$\Omega = \{1, 2, 3\}$

$\Sigma =$ "Menge aller Teilmenge von Ω"

$= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \Omega\}$

Würfel: $\Omega = \{1, 2, 3, 4, 5, 6\}$

Ereignis: $C \subseteq \Omega$

$A = \{1, 3, 5\}$

$B = \{3, 4, 5, 6\}$

- $P(C) = \frac{\# C}{\# \Omega}$

  erfüllt (1) & (2)

  "Laplace-Würfel"

  damit $P[A] = \frac{3}{6} = \frac{1}{2}$

  $P[B] = \frac{4}{6} = \frac{2}{3}$

- $P[\{1\}] = \frac{1}{12}$, $P[\{6\}] = \frac{3}{12}$

  $P[\{j\}] = \frac{2}{12} = \frac{1}{6}$, $j = 2, 3, 4, 5$

"unfairer Würfel"

erfüllt auch (1) + (2)

$$P[A] = \frac{1}{12} + \frac{1}{6} + \frac{1}{6} = \frac{5}{12}$$

$$P[B] = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{3}{12} = \frac{9}{12} = \frac{3}{4}$$

Beweise der Folgerungen

(i) $P[A \cup A^c] = P[\Omega] \underset{(1)}{=} 1$

$\underset{(2)}{=} P[A] + P[A^c]$

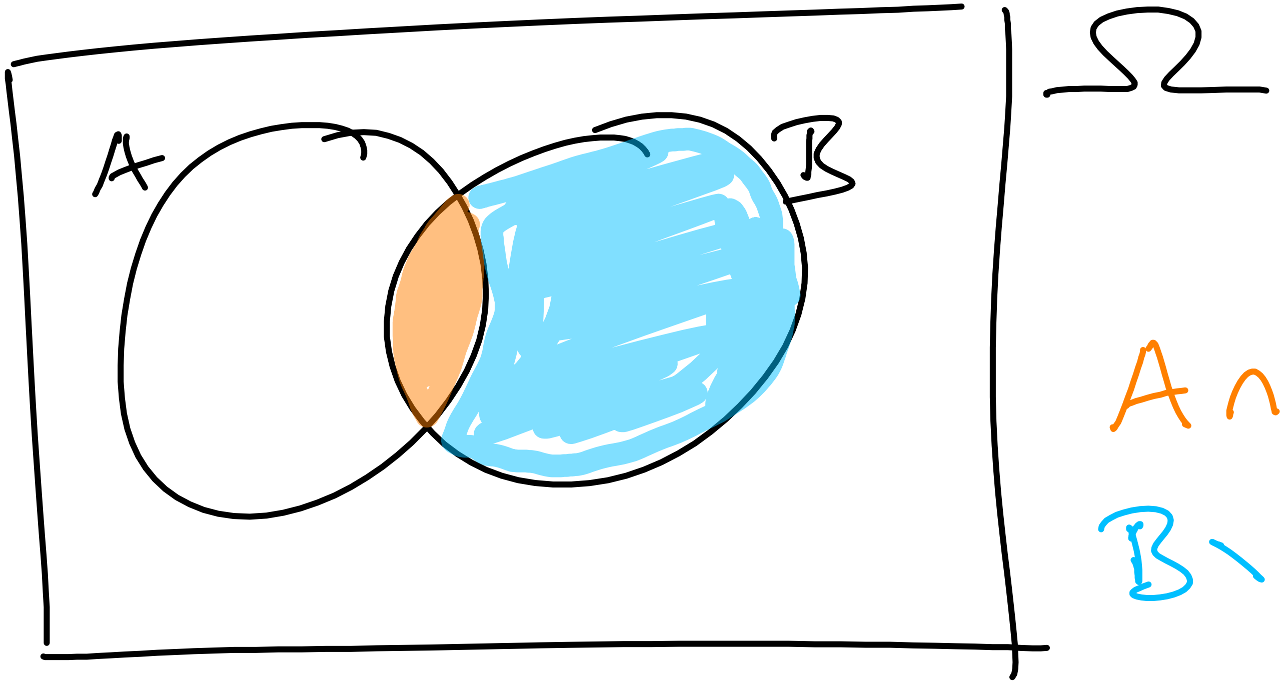
$\Rightarrow P[A^c] = 1 - P[A]$

(ii) $P[\emptyset] \underset{(i)}{=} 1 - P[\Omega] \underset{(1)}{=} 0$

(iii) $P[A \cup B] = P[A \cup (B \setminus A)]$

$\underset{(2)}{=} P[A] + P[B \setminus A] + P[A \cap B] - P[A \cap B]$

$= P[A] + P[\underbrace{(B \setminus A) \cup (A \cap B)}_{= B}] - P[A \cap B]$



$A \cap B$

$B \setminus A$

$$B \setminus A = \{\omega \mid \omega \in B \text{ und } \omega \notin A\}$$
$$= B \cap A^c$$

Beispiel: Würfel

$A = \{1,3,5\}$

$B = \{3,4,5,6\}$

$P[A \cup B] = P[\{1,3,4,5,6\}]$

$= \frac{5}{6}$ (Laplace-W.)

$P[A \cup B] = P[A] + P[B] - P[A \cap B]$

$= P[\{1,3,5\}] + P[\{3,4,5,6\}] - P[\{3,5\}]$

$= \frac{3}{6} + \frac{4}{6} - \frac{2}{6} = \frac{5}{6}$

Bsp: Bedingte Wahrscheinlichkeiten (Würfel)

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[\{3,5\}]}{P[\{3,4,5,6\}]}$$

$$= \frac{2/6}{4/6} = \frac{2}{4} = \frac{1}{2}$$

$$P[B|A] = \frac{P[\{3,5\}]}{P[\{1,3,5\}]} = \frac{2/6}{3/6} = \frac{2}{3}$$

Beweis zum Satz von Bayes:

$$P[A_j \mid B] = \frac{P(A_j \cap B)}{P[B]} \frac{P[A_j]}{P[A_j]}$$

$$= \frac{P[B \mid A_j]\, P[A_j]}{P[B]}$$

auch umgekehrt hilfreich

$$= \frac{P[B \mid A_j]\, P[A_j]}{\sum_{k=1}^{n} P[B \cap A_k]}$$

$$= \frac{P[B \mid A_j]\, P[A_j]}{\sum_{k=1}^{n} P[B \mid A_k]\, P[A_k]} \qquad \square$$

Beispiel: Diagnostischer Test

$P[A_1 | B] = ?$

$P[A_1] = 0{,}01$

$P[A_2] = P[A_1^c] = 0{,}99$

$P[B | A_1] = 0{,}98$

$P[B^c | A_2] = 0{,}95 \Rightarrow P[B | A_2] = 0{,}05$

$$P[A_1 | B] = \frac{P[B|A_1]\,P[A_1]}{P[B|A_1]\,P[A_1] + P[B|A_2]\,P[A_2]}$$

$$= \frac{0{,}98 \cdot 0{,}01}{0{,}98 \cdot 0{,}01 + 0{,}05 \cdot 0{,}99} = \frac{98}{98 + 495} \approx \underline{\underline{\frac{1}{6}}}$$

Wahrsch. dass Pers. krank, falls Test positiv

B_A : Anton wurde begnadigt

B_B : Brigitte —— " ——

B_C : Clemens —— " ——

G_B : Brigitte wird genannt

G_C : Clemens —— " ——

$$P[B_A | G_B] = ?$$

$$P[B_A] = P[B_B] = P[B_C] = \frac{1}{3}$$

$$P[G_B | B_A] = \frac{1}{2} = P[G_C | B_A]$$

$$P[G_B | B_B] = 0 = P[G_C | B_C]$$

$$(P[G_A] = 0) \quad P[G_B | B_C] = 1 = P[G_C | B_B]$$

$$P[B_A \mid G_B] = \frac{P[G_B \mid B_A] \cdot P[B_A]}{P[G_B \mid B_A]\,P[B_A] + P[G_B \mid B_B] \cdot P[B_B] + P[G_B \mid B_C]\,P[B_C]}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} = \frac{1/6}{1/6 + 1/3} = \frac{1}{3}$$

und damit $P[B_C \mid G_B] = \frac{2}{3}$!