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The umbilic set of Willmore surfaces

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Abstract: It is well known that the umbilic points of minimal surfaces in spaces of constant sectional curvature consist only of isolated points unless the surface is totally umbilic on some connected component, as for example the Hopf form is holomorphic. In this note, we prove that on connected not totally umbilic Willmore surfaces in codimension one the umbilic set is locally a one dimensional real-analytic submanifold without boundary or an isolated point, and we mention examples that one dimensional submanifolds of umbilic points indeed occur.

Keywords: Willmore surfaces, umbilic points, Cauchy-Riemann equation.

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1 Introduction

For an immersed surface $f : \Sigma \to \mathbb{R}^3$ the Willmore functional is defined by

$$\mathcal{W}(f) = \frac{1}{4} \int_\Sigma |H|^2 \, \mathrm{d}\mu_g,$$

where H denotes the scalar mean curvature of f , $g = f^* g_{euc}$ the pull-back metric and μ_g the induced area measure of f on Σ .

Critical points of the Willmore functional, called Willmore surfaces or immersions, satisfy the Euler-Lagrange equation

$$\Delta_g H + |A^0|_g^2 H = 0, \tag{1.1}$$

see i.e. [KuSch02] §2, where the laplacian along f is used and $A^0 = A - \frac{1}{2}\vec{\mathbf{H}}g$ is the tracefree second fundamental form of f . In particular, smooth Willmore immersions are real-analytic, see [Mo58].

A point is called umbilic, when the tracefree part of the second fundamental form A^0 vanishes, likewise the set of non-umbilic points is given by

$$\mathcal{N} := [A^0 \neq 0]. \tag{1.2}$$

As Willmore immersions are real-analytic, $\mathcal{N} = [A^0 \neq 0]$ is dense in Σ , if f is not totally umbilic on any connected component.

Minimal surfaces are particular examples of Willmore surfaces by (1.1). This is also true for minimal surfaces in the sphere S^3 or in the hyperbolic space $\mathcal{H}^3$ after applying a local conformal diffeomorphism to the euclidean space by conformal invariance of the Willmore functional, see [Wei78] and [Ch74], and as $\mathbb{R}^3, S^3$ and $\mathcal{H}^3$ all have constant sectional curvature. Now the umbilic points of minimal surfaces consist only of isolated points unless the surface is totally umbilic on some connected component. In codimension one, this can easily be seen by considering the scalar trace-free second fundamental form $h^0_{ij} := \langle A^0_{ij}, \nu \rangle$ for a smooth unit normal ν of f , the function

$$\varphi := h^0_{11} - i h^0_{12} \tag{1.3}$$

and the quadratic Hopf form $\mathcal{H} = (\varphi/2)(dz)^2$. The Hopf form is defined independent of the oriented conformal local chart and changes to its conjugate when switching the orientation of the local chart. In case of minimal surfaces, φ is holomorphic, see for example [La70] Lemma 1.2 for the sphere, hence its zeros, which are precisely the umbilic points, consist only of isolated points unless φ vanishes identically on some connected component.

The isolated character of umbilic points is no longer true on general Willmore surfaces. In [BaBo93], examples of Willmore tori with umbilic lines are given, and in [LaSi84], see also [DaSch24], examples of rotational symmetric Willmore surfaces with umbilic circles are given. Both examples form minimal surfaces in the hyperbolic 3-space and are therefore isothermic, see [Pa91] Theorem 2.2 or our remark after Proposition 2.1. Moreover an elementary construction with the theorem of Cauchy-Kowalewskaja gives non-isothermic open Willmore surfaces in codimension one with closed umbilic lines, see [Sk18]. The construction of various umbilic points on Willmore surfaces with the theorem of Cauchy-Kowalewskaja goes back to K. Voss, see [Pa91] remark before Proposition 2.1.

The aim of this article is to show that not more can happen on Willmore surfaces in codimension one, more precisely we prove the following theorem.

Theorem 3.1 *For any smooth not totally umbilic Willmore immersion $f : \Sigma \to \mathbb{R}^3$ of a connected surface Σ , the set of umbilic points is a closed set in Σ which is locally a one dimensional real-analytic submanifold of Σ without boundary or an isolated point.*

Therefore the set of umbilic points of f can be written

$$\Sigma - \mathcal{N} = [A^0 = 0] = \Gamma + \mathcal{A},$$

where Γ is a closed set and a one dimensional real-analytic submanifold of Σ without boundary and $\mathcal{A}$ is a closed set isolated points. □

2 Cauchy-Riemann equation

We consider a smooth immersion $f : \Sigma \to \mathbb{R}^3$ of a surface Σ , or more precisely in a local conformal chart a conformal immersion $f : B_1(0) \subseteq \mathbb{C} \to \mathbb{R}^3$ with pull-back metric

$g = f^* g_{euc} = e^{2u} g_{euc}$ and use the differential operators of the Wirtinger calculus

$$\partial_z = \frac{1}{2}(\partial_1 - i\partial_2) \quad \text{and} \quad \partial_{\bar z} = \frac{1}{2}(\partial_1 + i\partial_2).$$

We derive a Cauchy-Riemann equation for the pair of functions φ , defined in (1.3), and $\partial_z H$, and we start with an equation for φ .

Proposition 2.1 *For any conformal immersion $f : B_1(0) \subseteq \mathbb{C} \to M^3, M^3 \in \{\mathbb{R}^3, S^3, \mathcal{H}^3\}$, with pull-back metric $g = f^* g_{M^3} = e^{2u} g_{euc}$, we have*

$$\partial_{\bar z}\varphi = \frac{e^{2u}}{2}\partial_z H, \tag{2.1}$$

where H is the scalar mean curvature of f in M^3 .

Beweis:
From (1.3), we calculate

$$\begin{aligned} 2\partial_{\bar z}\varphi &= (\partial_1 h^0_{11} + \partial_2 h^0_{12}) - i(\partial_1 h^0_{12} - \partial_2 h^0_{11}) \\ &= (\partial_1 h^0_{11} + \partial_2 h^0_{21}) - i(\partial_1 h^0_{12} + \partial_2 h^0_{22}) = \\ &= e^{2u} g^{kl}(\partial_k h^0_{l1} - i\partial_k h^0_{l2}), \end{aligned} \tag{2.2}$$

as h^0 is symmetric and tracefree with respect to $g = e^{2u} g_{euc}$. The covariant derivatives with respect to $g = e^{2u} g_{euc}$ are defined by

$$\nabla_k h^0_{lm} = \partial_k h^0_{lm} - \Gamma^r_{kl} h^0_{rm} - \Gamma^r_{km} h^0_{lr},$$

where Γ denote the Christoffel symbols with respect to the metric $g = e^{2u} g_{euc}$ and are given by

$$\begin{aligned} \Gamma^r_{kl} &= \frac{1}{2} g^{rs}(\partial_k g_{ls} + \partial_l g_{sk} - \partial_s g_{kl}) = \\ &= g^{rs}(g_{ls}\partial_k u + g_{sk}\partial_l u - g_{kl}\partial_s u) = \delta^r_l \partial_k u + \delta^r_k \partial_l u - g_{kl} g^{rs} \partial_s u \end{aligned}$$

for the Kronecker symbols δ . We calculate

$$\begin{aligned} g^{kl}\Gamma^r_{kl} &= g^{kl}(\delta^r_l \partial_k u + \delta^r_k \partial_l u - g_{kl} g^{rs}\partial_s u) = \\ &= g^{kr}\partial_k u + g^{lr}\partial_l u - 2g^{rs}\partial_s u = 0 \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} g^{kl}\Gamma^r_{km} h^0_{lr} &= g^{kl}(\delta^r_m \partial_k u + \delta^r_k \partial_m u - g_{km} g^{rs}\partial_s u) h^0_{lr} = \\ &= g^{kl} h^0_{lm}\partial_k u + g^{kl} h^0_{lk}\partial_m u - \delta^l_m g^{rs} h^0_{lr}\partial_s u = \\ &= g^{kl} h^0_{lm}\partial_k u - g^{rs} h^0_{rm}\partial_s u = 0, \end{aligned} \tag{2.4}$$

as h^0 is symmetric and tracefree with respect to g . Together

$$g^{kl}\nabla_k h^0_{lm} = g^{kl}\partial_k h^0_{lm}$$

and plugging into (2.2)

$$2\partial_{\bar z}\varphi = e^{2u} g^{kl}(\nabla_k h^0_{l1} - i\nabla_k h^0_{l2}).$$

On the other hand by the Mainardi-Codazzi equation, as $\mathbb{R}^3, S^3, \mathcal{H}^3$ all have constant sectional curvature, see [dC] §6 Proposition 3.4 and §4 Lemma 3.4,

$$g^{kl}\nabla_k h^0_{lm} = g^{kl}\nabla_k(h_{lm} - \frac{1}{2}g_{lm}H) = g^{kl}(\nabla_m h_{kl} - \frac{1}{2}g_{lm}\partial_k H) = \frac{1}{2}\partial_m H,$$

hence

$$\partial_{\bar{z}}\varphi = \frac{e^{2u}}{4}(\partial_1 H - i\partial_2 H) = \frac{e^{2u}}{2}\partial_z H,$$

which is (2.1).

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Remark:
For minimal surfaces in the constant curvature spaces $\mathbb{R}^3, S^3$ or $\mathcal{H}^3$, we get from (2.1) that φ is holomorphic, hence the umbilic points, which are the zeros of φ , consist only of isolated points, if f is not totally umbilic on any connected component.

Moreover by the holomorphy of φ , we can choose locally around any non-umbilic $p \in \mathcal{N}$ a holomorphic function w with $(w')^2 = \varphi/2$. Then we have in the local conformal coordinate w that $\mathcal{H}_f = (dw)^2$ or likewise $\varphi \equiv 1$, hence $h^0_{12} \equiv 0$ and likewise $A_{12} \equiv 0$, and we have local conformal principal curvature coordinates, see [Pa91] Theorem 2.2. In this case f is called isothermic locally around p , and f is called isothermic, if it is isothermic locally around any non-umbilic point. The property that f is isothermic locally around p remains true for $\Phi \circ f$ for any conformal transformation Φ , as φ transforms by an elementary calculation by

$$\varphi_{\Phi\circ f} = |D\Phi(f)|\varphi_f.$$

□

Proposition 2.2 *For any conformal Willmore immersion* $f : B_1(0) \subseteq \mathbb{C} \to \mathbb{R}^3$ *, we have the Cauchy-Riemann equation*

$$\partial_{\bar{z}}(\varphi, \partial_z H)^T = M \cdot (\varphi, \partial_z H)^T \tag{2.5}$$

for some smooth matrix $M : B_1(0) \to \mathbb{C}^{2\times 2}$.

Proof:
(2.1) implies the first row of (2.5). Now f satisfies as Willmore immersion the Euler-Lagrange equation (1.1), hence by the definition in (1.3) that

$$\Delta_g H = -|A^0|^2_g H = -g^{jk}g^{lm}\langle A^0_{jl}, A^0_{km}\rangle H =$$
$$= -e^{-4u}\Big((h^0_{11})^2 + (h^0_{12})^2 + (h^0_{21})^2 + (h^0_{22})^2\Big)H = -2e^{-4u}|\varphi|^2 H,$$

as h^0 is symmetric and tracefree with respect to g . On the other hand using (2.3), we get

$$\Delta_g H = g^{kl}\nabla_k\nabla_l H = g^{kl}\partial_k\partial_l H - g^{kl}\Gamma^m_{kl}\partial_m H =$$
$$= e^{-2u}(\partial_1\partial_1 H + \partial_2\partial_2 H) = 4e^{-2u}\partial_{\bar{z}}\partial_z H.$$

Together

$$\partial_{\bar{z}}\partial_z H = \frac{1}{4}e^{2u}\Delta_g H = -\frac{1}{2}e^{-2u}\bar{\varphi}H\varphi,$$

which gives the second row of (2.5).

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3 The structure of the umbilic set

In this section we prove our main theorem.

Theorem 3.1 *For any smooth not totally umbilic Willmore immersion* $f : \Sigma \to \mathbb{R}^3$ *of a connected surface* Σ *, the set of umbilic points is a closed set in* Σ *which is locally a one dimensional real-analytic submanifold of* Σ *without boundary or an isolated point.*

Therefore the set of umbilic points of f *can be written*

$$\Sigma - \mathcal{N} = [A^0 = 0] = \Gamma + \mathcal{A}, \tag{3.1}$$

where Γ *is a closed set and a one dimensional real-analytic submanifold of* Σ *without boundary and* $\mathcal{A}$ *is a closed set isolated points.*

Proof:
By continuity, the set of umbilic points $[A^0 = 0]$ is closed in Σ . We consider a conformal Willmore immersion $f : B_1(0) \subseteq \mathbb{C} \to \mathbb{R}^3$ with pull-back metric $g = f^* g_{euc} = e^{2u} g_{euc}$ which has 0 as an umbilic point. Clearly $[A^0 = 0] = [\varphi = 0]$.

By the Cauchy-Riemann equation for $(\varphi, \partial_z H)$ in Proposition 2.2 and observing that φ is real-analytic and does not vanish identically, as f is real-analytic and not totally umbilic, there exists by [EsTr88] Lemmas 2.1 and 2.2 an integer $m \in \mathbb{N}_0$ such that $(\varphi, \partial_z H)(z) = z^m (\psi, \chi)(z)$ for some smooth $\psi, \chi : B_1(0) \to \mathbb{C}$ with $(\psi(0), \chi(0)) \neq 0$. As $\varphi, \partial_z H$ are real-analytic, ψ, χ are real-analytic up to the origin as well.

As the origin is considered to be an umbilic point, we have

$$[A^0 = 0] = [\varphi = 0] = [\psi = 0] \cup \{0\}.$$

If $\psi(0) \neq 0$, the origin is an isolated umbilic point.

If $\psi(0) = 0$, then $\chi(0) \neq 0$. Observing by (2.1) that

$$z^m \partial_{\bar z} \psi(z) = \partial_{\bar z} \varphi(z) = \frac{e^{2u(z)}}{2} \partial_z H(z) = z^m \frac{e^{2u(z)}}{2} \chi(z),$$

we get by continuity $\partial_{\bar z} \psi = (e^{2u}/2)\chi$ and

$$\partial_{\bar z} \psi(0) = \frac{e^{2u(0)}}{2} \chi(0) \neq 0.$$

Therefore $D\psi(0) \neq 0$ or likewise $\nabla Re(\psi)(0) \neq 0$ or $\nabla Im(\psi)(0) \neq 0$. Then by the implicit function theorem, the set $[Re(\psi) = 0]$ or the set $[Im(\psi) = 0]$ is a real-analytic curve locally around the origin, and the set of umbilic points

$$[\varphi = 0] = [\psi = 0] = \Big([Re(\psi) = 0] \cap [Im(\psi) = 0] \Big)$$

is contained in this real-analytic curve locally around the origin. Since φ is real-analytic, we see that either the origin is an isolated umbilic point or the whole real-analytic curve belongs in a neighbourhood to the set of umbilic points, and the theorem is proved.

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Appendix

A Constructions of Willmore immersions with umbilic lines

In this appendix, we give a brief outline of the construction of Willmore surfaces with umbilic lines with the theorem of Cauchy-Kowalewskaja for the reader's convenience, for more details see [Sk18].

We consider any real-analytic path $\gamma : I \to \mathbb{R}^3$ parametrized by arc-length on an interval $0 \in I \subseteq \mathbb{R}$ with a real-analytic unit tangent $\mathbf{t} = \gamma'$ and a real-analytic unit normal $\mathbf{n}$ on I such that γ parametrizes a principal curvature line in any surface $M \supseteq \Gamma$ with tangent space spanned by $\mathbf{t}, \mathbf{n}$ on I, that is the second fundamental form h_M of M is diagonal on I with respect to a basis $\mathbf{t}, \mathbf{n}$ of $T_\gamma M$ on I. For a smooth unit normal $\nu_M \perp \mathbf{t}, \mathbf{n}$, this reads

$$0 = h_M(\mathbf{t}, \mathbf{n}) = \langle \nu_M, \nabla^M_{\mathbf{t}} \mathbf{n} \rangle = \langle \nu_M, \mathbf{n}' \rangle \quad \text{on } I. \tag{A.1}$$

Indeed this exists, for example for any planar $\gamma : I \to \mathbb{R}^2$, we can put $\mathbf{T} := \mathbf{t}, \mathbf{N} := D_{\pi/2}\mathbf{T}$, where $D_{\pi/2}(x, y) = (-y, x)$ defines the counter-clockwise rotation with positive angle $\pi/2$ in the plane, and for any $|\alpha| \leq 1$ that

$$\mathbf{n} := \alpha \mathbf{N} + \sqrt{1 - \alpha^2} \cdot e_3. \tag{A.2}$$

Then $\mathbf{T}' = \kappa \mathbf{N}$ for the curvature κ and $\mathbf{N}' = -\kappa \mathbf{T} = -\kappa \mathbf{t} \perp \nu_M$, hence

$$\langle \nu_M, \mathbf{n}' \rangle = \langle \nu_M, -\alpha\kappa \mathbf{t} \rangle = 0,$$

which is (A.1). We see that M intersects respectively touches $\mathbb{R}^2$ in constant angle ϑ with $\cos\vartheta = |\langle \nu_M, e_3 \rangle| = |\langle \mathbf{n}, \mathbf{N} \rangle| = |\alpha| \leq 1, 0 \leq \vartheta \leq \pi/2$.

Putting $f_* : I \times]-\varepsilon, \varepsilon[\to \mathbb{R}^3$ by

$$f_*(s, t) := \gamma(s) + t\mathbf{n}(s),$$

this is real-analytic with Df_* of full rank for $\varepsilon > 0$ small after possibly making I smaller or for periodic data on $I = \mathbb{R}$. Clearly for a real-analytic unit normal ν_* of f_*, we have $\nu_*(s, 0) = \nu_M(\gamma(s))$ by appropriate choice, and for the second fundamental form h_* of f_* by (A.1) that

$$0 = \langle \mathbf{n}'(s), \nu_M(\gamma(s)) \rangle = \langle \partial_{12} f_*(s, 0), \nu_*(s, 0) \rangle = h_{*,12}(s, 0) \quad \text{for } s \in I. \tag{A.3}$$

For any $\psi : I \times]-\varepsilon, \varepsilon[\to \mathbb{R}$, we put $f : I \times]-\varepsilon, \varepsilon[\to \mathbb{R}^3$ with

$$f(s, t) := f_*(s, t) + \psi(s, t)\nu_*(s, t) \quad \text{for } s \in I, |t| < \varepsilon.$$

When

$$\psi(s, 0), \partial_2\psi(s, 0) = 0 \quad \text{for } s \in I, \tag{A.4}$$

then Df has full rank for $\varepsilon = \varepsilon_\psi$ small after possibly making I smaller or for periodic data on $I = \mathbb{R}$, and the pull-back metric $g = f^* g_{euc}$ satisfies

$$g_{ij}(s, 0) = \delta_{ij} \quad \text{for } s \in I, \tag{A.5}$$

$T_\gamma f = T_\gamma f_*$ and for a real-analytic unit normal ν of f that $\nu(s, 0) = \nu_*(s, 0)$. Further for the second fundamental form

$$h_{ij}(s, 0) := \langle \partial_{ij} f(s, 0), \nu(s, 0) \rangle = h_{*,ij}(s, 0) + \partial_{ij}\psi(s, 0).$$

As $\partial_{12}\psi(s,0) = 0$ by (A.4), we get by (A.3) that $h_{12}(s,0) = 0$. Then by (A.5), we see that f is umbilic on I , if and only if $h_{11}(s,0) = h_{22}(s,0)$. Observing that $\partial_{11}\psi(s,0) = 0$ by (A.4) and

$$h_{*,22}(s,0) = \langle \partial_{22} f_*(s,0), \nu_*(s,0) \rangle = 0,$$

this is true if and only if

$$\partial_{22}\psi(s,0) = h_{11}(s,0) = \langle \gamma''(s), \nu_*(s,0) \rangle \quad \text{for } s \in I. \tag{A.6}$$

Moreover f is a Willmore immersion, if by (1.1)

$$\Delta_g H_f + |h_f^0|_g^2 H_f = 0.$$

Now clearly with the Christoffel symbols Γ_{ij}^r of g , we have

$$H_f = \langle \Delta_g f, \nu \rangle = \langle g^{ij}(\partial_{ij} f - \Gamma_{ij}^r \partial_r f), \nu \rangle$$

and for the covariant derivative ∇ with repsect to g that

$$\nabla_k H_f = \langle g^{ij} \nabla_{kij} f, \nu \rangle + \langle \Delta_g f, \nabla_k \nu \rangle =$$

$$= \langle g^{ij} \partial_{ijk} f, \nu \rangle + B_1(f, Df, D^2 f) = g^{ij} \partial_{ijk}\psi \cdot \langle \nu_*, \nu \rangle + B_1(., \psi, D\psi, D^2\psi),$$

where B_1 is a real-analytic function depending on f_* and adjusts from line to line. Then

$$\Delta_g H_f = g^{kl} \nabla_l \nabla_k H_f = g^{ij} g^{kl} \partial_{ijkl}\psi \cdot \langle \nu_*, \nu \rangle + B_2(., \psi, D\psi, D^2\psi, D^3\psi),$$

and, as on the other hand clearly $|h_f^0|_g^2 H_f = B_3(., \psi, D\psi, D^2\psi)$ and $\langle \nu_*, \nu \rangle \equiv 1 \neq 0$ for $t = 0$, we see that f is a Willmore immersion if and only if

$$g^{ij} g^{kl} \partial_{ijkl}\psi + B_4(., \psi, D\psi, D^2\psi, D^3\psi) = 0.$$

As $g^{22} \neq 0$, we can eliminate $\partial_{2222}\psi$ and see that f is a Willmore immersion if and only if

$$\partial_{2222}\psi = B(., (D^m\psi, D^m\partial_1\psi)_{m=0,1,2,3}) \quad \text{on } I \times]-\varepsilon, \varepsilon[. \tag{A.7}$$

with B real-analytic function depending on f_* . Adding an intial condition for the third derivative

$$\partial_{222}\psi(s,0) = \psi_3(s) \quad \text{for } s \in I \tag{A.8}$$

with some real-analytic ψ_3 being periodic for periodic data, there exists by the theorem of Cauchy-Kowalewskaja a unique real-analytic solution ψ of (A.7), (A.4), (A.6) and (A.8) locally around 0 or for some $\varepsilon > 0$ for periodic data. Then f parametrizes a Willmore surface with all points on $I \times \{0\}$ being umbilic.

If for example γ is planar in $\mathbb{R}^2$ as above, and γ is not part of a circle, and M respectively f are not tangent to $\mathbb{R}^2 \times \{0\}$ along I , then f is not part of a 2-plane or a 2-sphere, hence f is not totally umbilic, and even closed umbilic lines can occur in the Theorem 3.1.

The examples of Willmore surfaces with umbilic lines in [BaBo93] and [DaSch24] mentioned in the introduction are both minimal surfaces in the hyperbolic 3-space and are therefore isothermic for example by the remark after Proposition 2.1. Now by Proposition

A.1 below, for isothermic, not totally umbilic Willmore immersions in codimension one all umbilic lines are contained in one 2-plane or one 2-sphere which is intersected orthogonally by the immersion. Choosing γ in the above construction as planar in $\mathbb{R}^2$ and not being part of a circle, and M respectively f are neither tangent, nor orthogonal to $\mathbb{R}^2 \times \{0\}$ along I, then the resulting Willmore immersion f is not isothermic. $\square$

Remarks:

1. For any real-analytic path $\gamma : I \to \mathbb{R}^3$ parametrized by arc-length, condition (A.1) reads, as $\mathbf{t}, \mathbf{n}, \nu_M$ form an orthnormal basis of $\mathbb{R}^3$ and $\langle \mathbf{n}', \mathbf{n} \rangle = 0$ by $|\mathbf{n}| = 1$, that
$$\mathbf{n}' = \langle \mathbf{n}', \mathbf{t} \rangle \mathbf{t} = -\langle \mathbf{n}, \mathbf{t}' \rangle \mathbf{t}. \tag{A.9}$$
This is a linear ordinary differential equation for $\mathbf{n}$ with real-analytic coefficients, and therefore admits for any initial data locally a real-analytic solution, in particular any γ locally admits many unit normals satisfying (A.1). For periodic and planar γ, we see by (A.2) that $\mathbf{n}$ is always periodic as well.

   For non-planar γ with $\kappa = |\mathbf{t}'| > 0$, we have $\mathbf{T} = \mathbf{t}$, the normal $\mathbf{N} := \mathbf{T}'/\kappa$ of γ and get the binormal $\mathbf{B}$ of γ as the unit vector such that $\mathbf{T}, \mathbf{N}, \mathbf{B}$ form a positive orthonormal basis of $\mathbb{R}^3$, that is $\mathbf{B} := \mathbf{T} \times \mathbf{N}$. We put
$$\mathbf{n} := \mathbf{N} \cos\vartheta + \mathbf{B} \sin\vartheta \tag{A.10}$$
for some real-analytic function ϑ on I. By the Frenet-Serret equations, we have for the torsion τ of γ that $\mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B}, \mathbf{B}' = -\tau\mathbf{N}$, hence
$$\pm\langle \nu_M, \mathbf{n}' \rangle =$$
$$= \langle -\mathbf{N}\sin\vartheta + \mathbf{B}\cos\vartheta, -\mathbf{N}(\sin\vartheta)\vartheta' + (-\kappa\mathbf{T} + \tau\mathbf{B})\cos\vartheta + \mathbf{B}(\cos\vartheta)\vartheta' - \mathbf{N}\tau\sin\vartheta \rangle =$$
$$= \langle -\mathbf{N}\sin\vartheta + \mathbf{B}\cos\vartheta, (-\mathbf{N}(\sin\vartheta) + \mathbf{B}(\cos\vartheta))(\vartheta' + \tau) \rangle = \vartheta' + \tau,$$
and (A.1) respectively (A.9) are equivalent to
$$\vartheta' = -\tau \quad \text{on } I. \tag{A.11}$$
For planar γ, there is $\tau \equiv 0$, hence (A.11) conincides with (A.2).

   In order to have $\mathbf{n}$ periodic with the same period, say L, as γ, we must have
$$\int_0^L \tau(s)\ \mathrm{d}s \in 2\pi\mathbb{Z}. \tag{A.12}$$

2. Choosing for example $\gamma(t) := (\cos t, \sin t, \sigma \sin(2t))$ for $t \in \mathbb{R}, \sigma \neq 0$, then γ has period 2π and parametrizes a closed real-analytic curve $\Gamma := \gamma(\mathbb{R}) \subseteq \mathbb{R}^3$, although the parametrization is not by arc length. As the curvature
$$\kappa = |\gamma' \times \gamma''| / |\gamma'|^3$$
by standard curve theory and
$$\gamma'(t) \times \gamma''(t) \cdot e_3 = (-\sin t, \cos t, 2\sigma\cos(2t)) \times (-\cos t, -\sin t, -4\sigma\sin(2t)) \cdot e_3 =$$

$$= (-\sin t, \cos t, 0) \times (-\cos t, -\sin t, 0) \cdot e_3 = 1 \neq 0,$$

we have $\kappa > 0$ on $\mathbb{R}$. Next as

$$\gamma(t + \pi/2) = (\cos(t + \pi/2), \sin(t + \pi/2), \sigma \sin(2t + \pi)) =$$

$$= (D_{\pi/2}(\cos t, \sin t), -\sigma \sin(2t)) = diag(D_{\pi/2}, -1)\gamma(t) \tag{A.13}$$

and $O := diag(D_{\pi/2}, -1)$ is orthogonal with $\det O = -1$, we get $|\gamma'(t + \pi/2)| = |\gamma'(t)|, (\mathbf{T}, \mathbf{N}, \mathbf{B})(t+\pi/2) = (O\mathbf{T}, O\mathbf{N}, -O\mathbf{B})(t), \kappa(t+\pi/2) = \kappa(t)$ and $\tau(t+\pi/2) = -\tau(t)$. Therefore

$$\int_0^{2\pi} \tau(t)|\gamma'(t)| \; \mathrm{d}t = 0, \tag{A.14}$$

which yields (A.12).

Further $\tau \not\equiv 0$, since otherwise $B \equiv B(0)$, hence $B(0) = B(\pi/2) = -OB(0)$ and $B(0) = \pm e_3$, but this is not true, as $\gamma'(0) = (0, 1, 2\sigma) \not\perp e_3$. In particular, Γ is not contained in any 2-plane.

Now if Γ were contained in some 2-sphere $S \subseteq \mathbb{R}^3$, we observe from

$$\gamma(0) = (1, 0, 0), \gamma(\pi) = (-1, 0, 0) \in S,$$

$$\gamma(\pi/2) = (0, 1, 0), \gamma(3\pi/2) = (0, -1, 0) \in S,$$

that the center of S lies in the $y - z-$plane and in the $x - z-$plane, hence the center lies on the $z-$axis, say it is $(0, 0, z_0)$. But this is not ture, as

$$|\gamma(t) - (0, 0, z_0)|^2 = 1 + \sigma^2 |\sin(2t) - z_0|^2$$

is not constant.

Since $\kappa > 0$, the torsion τ is real-analytic, and we can integrate (A.11) with some initial data and get a real-analytic normal $\mathbf{n}$ on $\mathbb{R}$ which satisfies (A.1) and which is periodic by (A.14) respectively by (A.12). Then our previous construction by the theorem of Cauchy-Kowalewskaja yields a Willmore immersion f with a closed umbilic line on $\mathbb{R} \times \{0\} \to \Gamma$, and the umbilic line Γ is not contained in any 2-plane or 2-sphere. In particular, f is not totally umbilic, and moreover f is not isothermic by Proposition A.1 below.

□

Finally, we prove the proposition, which was applied above. Our proof uses a theorem of Thomsen in [Th24].

Proposition A.1 *For any isothermic, not totally umbilic Willmore immersion $f : \Sigma \to \mathbb{R}^3$ of a connected surface Σ , all umbilic lines are contained in one 2-plane or one 2-sphere which is intersected orthogonally everywhere by f .*

Moreover if f indeed has some umbilic line, then after applying an appropriate Moebius transformation, f is minimal in the hyperbolic space outside its umbilic lines.

Proof:
Thomsen showed in [Th24], see also [Pa91] Theorem 2.2, that any umbilic free isothermic Willmore immersion of a connected surface in codimension one becomes after applying an appropriate Moebius transformation of $\mathbb{R}^3 \cup \{\infty\}$ minimal in a constant curvature space as the euclidian space $\mathbb{R}^3$, the sphere S^3 or the hyperbolic space $\mathcal{H}^3$. Applying this result to any component of the set of non-umbilic points $\mathcal{N} := [A^0 \neq 0] \neq \emptyset$ of f, we see that there is a Moebius transformation Φ of $\mathbb{R}^3 \cup \{\infty\}$ such that $\Phi \circ f$ is minimal in $\mathbb{R}^3, S^3$ or $\mathcal{H}^3$ on a non-empty open subset. By real-analyticity of Willmore immersions, see [Mo58], this extends in case of S^3 to whole of Σ and in case of $\mathbb{R}^3$ to Σ apart from the countable set $f^{-1}(\Phi^{-1}(\infty))$. As the umbilic points of minimal surfaces in S^3 or $\mathbb{R}^3$ consist only of isolated points, f cannot contain an umbilic line.

Thererfore if f indeed has some umbilic line, then $\Phi \circ f$ is minimal in the hyperbolic space $\mathcal{H}^3$ on a non-empty open subset. In this case, we equipp $\mathcal{H}^3_\pm := \{\pm x_3 > 0\}$ with the hyperbolic metric $x_3^{-2} g_{euc}$ and see that $\Phi \circ f : \Sigma - f^{-1}(\Phi^{-1}(\infty)) \to \mathbb{R}^3$ is minimal in the hyperbolic space on a non-emtpy open subset of $\Sigma_\pm := (\Phi \circ f)^{-1}(\mathcal{H}_\pm)$. Now $\{x_3 = 0\} \not\subseteq (\Phi \circ f)(\Sigma)$, since otherwise $\Phi \circ f$ parametrizes a non-emtpy open part of a plane, hence is totally umbilic on a non-emtpy open set, hence is totally umbilic on the whole of Σ again by real-analyticity, contrary to our assumption. Therefore we can invert at a point $p \in \{x_3 = 0\} - (\Phi \circ f)(\Sigma)$, which is an isometry in the hyperbolic space, and get $f^{-1}(\Phi^{-1}(\infty)) = \emptyset$, hence $\Phi \circ f : \Sigma \to \mathbb{R}^3$ is minimal in the hyperbolic space on a non-emtpy open subset of $\Sigma_\pm := (\Phi \circ f)^{-1}(\mathcal{H}_\pm)$. We replace $\Phi \circ f$ by f in order to simplify the notation. By elementary differential geometry, the hyperbolic mean curvature is given by

$$H_f^{\mathcal{H}^3} = f_3 H_f^{\mathbb{R}^3} + 2\nu_3^{\mathbb{R}^3} \quad \text{on } \Sigma_\pm, \tag{A.15}$$

where $\nu^{\mathbb{R}^3}$ is locally a smooth unit normal along f in $\mathbb{R}^3$. The right hand side is a real-analytic function up to the sign on the whole of Σ which vanishes on a non-emtpy open subset of $\Sigma_\pm$, hence vanishes everywhere on Σ, that is

$$f_3 H_f^{\mathbb{R}^3} + 2\nu_3^{\mathbb{R}^3} \equiv 0 \quad \text{on } \Sigma, \tag{A.16}$$

in particular f is minimal on the whole $\Sigma_+ \cup \Sigma_-$. Again as the umbilic points of minimal surfaces in $\mathcal{H}^3$ consist only of isolated points, $\Sigma_\pm$ cannot contain umbilic lines, hence

$$\cup\{\text{umbilic lines of } f\} \subseteq \Sigma_0 := \Sigma - (\Sigma_+ \cup \Sigma_-) = [f_3 = 0]. \tag{A.17}$$

Therefore all umbilic lines of f are contained in one 2-plane or one 2-sphere. Next we see by (A.16) that

$$\nu_3^{\mathbb{R}^3} = 0 \quad \text{on } \Sigma_0, \tag{A.18}$$

hence f intersects $\{x_3 = 0\}$ orthogonally, which proves the first part of the proposition.

As we already know that f is minimal in the hyperbolic space on $\Sigma_+ \cup \Sigma_- = \Sigma - \Sigma_0$, the second part follows when we prove equality in the first inclusion of (A.17), that is

$$\cup\{\text{umbilic lines of } f\} = \Sigma_0. \tag{A.19}$$

From (A.18), we see that Σ_0 is a smooth one-dimensional submanifold without boundary and

$$e_3 \in T_p f \text{ for any } p \in \Sigma_0.$$

Clearly as $\Sigma_0 = [f_3 = 0]$, we have $T\Sigma_0 \perp e_3$, and there is a smooth orthonormal basis $\mathbf{t}, \mathbf{n}$ of Tf along Σ_0 with $\mathbf{t} \in T\Sigma_0, \partial_{\mathbf{n}} f = e_3$. First for the scalar second fundamental form h_f of f in $\mathbb{R}^3$, we get by tangency of $\mathbf{t}$ at Σ_0 that

$$0 = \langle \partial_{\mathbf{t}} e_3, \nu \rangle = \langle \partial_{\mathbf{tn}} f, \nu \rangle = h_f(\mathbf{t}, \mathbf{n}) \quad \text{on } \Sigma_0.$$

Secondly differentiating (A.16), we get on $\Sigma_0 = [f_3 = 0]$ that

$$0 = \partial_{\mathbf{n}}(f_3 H_f^{\mathbb{R}^3} + 2\nu_3^{\mathbb{R}^3}) = (\partial_{\mathbf{n}} f_3) H_f^{\mathbb{R}^3} + 2\langle \partial_{\mathbf{n}} \nu, e_3 \rangle = H_f^{\mathbb{R}^3} + 2\langle \partial_{\mathbf{n}} \nu, \partial_{\mathbf{n}} f \rangle =$$

$$= H_f^{\mathbb{R}^3} - 2h_f(\mathbf{n}, \mathbf{n}) = h_f(\mathbf{t}, \mathbf{t}) - h_f(\mathbf{n}, \mathbf{n})$$

and $h_f(\mathbf{t}, \mathbf{t}) = h_f(\mathbf{n}, \mathbf{n})$ on Σ_0 . Together we get that $h_f = (1/2) H_f g$ is scalar on Σ_0 , hence $A^0 \equiv 0$ on Σ_0 , and Σ_0 contains only umbilic points. Combining with (A.17), this yields (A.19) and concludes the proof of the proposition.

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B Minimal surfaces in constant curvature spaces

For minimal surfaces in the constant curvature spaces $\mathbb{R}^n, S^n, \mathcal{H}^n$ in arbitrary codimension, we even have that the umbilic points consist only of isolated points.

Theorem B.1 *For any smooth not totally umbilic minimal immersion $f : \Sigma \to M^n, M^n \in \{\mathbb{R}^n, S^n, \mathcal{H}^n\}$ of a connected surface Σ , the set of umbilic points is a closed set in Σ which consists only of isolated points.*

Proof:
Again by continuity, the set of umbilic points $[A^0 = 0]$ is closed in Σ . Working locally, we consider a smooth conformal immersion $f : B_1(0) \subseteq \mathbb{C} \to M^n, M^n \in \{\mathbb{R}^n, S^n, \mathcal{H}^n\}$, with pull-back metric $g = f^* g_{M^n} = e^{2u} g_{euc}$, denote by $D^\perp$ the connection in the normal bundle of f and define the Wirtinger connections in the normal bundle of f by

$$D_z^\perp = \frac{1}{2}(D_1^\perp - i D_2^\perp) \quad \text{and} \quad D_{\bar{z}}^\perp = \frac{1}{2}(D_1^\perp + i D_2^\perp).$$

We consider the complex normal field along f defined by

$$\phi := A_{11}^0 - i A_{12}^0,$$

or more precisely the vector-valued Hopf form as the quadratic form $\mathcal{H} = (\phi/2)(dz)^2$. The Hopf form is defined independent of the oriented conformal local chart and changes to its conjugate when switching the orientation of the local chart. Clearly $[A^0 = 0] = [\phi = 0]$.

We calculate

$$2D_{\bar{z}}^\perp \phi = (D_1^\perp A_{11}^0 + D_2^\perp A_{12}^0) - i(D_1^\perp A_{12}^0 - D_2^\perp A_{11}^0)$$

$$= (D_1^\perp A_{11}^0 + D_2^\perp A_{21}^0) - i(D_1^\perp A_{12}^0 + D_2^\perp A_{22}^0) =$$

$$= e^{2u} g^{kl} (D_k^\perp A_{l1}^0 - i D_k^\perp A_{l2}^0), \tag{B.1}$$

as A^0 is symmetric and tracefree with respect to $g = e^{2u} g_{euc}$. Again by (2.3) and (2.4), we obtain for the covariant normal derivative that

$$g^{kl}\nabla_k^\perp A_{lm}^0 - g^{kl} D_k^\perp A_{lm}^0 = -g^{kl}\Gamma_{kl}^r A_{rm}^0 - g^{kl}\Gamma_{km}^r A_{lr}^0 = 0,$$

and plugging into (B.1)

$$2D_{\bar z}^\perp \phi = e^{2u} g^{kl}(\nabla_k^\perp A_{l1}^0 - i\nabla_k^\perp A_{l2}^0).$$

On the other hand by the Mainardi-Codazzi equation, as $\mathbb{R}^n, S^n, \mathcal{H}^n$ all have constant sectional curvature, see [dC] §6 Proposition 3.4 and §4 Lemma 3.4,

$$g^{kl}\nabla_k^\perp A_{lm}^0 = g^{kl}\nabla_k^\perp A_{lm} - \frac{1}{2} g^{kl} g_{lm}\nabla_k^\perp \vec{\mathbf{H}} = g^{kl}\nabla_m^\perp A_{kl} - \frac{1}{2}\nabla_m^\perp \vec{\mathbf{H}} = \frac{1}{2} D_m^\perp \vec{\mathbf{H}},$$

hence

$$D_{\bar z}^\perp \phi = \frac{e^{2u}}{4}(D_1^\perp \vec{\mathbf{H}} - iD_2^\perp \vec{\mathbf{H}}) = \frac{e^{2u}}{2} D_z^\perp \vec{\mathbf{H}}.$$

When f is minimal, this yields

$$D_{\bar z}^\perp \phi = 0.$$

As we work locally, we may assume that $f : B_1(0) \to U \subseteq M^n$ for a chart $\varphi : U \xrightarrow{\approx} V \subseteq \mathbb{R}^n$. Writing $\phi = (\phi^1, \ldots, \phi^n)$, the connection D in $M^n \supseteq U \cong V$ reads

$$D_k \phi^\alpha = \partial_k \phi^\alpha + \bar\Gamma_{k\beta}^\alpha \phi^\beta$$

for appropriate symbols $\bar\Gamma$, hence

$$0 = D_{\bar z}^\perp \phi = \partial_{\bar z}^\perp \phi + \frac{1}{2}(\bar\Gamma_{1\beta}^\alpha + i\bar\Gamma_{2\beta}^\alpha)^\perp \phi^\beta$$

or likewise $\partial_{\bar z}^\perp \phi = M\phi$ for some smooth matrix $M : B_1(0) \to \mathbb{C}^{n\times n}$.

The normal derivative on the left side is no problem, as we see for any complex normal field $N : B_1(0) \to \mathbb{C}^n$ by observing $\langle N, \partial_k f\rangle \equiv 0$ that

$$\partial_{\bar z} N - \partial_{\bar z}^\perp N = g^{kl}\langle \partial_z N, \partial_k f\rangle \partial_l f = -g^{kl}\langle N, \partial_{\bar z}\partial_k f\rangle \partial_l f = M_f \cdot N$$

for some smooth matrix $M_f : B_1(0) \to \mathbb{C}^{n\times n}$, hence we obtain the Cauchy-Riemann equation $\partial_{\bar z}\phi = \tilde M \phi$ for some smooth matrix $\tilde M : B_1(0) \to \mathbb{C}^{n\times n}$.

Again by [EsTr88] Lemmas 2.1 and 2.2, as ϕ is real-analytic and does not vanish identically, as f is real-analytic and not totally umbilic, we get $\phi(z) = z^m \Psi(z)$ for some $m \in \mathbb{N}_0$ and some real-analytic $\Psi : B_1(0) \to \mathbb{C}^n$ with $\Psi(0) \neq 0$. Therefore $\phi(z) \neq 0$ for $z \neq 0$ close to the origin, and the origin can be at most an isolated umbilic point of f, which concludes the proof of the theorem.

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