Möbius transformation, which keeps the rotational symmetry, minimal in hyperbolic space outside its umbilic lines. As a consequence such immersions are isothermic.

Keywords: Willmore surfaces, free elastica, hyperbolic space.

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1 Introduction

For an immersed surface $f : \Sigma \to \mathbb{R}^3$ the Willmore functional is defined by

$$\mathcal{W}(f) = \frac{1}{4} \int_\Sigma |H|^2 \, d\mu_g,$$

where H denotes the scalar mean curvature of f, $g = f^* g_{euc}$ the pull-back metric and μ_g the induced area measure of f on Σ.

Critical points of the Willmore functional, called Willmore surfaces or immersions, satisfy the Euler-Lagrange equation

$$\Delta_g H + |A^0|_g^2 H = 0, \tag{1.1}$$

see i.e. [KuSch02] §2, where the laplacian along f is used and $A^0 = A - \frac{1}{2}\vec{\mathbf{H}}g$ is the tracefree second fundamental form of f . In particular, smooth Willmore immersions are real-analytic, see [Mo58].

A point is called umbilic, when the tracefree part of the second fundamental form A^0 vanishes. In [Sch17], the second author proved that the set of umbilic points of not totally umbilic Willmore immersions of a connected surface in codimension one consists of a closed set of isolated points and of a closed set which is a one dimensional real-analytic submanifold of Σ without boundary.

These one dimensional submanifolds indeed occur, as examples of Willmore tori with umbilic lines in [BaBo93] and constructions with the theorem of Cauchy-Kowalewskaja show. The Willmore tori in [BaBo93] are minimal surfaces in the hyperbolic 3-space and are therefore isothermic, see [Pa91] Theorem 2.2, that is local conformal principal curvature coordinates can be introduced around any non-umbilic point. The constructions with the theorem of Cauchy-Kowalewskaja show that there are also non-isothermic examples of open Willmore surfaces with umbilic lines.

In case of rotational symmetry of f , the umbilic lines are circles, which do not accumulate by real analyticity unless f is totally umbilic on some component.

That there are indeed rotational symmetric not totally umbilic Willmore immersions of a connected surface with countably many umbilic circles can be seen with the classification by Langer and Singer in [LaSi84b] of all rotational symmetric Willmore immersions via their regular profile curves in the upper half plane $\mathcal{H} := \{(x,y) \in \mathbb{R}^2 \mid y > 0 \}$ equipped with the hyperbolic metric $y^{-2}g_{euc}$. The profile curve is the intersection of the immersion with $\mathcal{H}$.

In this article, we prove that any rotational symmetric Willmore immersion of a connected surface in codimension one with an umbilic line is after applying an appropriate Möbius transformation, which keeps the rotational symmetry, minimal in hyperbolic space outside its umbilic lines, in particular it is isothermic, as the examples in [BaBo93]. Here we consider two copies of the hyperbolic 3-space $\mathcal{H}^3_\pm := \{(x,y,z) \in \mathbb{R}^3 \mid \pm x > 0 \}$ equipped with the hyperbolic metric $x^{-2}g_{euc}$. In particular adding the infinite hyperbolic plane $\mathcal{H}^3_0 := \{0\} \times \mathbb{R}^2$, we have $\mathbb{R}^3 = \mathcal{H}^3_+ + \mathcal{H}^3_0 + \mathcal{H}^3_-$.

Theorem 1.1 *Any smooth rotational symmetric not totally umbilic Willmore immersion of a connected surface in* $\mathbb{R}^3$ *with an umbilic line is after applying an appropriate Möbius transformation, which keeps the rotational symmetry, minimal in hyperbolic space outside its umbilic lines, in particular it is isothermic. Moreover the union of the umbilic lines is exactly the intersection with the infinite hyperbolic plane, and this intersection is orthogonal everywhere.* □

For the proof, we use the Willmore equation in (1.1) transformed to the hyperbolic space in §A and the unique continuation of solutions of elliptic equations in [Ar57] or [Co56].

2 Rotational symmetric immersions

Rotational symmetric immersions of a connected surface in $\mathbb{R}^3$ not touching the rotational axis, which we assume being the $x-$axis, admit a regular profile curve in the upper half plane $\gamma :]a,b[\to \mathcal{H} := \{(x,y) \in \mathbb{R}^2 \mid y > 0 \}$ with $|\gamma'| \neq 0, -\infty \leq a < b \leq \infty$, up to reparametrization, and we get an equivalent rotationally symmetric immersion by putting $f_\gamma :]a,b[\times S^1 =: \Sigma_{]a,b[} \to \mathbb{R}^3$ with

$$f_\gamma(t, e^{i\alpha}) := (\gamma^1(t), \gamma^2(t)\cos\alpha, \gamma^2(t)\sin\alpha) \quad \text{for } t \in]a,b[, \alpha \in \mathbb{R}. \tag{2.1}$$

The principal curvatures of f in $\mathbb{R}^3 - (x - \text{axis})$ equipped with the product metric $\frac{dx^2+dy^2}{y^2} + \mathrm{d}\theta^2 = |(y,z)|^{-2} g_{euc}$ are given by the curvature κ of γ in the hyperbolic plane $\mathcal{H}$ equipped with the hyperbolic metric $y^{-2} g_{euc}$, called the hyperbolic curvature of γ, and 0, as the circles $(x, y\cos\alpha, y\sin\alpha), \alpha \in \mathbb{R}$, are geodesics in the metric $|(y,z)|^{-2} g_{euc}$, see [LaSi84a] proof of the Theorem. Therefore a point $(t, e^{i\alpha})$ is an umbilic point of f with respect to the metric $|(y,z)|^{-2} g_{euc}$, if and only if $\kappa(t) = 0$. By conformal invariance of umbilic points, see [Ch74] or Proposition A.1, we see that the umbilic points of f in $\mathbb{R}^3$ correspond exactly to the points of vanishing hyperbolic curvature of γ.

Moreover, Langer and Singer determined in [LaSi84b] Table 2.7 (c) the curvature functions in terms of hyperbolic arc length of all profile curves which induce rotational symmetric Willmore immersions. These profile curves are called free elastica when parametrized by hyperbolic arc length. All of these families apart from the last, have either identically vanishing hyperbolic curvature, that is are geodesics, or have never vanishing hyperbolic curvature. The last family in Table 2.7 (c), called the wavelike free elastica, have a periodic, not identically vanishing curvature function in terms of hyperbolic arc length with countably many, not accumulating zeros. These wavelike free elastica yield rotational symmetric not totally umbilic Willmore immersions with countably many umbilic circles.

We turn to the
Proof of Theorem 1.1:
To this end, we consider any smooth rotational symmetric Willmore immersion of a connected surface in $\mathbb{R}^3$ with an umbilic line. After reparametrization, the immersion is of the form f_γ in (2.1) for its profile curve γ in $\mathcal{H}$. As $f := f_\gamma$ has an umbilic line by assumption, the hyperbolic curvature κ of γ vanishes at some point by [LaSi84a], say $\kappa(t_0) = 0$ for some $a < t_0 < b$. After applying an appropriate Möbius transformation of $\mathcal{H}$, which is an isometry for the hyperbolic metric, hence leaves the curvature of γ unchanged, we may assume that γ intersects at t_0 the $y-$axis, which is a geodesic in the hyperbolic metric, orthogonally in $(0,1)$. Then f is a rotational symmetric Willmore immersion which intersects the vertical plane $\{x_1 = 0\}$ orthogonally in the unit circle $\Gamma := \partial B_1(0) \cap \{x_1 = 0\}$. Selecting a smooth unit normal $\nu_f^{\mathbb{R}^3}$ of f in $\mathbb{R}^3$ at least locally, this means

$$f_1 = 0 \text{ and } \nu_{f,1}^{\mathbb{R}^3} = 0 \quad \text{on } \Gamma. \tag{2.2}$$

In particular e_1 is tangent at f along Γ, and we get a smooth vector field $\mathbf{n}$ along Γ with

$$\partial_{\mathbf{n}} f = e_1, \mathbf{n} \perp T\Gamma. \tag{2.3}$$

Clearly, $\mathbf{n}$ is a unit vector with respect to the pull-back metric $g^{\mathbb{R}^3}$ of f as immersion into the euclidian space $\mathbb{R}^3$, and choosing a smooth unit tangent vector $\mathbf{t}$ along Γ at least locally, we see that $\mathbf{t}, \mathbf{n}$ form a smooth orthonormal basis with respect to $g^{\mathbb{R}^3}$ of $T\Sigma$ locally on Γ.

The scalar mean curvature of f in the hyperbolic space $\mathcal{H}^3_\pm := \{\pm x_1 > 0\}$ equipped with the hyperbolic metric $x_1^{-2}\delta_{ij}$ and with respect to the smooth hyperbolic unit normal

$\nu_f^{\mathcal{H}^3_\pm}$ with $\nu_f^{\mathbb{R}^3} \cdot \nu_f^{\mathcal{H}^3} > 0$ is given by Proposition A.1 as

$$H_f^{\mathcal{H}^3} = f_1 H_f^{\mathbb{R}^3} + 2\nu_{f,1}^{\mathbb{R}^3},$$

and it therefore extends to a real-analytic function throughout $\Sigma := \Sigma_{]a,b[}$. Moreover, as f is Willmore it satisfies by Proposition A.1 and (1.1) the elliptic equation

$$\Delta_{g^{\mathbb{R}^3}} H_f^{\mathcal{H}^3} + |A_f^{\mathbb{R}^3,0}|^2_{g^{\mathbb{R}^3}} H_f^{\mathcal{H}^3} = f_1(\Delta_{g^{\mathbb{R}^3}} H_f^{\mathbb{R}^3} + |A_f^{\mathbb{R}^3,0}|^2_{g^{\mathbb{R}^3}} H_f^{\mathbb{R}^3}) = 0 \quad \text{on } \Sigma. \tag{2.4}$$

By (2.2) obviously

$$H_f^{\mathcal{H}^3} = 0 \quad \text{on } \Gamma \subseteq \{f_1 = 0\}.$$

Further using (2.3) and the scalar second fundamental form $h_f^{\mathbb{R}^3} := \langle A_f^{\mathbb{R}^3}, \nu_f^{\mathbb{R}^3} \rangle$ of f in $\mathbb{R}^3$, we get

$$\partial_{\mathbf{n}} H_f^{\mathcal{H}^3} = \partial_{\mathbf{n}}(f_1 H_f^{\mathbb{R}^3} + 2\nu_{f,1}^{\mathbb{R}^3}) = (\partial_{\mathbf{n}} f_1) H_f^{\mathbb{R}^3} + 2\langle \partial_{\mathbf{n}} \nu_f^{\mathbb{R}^3}, e_1 \rangle =$$

$$= H_f^{\mathbb{R}^3} + 2\langle \partial_{\mathbf{n}} \nu_f^{\mathbb{R}^3}, \partial_{\mathbf{n}} f \rangle = H_f^{\mathbb{R}^3} - 2h_f^{\mathbb{R}^3}(\mathbf{n},\mathbf{n}) = h_f^{\mathbb{R}^3}(\mathbf{t},\mathbf{t}) - h_f^{\mathbb{R}^3}(\mathbf{n},\mathbf{n}) = 0 \quad \text{on } \Gamma, \tag{2.5}$$

as Γ consists only of umbilic points of f . As $\mathbf{n}$ is not tangent, but even orthogonal at Γ , we conclude with the elliptic equation (2.4) and the unique continuation, see [Ar57] Remark 2 and 3 or [Co56] Satz 5, that

$$f_1 H_f^{\mathbb{R}^3} + 2\nu_{f,1}^{\mathbb{R}^3} \equiv 0 \text{ on } \Sigma. \tag{2.6}$$

Therefore f is minimal on $\Sigma_\pm := [\pm f_1 > 0]$ in the hyperbolic space $\mathcal{H}^3_\pm$ and further

$$\nu_{f,1}^{\mathbb{R}^3} = 0 \quad \text{on } \Sigma_0 := [f_1 = 0] = \Sigma - (\Sigma_+ \cup \Sigma_-), \tag{2.7}$$

that is the intersection of f with the vertical plane, which is the infinite plane for the hyperbolic space $\mathcal{H}^3_\pm$, is orthogonal everywhere.

As on minimal surfaces in the constant curvature space $\mathcal{H}^3_\pm$ the function

$$\varphi_f^{\mathcal{H}^3} := \left\langle A_{f,11}^{\mathcal{H}^3,0} - iA_{f,12}^{\mathcal{H}^3,0}, \nu_f^{\mathcal{H}^3} \right\rangle \quad \text{on } \Sigma_\pm$$

is holomorphic in local conformal charts, see for example [Sch17] Proposition 2.1 or [La70] Lemma 1.2 for minimal immersions in the shpere S^3 , and as by Proposition A.1 that

$$[\varphi_f^{\mathcal{H}^3} = 0] = [A^{\mathcal{H}^3,0} = 0] = [A^{\mathbb{R}^3,0} = 0],$$

we see that $\varphi_f^{\mathcal{H}^3}$ has only isolated zeros on $\Sigma_\pm$, since it does not vanish identically, as f is not totally umbilic, hence by rotational symmetry, f is even umbilic free on $\Sigma_\pm$ or likewise

$$\{\text{umbilic points of } f\} = \cup\{\text{umbilic lines of } f\} \subseteq \Sigma_0. \tag{2.8}$$

To prove equality, that is that Σ_0 consists only of umbilic lines, we first observe from (2.7) that Σ_0 is a smooth one-dimensional submanifold of Σ without boundary and

$$e_1 \in T_p f \text{ for any } p \in \Sigma_0.$$

Clearly as $\Sigma_0 = [f_1 = 0]$, we have $T\Sigma_0 \perp e_1$, and there is as above a smooth orthonormal basis $\mathbf{t}, \mathbf{n}$ of Tf locally on Σ_0 with $\mathbf{t} \in T\Sigma_0, \partial_{\mathbf{n}} f = e_1$. First we get by tangency of $\mathbf{t}$ at Σ_0 that

$$0 = \langle \partial_{\mathbf{t}} e_1, \nu_f^{\mathbb{R}^3} \rangle = \langle \partial_{\mathbf{tn}} f, \nu_f^{\mathbb{R}^3} \rangle = h_f^{\mathbb{R}^3}(\mathbf{t}, \mathbf{n}) \quad \text{on } \Sigma_0.$$

Secondly differentiating (2.6), we get as in (2.5) that

$$0 = \partial_{\mathbf{n}}(f_1 H_f^{\mathbb{R}^3} + 2\nu_{f,1}^{\mathbb{R}^3}) = (\partial_{\mathbf{n}} f_1) H_f^{\mathbb{R}^3} + 2\langle \partial_{\mathbf{n}} \nu_f^{\mathbb{R}^3}, e_1 \rangle =$$

$$= H_f^{\mathbb{R}^3} + 2\langle \partial_{\mathbf{n}} \nu_f^{\mathbb{R}^3}, \partial_{\mathbf{n}} f \rangle = H_f^{\mathbb{R}^3} - 2h_f^{\mathbb{R}^3}(\mathbf{n}, \mathbf{n}) = h_f^{\mathbb{R}^3}(\mathbf{t}, \mathbf{t}) - h_f^{\mathbb{R}^3}(\mathbf{n}, \mathbf{n}) \quad \text{on } \Sigma_0,$$

hence $h_f^{\mathbb{R}^3}(\mathbf{t}, \mathbf{t}) = h_f^{\mathbb{R}^3}(\mathbf{n}, \mathbf{n})$ on Σ_0 . Together we get that $h_f = (1/2) H_f g$ is scalar on Σ_0 , hence $A^0 \equiv 0$ on Σ_0 , and Σ_0 contains only umbilic points and consists therefore only of umbilic lines. As we already know that the umbilic lines are contained in Σ_0 , we get that the union of the umbilic lines is exactly Σ_0 which is the intersection of f with the vertical plane, and this intersection is orthognal everywhere by (2.7).

As f is minimal in $\mathcal{H}^3_\pm$ on $(\Sigma_+ \cup \Sigma_-) = \Sigma - \Sigma_0$, that is outside of Σ_0 , it is minimal outside its umbilic lines, hence isothermic, see [Pa91] Theorem 2.2, which concludes the proof of the theorem.

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3 Wavelike free elastica

In this section, we reprove some statements of [LaSi84a] about wavelike free elastica in the hyperbolic plane, see also [Ei16] Lemma 6.1 and [EiGr17] Lemma 6.13. The monotonicity in (3.4) below was used in [EiSch24].

Proposition 3.1 *Let γ be a wavelike free elastica in $\mathcal{H}$, that is a free elastica in the last row of [LaSi84b] Table 2.7 (c). Then γ has no selfintersections and after applying an appropriate Möbius transformation of $\mathcal{H}$, it crosses perpendicularly the vertical geodesic $\{x = 0\} \subseteq \mathcal{H}$ at its inflection points, and these are*

$$\{(0, \lambda^k) \mid k \in \mathbb{Z}\} \text{ for some } \lambda = \lambda_\gamma > 1. \tag{3.1}$$

Proof:
By [LaSi84b] Table 2.7 (c), the hyperbolic curvature κ of γ is periodic, say with period $2p > 0$, more precisely

$$\kappa(s + p) = -\kappa(s) \text{ for } s \in \mathbb{R}, \tag{3.2}$$

and the points of vanishing hyperbolic curvature, which are the inflection points of γ , are exactly $(p/2) + kp$ with $k \in \mathbb{Z}$. To simplify the notation, we replace κ by $\kappa(. - p/2)$, hence the zeros of κ are exactly kp with $k \in \mathbb{Z}$. As γ is a free elastica, we see from [LaSi84a] that the corresponding rotational symmetric immersion f_γ with regular profile curve γ as in (2.1) is a rotational symmetric Willmore immersion. Then by Theorem 1.1 after applying an appropriate Möbius transformation of $\mathcal{H}$, which is an isometry for the hyperbolic metric, hence leaves κ unchanged, f_γ is minimal in hyperbolic space outside its umbilic lines, and the union of the umbilic lines of f_γ is exactly the intersection with the vertical plane $\{x = 0\}$, and this intersection is orthogonal everywhere. As the umbilic lines of f_γ correspond exactly to the points of vanishing hyperbolic curvature of γ by [LaSi84a] proof of the Theorem, we see that γ intersects the vertical geodesic $\{x = 0\} \subseteq \mathcal{H}$ exactly at its inflection points, which are $\{\gamma(kp) \mid k \in \mathbb{Z}\}$, and each intersection is orthogonal.

After stretching, we may assume that $\gamma(0) = (0,1)$ and $\gamma(p) = (0,\lambda) = \lambda\gamma(0)$ for some $\lambda > 0$. Putting $\bar{\gamma}(s) := \lambda^{-1}(-\gamma_1(s+p), \gamma_2(s+p))$ for $s \in \mathbb{R}$, we see by (3.2) and as $(x,y) \mapsto (-x,y)$ is an orientation reversing isometry of $\mathcal{H}$ that $\bar{\gamma}$ and γ have the same hyperbolic curvature κ . By construction

$$\bar{\gamma}(0) = \lambda^{-1}(-\gamma_1(p), \gamma_2(p)) = \lambda^{-1}(-0, \lambda) = (0,1) = \gamma(0).$$

Next as γ intersects the $y-$axis orthognally in kp for $k \in \mathbb{Z}$, we get that $\gamma'(kp)$ are horizontal, in particular

$$\bar{\gamma}_2'(0) = \lambda^{-1}\gamma_2'(p) = 0 = \gamma_2'(0). \tag{3.3}$$

Since both $\bar{\gamma}$ and γ are parametrized by hyperbolic arc length and the crossing of the $y-$axis by γ changes direction from kp to $(k+1)p$, we get $\bar{\gamma}_1'(0) = \gamma_1'(0)$, hence by the fundamental theorem for curves that $\bar{\gamma} = \gamma$. Therefore

$$\lambda^{-1}\gamma_2(kp) = \bar{\gamma}_2((k-1)p) = \gamma_2((k-1)p)$$

and by induction

$$\gamma_2(kp) = \lambda^k \gamma_2(0) = \lambda^k \quad \text{for } k \in \mathbb{Z},$$

which yields (3.1) apart from $\lambda \neq 1$, after possibly replacing λ by λ^{-1} .

It remains to prove that γ has no selfintersections and that $\lambda \neq 1$. To this end we show that

$$s \mapsto |\gamma(s)| \text{ is strictly monotone on } \mathbb{R}. \tag{3.4}$$

Indeed, this implies that γ has no selfintersections and that $\lambda = |\gamma(p)| \neq |\gamma(0)| = 1$. We first prove that

$$|\gamma| \text{ has neither a local maximum, nor a local minimum on }]0,p[. \tag{3.5}$$

If $|\gamma|$ has a local maximum or a local minimum in $s_0 \in]0,p[$, then $\partial B_{|\gamma(s_0)|}(0) \cap \mathcal{H}$ touches γ at s_0 or likewise passing to the rotational symmetric surfaces, $\partial B_{|\gamma(s_0)|}(0)$ touches f_γ in the point $\gamma(s_0) \in \{x_1, x_2 > 0\}$, as $s_0 \notin \{kp \mid k \in \mathbb{Z}\}$. Now we already know that f_γ is minimal in the hyperbolic space $\{x_1 > 0\}$, and, as the sphere $\partial B_{|\gamma(s_0)|}(0)$ intersects the vertical plane $\{x_1 = 0\}$ orthogonally, it is also minimal in the hyperbolic space $\{x_1 > 0\}$. Then by standard maximum principle, f_γ and $\partial B_{|\gamma(s_0)|}(0)$ locally coincide, hence γ is locally contained in the geodesic $\partial B_{|\gamma(s_0)|}(0) \cap \mathcal{H}$. But this is impossible, as $\kappa \not\equiv 0$ an any non-empty open subset, hence establishes (3.5).

Now if $|\gamma(a)| = |\gamma(b)|$ for some $0 < a < b < p$, then $|\gamma|$ has a local maximum or a local minimum on $]a,b[\subseteq]0,p[$, which is excluded by (3.5). This implies that $|\gamma|$ is injective on $]0,p[$, hence by continuity, it is strictly monotone on $[0,p]$. Since $|\gamma(.+p)| = \lambda|\gamma|$, we see that $|\gamma|$ is strictly monotone on any $[(k-1)p, kp]$ for $k \in \mathbb{Z}$ with the same type of monotonicity. This yields strict monotonicity of $|\gamma|$ on the whole of $\mathbb{R}$, which is (3.4), and the proposition is proved.

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Appendix

A The Willmore equation in the hyperbolic space $\mathcal{H}^3$

In this appendix, we transform the Willmore equation in (1.1) in the hyperbolic space. We use an auxiliary proposition from [KuSch12].

Proposition A.1 *For an immersion* $f : \Sigma \to \mathcal{H}^3 \subseteq \mathbb{R}^3$ *of a surface* Σ *into the hyperbolic space* $\mathcal{H}^3 := \{x_3 > 0\}$ *equipped with the hyperbolic metric* $g^{\mathcal{H}^3} := x_3^{-2} g^{\mathbb{R}^3}$ *, we see for the tracefree second fundamental forms and mean curvatures* $A_f^{\mathbb{R}^3,0}, \vec{\mathbf{H}}_f^{\mathbb{R}^3}$ *respectively* $A_f^{\mathcal{H}^3,0}, \vec{\mathbf{H}}_f^{\mathcal{H}^3}$ *, for the scalar mean curvatures* $H_f^{\mathbb{R}^3}$ *respectively* $H_f^{\mathcal{H}^3}$ *with respect to smooth unit normals* $\nu_f^{\mathbb{R}^3}$ *respectively* $\nu_f^{\mathcal{H}^3}$ *along* f *with* $\nu_f^{\mathbb{R}^3} \cdot \nu_f^{\mathcal{H}^3} > 0$ *and for the pull-back metrics* $g^{\mathbb{R}^3}$ *respectively* $g^{\mathcal{H}^3}$ *as immersions into* $\mathbb{R}^3$ *respectively* $\mathcal{H}^3$

$$\begin{aligned} A_f^{\mathcal{H}^3,0} &= A_f^{\mathbb{R}^3,0}, \\ \vec{\mathbf{H}}_f^{\mathcal{H}^3} &= f_3^2 \vec{\mathbf{H}}_f^{\mathbb{R}^3} + 2 f_3 e_3^{\perp}, \\ H_f^{\mathcal{H}^3} &= f_3 H_f^{\mathbb{R}^3} + 2\nu_{f,3}^{\mathbb{R}^3}, \\ f_3^{-2}(\Delta_{g^{\mathcal{H}^3}} H_f^{\mathcal{H}^3} + |A_f^{\mathcal{H}^3,0}|^2_{g^{\mathcal{H}^3}} H_f^{\mathcal{H}^3}) &= \Delta_{g^{\mathbb{R}^3}} H_f^{\mathcal{H}^3} + |A_f^{\mathbb{R}^3,0}|^2_{g^{\mathbb{R}^3}} H_f^{\mathcal{H}^3} = \\ &= f_3(\Delta_{g^{\mathbb{R}^3}} H_f^{\mathbb{R}^3} + |A_f^{\mathbb{R}^3,0}|^2_{g^{\mathbb{R}^3}} H_f^{\mathbb{R}^3}). \end{aligned} \tag{A.1}$$

Proof:
As the hyperbolic metric $x_3^{-2} g_{euc}$ is conformal to the euclidian metric, the first equation is immediate from conformal inyariance in [Ch74] or [KuSch12] Proposition 1.2.1 (1.2.2) and the pull-back metrics are related by $g^{\mathcal{H}^3} = f_3^{-2} g^{\mathbb{R}^3}$. Next by [KuSch12] Proposition 1.2.1 (1.2.1) for the second fundamental forms A_f that

$$A_{f,ij}^{\mathcal{H}^3} - A_{f,ij}^{\mathbb{R}^3} = g_{ij}^{\mathcal{H}^3}\ grad^{\perp}_{g^{\mathcal{H}^3}} \log f_3 = g_{ij}^{\mathcal{H}^3} f_3^2 (\nabla^{\mathbb{R}^3,\perp} \log x_3) \circ f = f_3 g_{ij}^{\mathcal{H}^3} e_3^{\perp}$$

and

$$\vec{\mathbf{H}}_f^{\mathcal{H}^3} = g^{\mathcal{H}^3,ij} A_{f,ij}^{\mathcal{H}^3} = g^{\mathcal{H}^3,ij} A_{f,ij}^{\mathbb{R}^3} + 2 f_3 e_3^{\perp} = f_3^2 \vec{\mathbf{H}}_f^{\mathbb{R}^3} + 2 f_3 e_3^{\perp},$$

which is the second equation. We see for any euclidian unit normal $\nu_f^{\mathbb{R}^3}$ along f , that $\nu_f^{\mathcal{H}^3} = f_3 \nu_f^{\mathbb{R}^3}$ is an hyperbolic unit normal along f with $\nu_f^{\mathbb{R}^3} \cdot \nu_f^{\mathcal{H}^3} > 0$, hence for the scalar mean curvatures

$$H_f^{\mathcal{H}^3} = f_3^{-2} \langle \vec{\mathbf{H}}_f^{\mathcal{H}^3}, \nu_f^{\mathcal{H}^3} \rangle_{\mathbb{R}^3} = f_3 H_f^{\mathbb{R}^3} + 2 \langle e_3^{\perp}, \nu_f^{\mathbb{R}^3} \rangle_{\mathbb{R}^3} = f_3 H_f^{\mathbb{R}^3} + 2 \nu_{f,3}^{\mathbb{R}^3},$$

which is the third equation. Then from the conformal invariance of the laplacian and from the first equation

$$f_3^{-2}(\Delta_{g^{\mathcal{H}^3}} H_f^{\mathcal{H}^3} + |A_f^{\mathcal{H}^3,0}|^2_{g^{\mathcal{H}^3}} H_f^{\mathcal{H}^3}) = \Delta_{g^{\mathbb{R}^3}} H_f^{\mathcal{H}^3} + |A_f^{\mathbb{R}^3,0}|^2_{g^{\mathbb{R}^3}} H_f^{\mathcal{H}^3},$$

which is the first identity in the fourth equation. Omitting the upper index $\mathbb{R}^3$ for simplicity of the notation, we calculate

$$\begin{aligned} \Delta_{g^{\mathbb{R}^3}} H_f^{\mathcal{H}^3} &= \Delta_g (f_3 H_f + 2\nu_{f,3}) = \\ &= (\Delta_g f_3) H_f + 2 g^{ij} \partial_i f_3 \partial_j H_f + f_3 \Delta_g H_f + 2 \Delta_g \nu_{f,3}. \end{aligned} \tag{A.2}$$

Firstly

$$\Delta_g f_3 = \vec{\mathbf{H}}_f \cdot e_3 = H_f \nu_{f,3}. \tag{A.3}$$

Next as $|\nu_f| = 1$, we have that $\nabla \nu_f \perp \nu_f$, that is $\nabla \nu_f$ is tangential, and more precisely for the scalar second fundamental form $h_{kl} := \langle A_{f,kl}, \nu_f \rangle$ that

$$\partial_j \nu_f = g^{kl} \langle \partial_j \nu_f, \partial_k f \rangle \partial_l f = -g^{kl} \langle \nu_f, \partial_{jk} f \rangle \partial_l f = -g^{kl} h_{jk} \partial_l f.$$

Further by use of the Mainardi-Codazzi equation

$$\Delta_g \nu_f = g^{ij} \nabla_i \nabla_j \nu_f = -g^{ij} g^{kl} (\nabla_i h_{jk}) \partial_l f - g^{ij} g^{kl} h_{jk} A_{il} =$$

$$= -g^{kl} \partial_k H_f \partial_l f - g^{ij} g^{kl} h_{jk} h_{il} \nu_f = -g^{kl} \partial_k H_f \partial_l f - |A_f|_g^2 \nu_f,$$

hence

$$\Delta_g \nu_{f,3} = \langle \Delta_g \nu_f, e_3 \rangle = -g^{ij} \partial_i f_3 \partial_j H_f - |A_f|_g^2 \nu_{f,3}.$$

Combining with (A.2) and (A.3), we obtain

$$\Delta_{g^{\mathbb{R}^3}} H_f^{\mathcal{H}^3} = f_3 \Delta_g H_f + H_f^2 \nu_{f,3} - 2|A_f|_g^2 \nu_{f,3}.$$

As $2|A_f|_g^2 - H_f^2 = 2|A_f^0|_g^2$ by the Gauß equations, we get

$$\Delta_{g^{\mathbb{R}^3}} H_f^{\mathcal{H}^3} = f_3 \Delta_g H_f - 2|A_f^0|_g^2 \nu_{f,3}.$$

Then using the third equation

$$\Delta_{g^{\mathbb{R}^3}} H_f^{\mathcal{H}^3} + |A_f^{\mathbb{R}^3,0}|^2_{g^{\mathbb{R}^3}} H_f^{\mathcal{H}^3} =$$

$$= f_3 \Delta_{g^{\mathbb{R}^3}} H_f^{\mathbb{R}^3} - 2|A_f^{\mathbb{R}^3,0}|^2_{g^{\mathbb{R}^3}} \nu_{f,3}^{\mathbb{R}^3} + |A_f^{\mathbb{R}^3,0}|^2_{g^{\mathbb{R}^3}} (f_3 H_f^{\mathbb{R}^3} + 2\nu_{f,3}^{\mathbb{R}^3}) =$$

$$= f_3 (\Delta_{g^{\mathbb{R}^3}} H_f^{\mathbb{R}^3} + |A_f^{\mathbb{R}^3,0}|^2_{g^{\mathbb{R}^3}} H_f^{\mathbb{R}^3}),$$

which is the second identity in the fourth equation.

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