Universität Tübingen, Fachbereich Mathematik
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Group Representations in Physics

Homework Assignment 3 (due on 15 Nov 2017)

Problem 13

Let $\varphi : G \to \mathrm{U}(n)$ be a unitary irreducible representation of a group. Show: If G is abelian then $n = 1$.

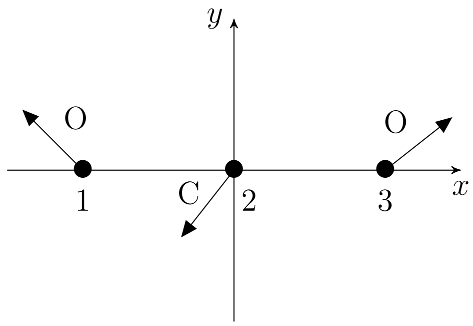
Problem 14

Let G be a finite group and $\Gamma : G \to \mathrm{GL}(V)$ a finite dimensional representation. Prove that $|\det \Gamma(g)| = 1 \ \forall\, g \in G$.

Problem 15

CO_2 is a linear molecule; in its ground state the carbon atom sits in the middle between the two oxygen atoms. The symmetry group of this system is isomorphic to the Klein four group $V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and has the following elements: the identity (e), reflections (σ_x and σ_y) across the x- and y-axis, respectively, and a rotation (R) by 180° about the origin.

A coplanar vibration entails displacements of the 3 atoms in a fixed plane. It can be characterised by a vector $(x_1, y_1, x_2, y_2, x_3, y_3) \in \mathbb{R}^6$.



Determine the action of the symmetry group on the canonical basis of $\mathbb{R}^6$. Write down the resulting six dimensional representation of V_4. Is this representation irreducible?

Problem 16

Let D_4 be the symmetry group of a square. We denote by R the rotation by $\frac{\pi}{2}$ and by σ the reflection across the diagonal through the lower left and upper right corner. We write all group elements as $R^k\sigma^\ell$ for some k and ℓ. (Why is this possible and which values do k and ℓ take?)

a) Find all conjugacy classes.
   HINT: Determine $\sigma R \sigma$ first, this simplifies calculations a lot.

b) Determine all normal subgroups and the isomorphism types of the corresponding quotient groups (i.e. name known groups to which they are isomorphic).

c) Is D_4 isomorphic to a direct product of non-trivial subgrous?

We now determine all irreducible representations of D_4 (up to equivalence):

d) What are the dimensions of the irreducible representations?

e) Find all one dimensional irreducible representations.
   HINT: First consider irreducible representations of quotient groups, cf. Problem 9 and the remarks on (non-)faithful representations in Section 2.1.

f) Determine the character table and the remaining representation(s).

Problem 17

Let G be a finite group, $|G| = n$. We number the group elements, $G = \{g_j, j = 1 \dots n\}$, denote by m the number of conjugacy classes c (with n_c elements) and by p the number of non-equivalent irreducible representations Γ^i of G (with dimensions d_i).

Consider the matrix U with entries $u_{ja} = \sqrt{\frac{d_{i_a}}{n}}\,\Gamma^{i_a}(g_j)_{\mu_a\nu_a}$ with a triple $a = (i_a, \mu_a, \nu_a)$.

Employ the results of Sections 2.5 and 2.6 in order to solve the following problems.

a) Determine the dimensions of U and express the orthogonality relation for irreducible representations (Theorem 6) in terms of U.

b) Show:

   (i) $\sum_{i\le p} d_i \operatorname{tr}\left(\Gamma^i(g_j)\Gamma^i(g_k)^\dagger\right) = n\delta_{jk}$,

   (ii) $\sum_{g\in c} d_i\,\Gamma^i(g) = n_c\,\chi^i_c\,\mathbb{1}$ and

   (iii) $\sum_{i\le p} n_c\,\chi^i_c\,\overline{\chi^i_{c'}} = n\delta_{cc'}$.

c) Conclude that $m = p$.